

## STATISTICAL CAUSALITY AND EXTREMAL MEASURES

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**ABSTRACT.** In this paper we consider the concept of statistical causality in continuous time between flows of information, represented by filtrations. Then we relate the given concept of causality to the equivalent change of measure that plays an important role in mathematical finance. We give necessary and sufficient conditions, in terms of statistical causality, for extremality of measure in the set of martingale measures. Also, we have considered the extremality of measure which involves the stopping time and the stopped processes, and obtained similar results. Finally, we show that the concept of unique equivalent martingale measure is strongly connected to the given concept of causality and apply this result to the continuous market model.

### 1. Introduction

The results of this paper are mainly concerned with a connection between the concept of statistical causality, extremal measures and equivalent martingale measures.

The notion of causality is considered in continuous time which unifies the nonlinear Granger's causality with some related concepts. The Granger causality is focused on discrete time stochastic processes (time series). But, in econometric practise continuous time models are more and more frequent. In this paper we will discuss the continuous time processes because, the systems to which can be applied tests of causality, usually take place in continuous time (see [6–8, 14, 15, 18]). In the case of economy, it may be difficult to capture a relations of causality in discrete-time model and it may depend on the length of interval between each sampling. Construction of the methods for testing causality involving sampling at irregularly (or randomly) spaced times of observation will be possible if we are using a continuous time model. The internal

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consistency of economic theories and the statistical approach to causality analysis between stochastic processes evolve rapidly, therefore using the continuous time framework is very fruitful (see [2]).

The paper is organized as follows. After Introduction, in Section 2 we present a generalization of a causality concept “ $\mathbf{G}$  is a cause of  $\mathbf{E}$  within  $\mathbf{H}$ ” which is based on Granger’s definition of causality (see [8]).

The given concept of causality can be connected to the orthogonality of martingales (see [23]) and stable subspaces (see [19]). Also, definitions of the weak solutions and local weak solutions of the stochastic differential equations driven with semimartingales can be expressed in terms of the given concept of causality (see [18, 20]). Weak uniqueness of those solutions are equivalent with this notion. If the  $\sigma$ -algebra increases, the preservation of the martingale property is equivalent to the concept of causality as it is proved in [1].

In Section 3, that contains the main results, we consider the connection between the given concept of causality and the equivalent change of measure. Also, we establish equivalence between the concept of causality and extremality of the measure on the set of measures with respect to which the process  $M_t = P(A | \mathcal{G}_t)$ ,  $A \in (\mathcal{G}_\infty)$  is martingale. Similar theorem we proved for the bounded stopping time  $\tau$  and set of measures  $\mathcal{M}^\tau$  with respect to which the stopped process  $M_{t \wedge \tau} = P(A | \mathcal{G}_{t \wedge \tau})$  is martingale.

In Section 4 we show the connection between the given concept of causality and unique equivalent martingale measure. This directly leads to application, namely, we relate the completeness problem in the continuous market model to the notion of statistical causality.

## 2. Preliminaries and notation

A probabilistic model for a time-dependent system is described by  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  where  $(\Omega, \mathcal{F}, P)$  is a probability space and  $\{\mathcal{F}_t, t \in I, I \subseteq R^+\}$  is a “framework” filtration, i.e.,  $(\mathcal{F}_t)$  is a set of all events in the model up to and including time  $t$  and  $(\mathcal{F}_t)$  is a sub- $\sigma$ -algebra of  $\mathcal{F}$ . We can say that the information available at time  $t$  is described by  $(\mathcal{F}_t)$  which satisfies the “usual conditions” of right continuity and completeness.  $(\mathcal{F}_\infty)$  is the smallest  $\sigma$ -algebra containing all the  $(\mathcal{F}_t)$  (even if  $\sup I < +\infty$ ), i.e.,  $\mathcal{F}_\infty = \bigvee_{t \in I} \mathcal{F}_t$ .

An analogous notation will be used for filtrations  $\mathbf{H} = \{\mathcal{H}_t\}$ ,  $\mathbf{G} = \{\mathcal{G}_t\}$  and  $\mathbf{E} = \{\mathcal{E}_t\}$ . It is said that the filtration  $\mathbf{G}$  is a subfiltration of  $\mathbf{F}$  and written as  $\mathbf{G} \subseteq \mathbf{F}$ , if  $\mathcal{G}_t \subseteq \mathcal{F}_t$  for each  $t$ . The subfiltration  $\mathbf{G}$  represents the reduced information.

The natural filtration of the process  $X = \{X_t, t \in I\}$  is given by  $\mathbf{F}^X = \{\mathcal{F}_t^X, t \in I\}$ , where  $\mathcal{F}_t^X = \sigma\{X_u, u \in I, u \leq t\}$ , is the smallest  $\sigma$ -algebra with respect to which the random variables  $X_u, u \leq t$  are measurable.

Let  $\mathbf{E}, \mathbf{G}$  and  $\mathbf{H}$  be arbitrary filtrations. We can say that “ $\mathbf{G}$  is a cause of  $\mathbf{E}$  within  $\mathbf{H}$ ” if for every  $t$ ,  $(\mathcal{E}_\infty)$  and  $(\mathcal{G}_t)$  are conditionally independent with

respect to  $\mathcal{H}_t$ , i.e.,

$$(1) \quad \mathcal{E}_\infty \perp \mathcal{H}_t | \mathcal{G}_t.$$

The essence of (1) is that  $(\mathcal{G}_t)$  contains all the information from the  $(\mathcal{H}_t)$  needed for predicting  $(\mathcal{E}_\infty)$ . According to Corollary 2.1.1 in [18] (1) is equivalent to  $\mathcal{E}_\infty \perp \mathcal{H}_t \vee \mathcal{G}_t | \mathcal{G}_t$ . The last relation means that the condition  $\mathbf{E} \subseteq \mathbf{H}$  does not represent essential restriction. Thus, it was natural to introduce the following definition of causality between filtrations.

**Definition 1** (see [15] and [18]). It is said that  $\mathbf{G}$  is a cause of  $\mathbf{E}$  within  $\mathbf{H}$  relative to  $P$  (and written as  $\mathbf{E} \prec \mathbf{G}; \mathbf{H}; P$ ) if  $\mathcal{E}_\infty \subseteq \mathcal{H}_\infty$ ,  $\mathbf{G} \subseteq \mathbf{H}$  and if  $(\mathcal{E}_\infty)$  is conditionally independent of  $(\mathcal{H}_t)$  given  $(\mathcal{G}_t)$  for each  $t$ , i.e.,  $\mathcal{E}_\infty \perp \mathcal{H}_t | \mathcal{G}_t$  (i.e.,  $\mathcal{E}_u \perp \mathcal{H}_t | \mathcal{G}_t$  holds for each  $t$  and each  $u$ ), or

$$(2) \quad (\forall A \in \mathcal{E}_\infty) \quad P(A | \mathcal{H}_t) = P(A | \mathcal{G}_t).$$

If there is no doubt about  $P$ , we omit “relative to  $P$ ”.

Intuitively,  $\mathbf{E} \prec \mathbf{G}; \mathbf{H}; P$  means that  $(\mathcal{H}_t)$  does not provide additional information over  $(\mathcal{G}_t)$  for arbitrary  $t$ .

The definition similar to Definition 1 was first given in [14]: “It is said that  $\mathbf{G}$  entirely causes  $\mathbf{E}$  within  $\mathbf{H}$  relative to  $P$  (and written as  $\mathbf{E} \prec \mathbf{G}; \mathbf{H}; P$ ) if  $\mathbf{E} \subseteq \mathbf{H}$ ,  $\mathbf{G} \subseteq \mathbf{H}$  and if  $\mathcal{E}_\infty \perp \mathcal{H}_t | \mathcal{G}_t$  for each  $t$ ”. However, this definition contains the condition  $\mathbf{E} \subseteq \mathbf{H}$ , or equivalently  $\mathcal{E}_t \subseteq \mathcal{H}_t$  for each  $t$ , (instead of  $\mathcal{E}_\infty \subseteq \mathcal{H}_\infty$ ) which does not have intuitive justification. Since Definition 1 is more general than the definition given in [14], all results related to causality in the sense of Definition 1 will be true and in the sense of the definition from [14], when we add the condition  $\mathbf{E} \subseteq \mathbf{H}$  to them.

It should be mentioned that the definition of causality from [14] is equivalent to definition of strong global noncausality as given in [6]. So, the Definition 1 is a generalization of the notion of strong global noncausality.

If  $\mathbf{G}$  and  $\mathbf{H}$  are such that  $\mathbf{G} \prec \mathbf{G}; \mathbf{H}; P$ , we shall say that  $\mathbf{G}$  is its own cause within  $\mathbf{H}$  (compare with [14]). It should be noted that “ $\mathbf{G}$  is its own cause” sometimes occurs as a useful assumption in the theory of martingales and stochastic integration (see [1]). The concept of being “its own cause” is equivalent to the hypothesis  $(\mathcal{H})$  introduced in [1]. It also, should be mentioned that the notion of subordination (as introduced in [21]) is equivalent to the notion of being “its own cause” as defined here.

If  $\mathbf{G}$  and  $\mathbf{H}$  are such that  $\mathbf{G} \prec \mathbf{G}; \mathbf{G} \vee \mathbf{H}$  (where  $\mathbf{G} \vee \mathbf{H}$  is a family determined by  $(\mathcal{G} \vee \mathcal{H})_t = \mathcal{G}_t \vee \mathcal{H}_t$ ), we shall say that  $\mathbf{H}$  does not cause  $\mathbf{G}$ . It is clear that the interpretation of Granger-causality is now that  $\mathbf{H}$  does not cause  $\mathbf{G}$  if  $\mathbf{G} \prec \mathbf{G}; \mathbf{G} \vee \mathbf{H}$  (see [14]).

These definitions can be applied to stochastic process if we consider corresponding induced filtrations. For example,  $(\mathcal{F}_t)$ -adapted stochastic process  $X_t$  is its own cause if  $(\mathcal{F}_t^X)$  is its own cause within  $(\mathcal{F}_t)$ , i.e., if

$$\mathbf{F}^X \prec \mathbf{F}^X; \mathbf{F}; P.$$

The following result shows that a process  $X$  which is its own cause is completely described by its behavior relative to  $\mathbf{F}^X$ .

**Proposition 2.1** ([18]).  $X = \{X_t, t \in [0, T]\}$  is a Markov process relative to filtration  $\mathbf{F} = \{\mathcal{F}_t, t \in [0, T]\}$  on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  if and only if  $X$  is a Markov process (relative to  $\mathbf{F}^X$ ) and the process is its own cause within  $\mathbf{F}$  with respect to  $P$ .

As a consequence, Brownian motion  $\mathbf{W} = \{W_t, t \in I\}$  with respect to the filtration  $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$  on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  is its own cause within  $\mathbf{F} = \{\mathcal{F}_t, t \in I\}$  with respect to the probability  $P$ .

We will need later the following definition that concerns the notion of extremal measure, which is very important in the theory of stochastic processes.

**Definition 2** ([11]). A probability measure  $P$  of  $\mathcal{M}$  is called extremal if whenever  $P = \alpha Q + (1 - \alpha)R$  with  $0 < \alpha < 1$  and  $Q, R \in \mathcal{M}$ , then  $P = Q = R$ .

In other words, for extremal point of a certain convex set of probability measures there exists no decomposition other than the trivial one. If  $\mathcal{M}$  consists of a single element  $P$  (singleton) this measure is extremal and this triviality is fundamental in applications (see [5, 9]).

Also, we consider the concept of absolutely continuous and locally absolutely continuous measures (see for example [10, 13, 22]). Let  $P$  be a measure on the  $\sigma$ -algebra  $(\mathcal{F}_t)$ . We say that  $P$  is absolutely continuous if  $P(N) = 0$  for every evanescent set  $N$ . We say that a measure  $Q$  is locally absolutely continuous with respect to a measure  $P$  if  $Q(t) \ll P(t)$  for every  $t$  where  $Q(t)$  is the restriction of  $Q$  and  $P(t)$  is the restriction of  $P$  to  $(\mathcal{F}_t)$ . We shall denote this relation by  $Q \ll^{\text{loc}} P$ . If  $Q \ll^{\text{loc}} P$  and  $P \ll^{\text{loc}} Q$ , then we shall say that  $P$  and  $Q$  are locally equivalent. We shall denote it by  $Q \sim^{\text{loc}} P$ .

The statistical concept of causality is focused on measurements taken over time, and how they may influence one another. In many situation we observe some system up to some random time, for example till the time when something happens for the first time. Now, we extend Definition 1 from fixed times to stopping times, i.e., we define causality using the truncated filtrations, which is specially applicable for the truncated (stopped) processes. Precisely, we now give the characterization of causality using stopping times - a class of random variables that plays the essential role in the theory of martingales.

The  $\sigma$ -field  $(\mathcal{F}_\tau) = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t\}$  is usually interpreted as the set of events that occurs before or at time  $\tau$ .

If  $X$  is a stochastic process and if  $\tau$  is a stopping time, we define the process stopped at time  $\tau$ , denoted by  $X^\tau$ , by

$$X_t^\tau = X^\tau(t) = \{X_{\tau \wedge t} \mid t \in \mathbf{R}_+\}.$$

Note that if  $X$  is adapted and cadlag and if  $\tau$  is a stopping time, then

$$X_t^\tau = X_{t \wedge \tau} = X_t \chi_{\{t < \tau\}} + X_\tau \chi_{\{t \geq \tau\}}$$

is also adapted. A martingale stopped at a stopping time is still a martingale. Lets mention that the truncated filtration  $(\mathcal{F}_{t \wedge \tau})$  is defined as

$$\mathcal{F}_{t \wedge \tau} = \mathcal{F}_t \cap \mathcal{F}_\tau = \begin{cases} \mathcal{F}_t, & t < \tau, \\ \mathcal{F}_\tau, & t \geq \tau. \end{cases}$$

Natural filtration for the stopped martingale  $X_{t \wedge \tau}$  is  $\mathbf{F}^{X^\tau} = (\mathcal{F}_{t \wedge \tau}^X)$ , with respect to which the process  $X_{t \wedge \tau}$  is completely described. So, we can use the definition of causality in continuous time which involves the stopping times.

**Definition 3** ([16]). Let  $\mathbf{F} = \{\mathcal{F}_t\}$ ,  $\mathbf{H} = \{\mathcal{H}_t\}$  and  $\mathbf{E} = \{\mathcal{E}_t\}$ ,  $t \in I$ , be given filtrations on the probability space  $(\Omega, \mathcal{F}, P)$  and let  $\tau$  be a stopping time with respect to filtration  $\mathbf{E}$ . The filtration  $\mathbf{H}^\tau$  entirely causes  $\mathbf{E}^\tau$  within  $\mathbf{F}^\tau$  relative to  $P$  (and written as  $\mathbf{E}^\tau \ll \mathbf{H}^\tau; \mathbf{F}^\tau; P$ ) if  $\mathbf{E}^\tau \subseteq \mathbf{F}^\tau$ ,  $\mathbf{H}^\tau \subseteq \mathbf{F}^\tau$  and if  $\mathcal{E}_\tau$  is conditionally independent of  $\mathcal{F}_{t \wedge \tau}$  given  $\mathcal{H}_{t \wedge \tau}$  for each  $t$ , i.e.,  $(\forall t) \mathcal{E}_\tau \perp \mathcal{F}_{t \wedge \tau} \mid \mathcal{H}_{t \wedge \tau}$ , or

$$(3) \quad (\forall t \in I)(\forall A \in \mathcal{E}_\tau) \quad P(A \mid \mathcal{F}_{t \wedge \tau}) = P(A \mid \mathcal{H}_{t \wedge \tau}).$$

The concept of causality given in Definition 3 is defined up to some specified stopping time  $\tau$ . It includes the stopped filtrations. The relation (3) does not consider the causality up to infinite horizon, so it does not imply (2).

### 3. Causality and extremal measures

The equivalent changes of measure play an important role in arbitrage pricing theory. Two important concepts in the mathematical theory of contingent claim pricing are the absence of arbitrage and the notion of risk neutral pricing. Both of these notions are expressed in mathematical terms using the notion of equivalent change of measure.

Lets consider a set of right continuous modifications of processes

$$(4) \quad H = \{M_t \mid M_t = P(A \mid \mathcal{G}_t), A \in \mathcal{G}_\infty\}.$$

Next theorem gives relation between the equivalent changes of measure and the given concept of causality.

**Theorem 3.1.** *Let  $\mathbf{F}$  and  $\mathbf{G}$  be filtrations on  $(\Omega, \mathcal{F})$ . Suppose that set  $H$  is of the form (4) and let  $\mathcal{M}$  be a set of probability measures  $Q$  on  $\mathcal{F}$  for which  $Q = P$  on  $\mathcal{F}_0$  and elements of  $H$  are  $(\mathcal{F}_t, Q)$ -martingales. Then, the following statements are equivalent.*

- (a)  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$ .
- (b)  $Q \in \mathcal{M}; Q \ll^{\text{loc}} P \implies Q = P$ .
- (c)  $Q \in \mathcal{M}; Q \ll P \implies Q = P$ .
- (d)  $Q \in \mathcal{M}; Q \sim P \implies Q = P$ .
- (e)  $Q \in \mathcal{M}; Q \sim P; \frac{dQ}{dP} \in \mathcal{G}_\infty \implies Q = P$ .

*Proof.* (a)  $\implies$  (b) Let  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$  holds, i.e.,  $\forall A \in \mathcal{G}_\infty, P(A \mid \mathcal{F}_t) = P(A \mid \mathcal{G}_t)$ . This is equivalent to the hypothesis  $(\mathcal{H})$  in [1], so elements of the set  $H$

are  $(\mathcal{F}_t, P)$ -martingales. Let for the measure  $Q \in \mathcal{M}$ ,  $Q \ll^{\text{loc}} P$  holds. According to Proposition 6.23 in [13], every element of the set  $H$  is  $(\mathcal{F}_t, Q)$ -martingale.

Since,  $Q \ll^{\text{loc}} P$ , the right regular version of the density process  $L_t$  is just the cadlag modification of  $E(L_\infty | \mathcal{F}_t)$  where  $L_\infty = \frac{dQ}{dP}$  is the Radon-Nikodym derivative. By an assumption of the theorem  $Q = P$  on  $\mathcal{F}_0$ , so  $L_0 = 1$ . From the definition of the conditional expectation

$$\int_F L(t)dP = \int_F L(\infty)dP = Q(F), \quad F \in \mathcal{F}_t.$$

Then for  $M_t \in H$  it follows that  $ML$  is an  $(\mathcal{F}_t, P)$ -martingale. Indeed, for  $s < t, F \in \mathcal{F}_s$

$$\begin{aligned} \int_{\mathbf{F}} M(t)L(t)dP &= \int_{\mathbf{F}} M(t)\frac{dQ}{dP}dP = \int_{\mathbf{F}} M(t)dQ = \int_F M(s)dQ \\ &= \int_F M(s)L(s)dP, \end{aligned}$$

where  $M_t$  is  $(\mathcal{F}_t, Q)$ -martingale. So,

$$M_t L_t = E_P(M_\infty L_\infty | \mathcal{F}_t) = E_P(M_\infty \frac{dQ}{dP} | \mathcal{F}_t) = E_Q(M_\infty | \mathcal{F}_t) = M_t.$$

Hence  $L = 1$ . This implies  $Q = P$ .

(b) $\implies$ (c)  $\implies$  (d)  $\implies$  (e) is trivial.

(e)  $\implies$  (a) Suppose that  $Q \in \mathcal{M}; Q \sim P; \frac{dQ}{dP} \in \mathcal{G}_\infty \implies Q = P$  holds. Hence  $P \in \mathcal{M}$ , so all elements of the set  $H$  are  $(\mathcal{F}_t, P)$ -martingales, i.e.,

$$\begin{aligned} E(M_\infty | \mathcal{F}_t) &= M_t, \\ E(P(A | \mathcal{G}_\infty) | \mathcal{F}_t) &= P(A | \mathcal{G}_t), \\ E(E(\chi_A | \mathcal{G}_\infty) | \mathcal{F}_t) &= P(A | \mathcal{G}_t), \\ P(A | \mathcal{F}_t) &= P(A | \mathcal{G}_t), \quad A \in \mathcal{G}_\infty, \end{aligned}$$

where  $\chi_A$  is  $(\mathcal{G}_\infty)$ -measurable indicator function of the set  $A$ . Obviously,  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$  holds. □

The concept of extremal measures is very important in applications. In [18] it is proved that for the weak solution  $(\Omega, \mathcal{F}, \mathcal{F}_t, P, X_t, Z_t)$  of the stochastic differential equation driven with semimartingale  $Z_t$ , the measure  $P$  is extremal if and only if  $\mathbf{F}^{Z, X} \ll \mathbf{F}^{Z, X}; \mathbf{F}; P$  holds. Also, in [14] it is shown that the causality concept is closely connected to extremal solution of the martingale problem.

Now we consider a set  $H$  of the form (4), where  $\mathcal{M}$  is a set of probability measures  $Q$  on  $\mathcal{F}$  for which  $Q = P$  on  $\mathcal{F}_0$  and elements of  $H$  are  $(\mathcal{F}_t, Q)$ -martingales. Then we have that the following result holds.

**Theorem 3.2.** *The measure  $P \in \mathcal{M}$  is extremal measure in the set  $\mathcal{H}$  if and only if  $\mathbf{G}$  is its own cause within  $\mathbf{F} = \{\mathcal{F}_t\}$ , i.e., if and only if  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$  holds.*

*Proof.* Suppose that  $P$  is an extremal measure in the set  $\mathcal{M}$ . Then elements of the set  $H$  defined by (3.3), are  $(\mathcal{F}_t, P)$ -martingales, so

$$(5) \quad E(M_\infty | \mathcal{F}_t) = M_t, \quad \forall M_t.$$

Since  $\chi_A$  is  $(\mathcal{G}_\infty)$ -measurable indicator function of the set  $A$ , from  $M_t \in H$  it follows that

$$(6) \quad \begin{aligned} E(M_\infty | \mathcal{F}_t) &= E(P(A | \mathcal{G}_\infty) | \mathcal{F}_t) = E(E(\chi_A | \mathcal{G}_\infty) | \mathcal{F}_t) \\ &= E(\chi_A | \mathcal{F}_t) = P(A | \mathcal{F}_t). \end{aligned}$$

According to (5) and (6), for all  $A \in \mathcal{G}_\infty$  we have

$$\begin{aligned} E(M_\infty | \mathcal{F}_t) &= M_t, \\ P(A | \mathcal{F}_t) &= P(A | \mathcal{G}_t) \quad \forall A \in \mathcal{G}_\infty. \end{aligned}$$

So,  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$  holds.

Conversely, suppose that  $\mathbf{G} \ll \mathbf{G}; \mathbf{F}; P$  holds. Also, suppose that for measure  $P$  holds  $P = aP_1 + (1 - a)P_2$  where  $a \in (0, 1)$  and the measures  $P_1, P_2 \in \mathcal{M}$ . Following the idea in [11] (the proof of Proposition 11.14), as  $P_2 \geq 0$  obviously  $P_1 \ll P$ , so one can define the Radon-Nikodym derivative  $L_\infty = \frac{dP_1}{dP}$  and  $L_t$  is the cadlag modification of  $E(L_\infty | \mathcal{F}_t)$ . By assumption of the theorem it follows that  $L_0 = 1$ .

From  $P_1 \in \mathcal{M}$  and  $P_1 \ll P$  it follows that elements of the set  $H$  are  $(\mathcal{F}_t, P_1)$ -martingales, i.e.,  $M_t = E_{P_1}(M_\infty | \mathcal{F}_t)$ . Obviously, for  $M_t \in H$ , the process  $(ML)_t$  is a  $(\mathcal{F}_t, P)$ -martingale, i.e., for  $s < t$  and  $F \in \mathcal{F}_s \subseteq \mathcal{F}_t$  we have

$$\begin{aligned} \int_F M(t)L(t)dP &= \int_F M(t)\frac{dQ}{dP}dP = \int_F M(t)dQ = \int_F M(s)dQ \\ &= \int_F M(s)\frac{dQ}{dP}dP = \int_F M(s)L(s)dP \end{aligned}$$

and we have

$$M_tL_t = E_P(M_\infty L_\infty | \mathcal{F}_t) = E_P(M_\infty \frac{dP_1}{dP} | \mathcal{F}_t) = E_{P_1}(M_\infty | \mathcal{F}_t) = M_t.$$

Hence,  $L_t = 1$ . Then  $L_t = L_0 = 1$ , so  $P = P_1$ . In the similar way, we can prove that  $P = P_2$ , so it follows that  $P$  is an extremal measure in the set  $\mathcal{M}$ .  $\square$

Let now consider a set

$$(7) \quad K = \{X_t | X_t = P(A | \mathcal{F}_t^X), A \in \mathcal{F}_\infty^X\}$$

and let  $\mathcal{K}$  be the set of all measures  $Q$  where  $P = Q$  on  $(\mathcal{F}_0)$  and under which elements of the set  $K$  are  $(\mathcal{F}_t, P)$ -martingales. As a consequence of Theorem 3.2 we have that the next lemma holds.

**Lemma 3.3.** *The measure  $P \in \mathcal{K}$  is extremal measure in the set  $\mathcal{K}$  if and only if  $\mathbf{F}^X$  is its own cause within  $\mathbf{F} = \{\mathcal{F}_t\}$ , i.e., if and only if  $\mathbf{F}^X \ll \mathbf{F}^X; \mathbf{F}; P$  holds.*

Let  $\tau$  be a  $(\mathcal{G}_t)$ -stopping time and  $H^\tau$  be a set of right continuous martingales of the form  $M_{t \wedge \tau} = M^\tau(t) = P(A | \mathcal{G}_{t \wedge \tau})$  for all  $A \in \mathcal{G}_\tau$ , i.e.,

$$(8) \quad H^\tau = \{M_{t \wedge \tau} \mid M_{t \wedge \tau} = P(A | \mathcal{G}_{t \wedge \tau}), A \in \mathcal{G}_\tau\}.$$

Suppose that  $\mathcal{M}^\tau$  is the set of all measures  $Q$  where  $P = Q$  on  $\mathcal{F}_0$  and under which elements of the set  $H^\tau$  are  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingales.

The following result shows an equivalence between the given concept of causality and extremality of the measure on the set  $\mathcal{M}^\tau$ .

**Theorem 3.4.** *Let  $\tau$  be a bounded  $(\mathcal{G}_t)$ -stopping time. Then, the measure  $P \in \mathcal{M}^\tau$  is extremal measure in the set  $\mathcal{M}^\tau$  if and only if  $\mathbf{G}^\tau$  is its own cause within  $\mathbf{F}^\tau = \{\mathcal{F}_{t \wedge \tau}\}$ , i.e.,  $\mathbf{G}^\tau \prec \mathbf{G}^\tau; \mathbf{F}^\tau; P$  holds.*

*Proof.* Suppose that  $P$  is an extremal measure on the set of measures  $\mathcal{M}^\tau$  and all elements of the set  $H^\tau$  are  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingales. So, for all  $M^\tau(t) = M_{t \wedge \tau} \in H^\tau$  we have

$$(9) \quad E_P(M^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = M_{t \wedge \tau}.$$

For all  $A \in \mathcal{G}_\tau$  where  $\chi_A$  is indicator function of the set  $A$  is

$$(10) \quad \begin{aligned} E_P(M^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) &= E_P(P(A \mid \mathcal{G}_\tau) \mid \mathcal{F}_{t \wedge \tau}) = E_P(E_P(\chi_A \mid \mathcal{G}_\tau) \mid \mathcal{F}_{t \wedge \tau}) \\ &= E_P(\chi_A \mid \mathcal{F}_{t \wedge \tau}) = P(A \mid \mathcal{F}_{t \wedge \tau}), \end{aligned}$$

where  $\chi_A$  is  $(\mathcal{G}_\tau)$ -measurable function. Therefore, according to (9) and (10)

$$\begin{aligned} E_P(M^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) &= M_{t \wedge \tau}, \\ P(A \mid \mathcal{F}_{t \wedge \tau}) &= P(A \mid \mathcal{G}_{t \wedge \tau}), \quad \forall A \in \mathcal{G}_\tau \end{aligned}$$

and the causality relation holds, i.e.,  $\mathbf{G}^\tau \prec \mathbf{G}^\tau; \mathbf{F}^\tau; P$ .

Conversely, assume that  $\mathbf{G}^\tau \prec \mathbf{G}^\tau; \mathbf{F}^\tau; P$  holds and the measure  $P \in \mathcal{M}^\tau$ . We need to prove that the measure  $P$  is extremal in the set  $\mathcal{M}^\tau$ . Suppose that

$$P = aQ + (1 - a)R,$$

where  $Q, R \in \mathcal{M}^\tau$  and  $a \in (0, 1)$ . Measures  $Q, R \in \mathcal{M}^\tau$  so elements of the set  $H^\tau$  are  $(\mathcal{F}_{t \wedge \tau}, Q)$ -martingales. As  $R \geq 0$  obviously  $Q \ll P$  so one can define the Radon-Nikodym derivative

$$(11) \quad L(t \wedge \tau) = L(t)\chi\{t < \tau\} + L(\tau)\chi\{t \geq \tau\}.$$

Now we differ two cases.

1. If  $t < \tau$ , then equality (11) is of the form  $L(t \wedge \tau) = \frac{dQ(t)}{dP}$ . As the stopped martingales are martingales,  $L^\tau = L(t \wedge \tau)$  is a  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingale, i.e.,

$$E_P(L(t \wedge \tau) \mid \mathcal{F}_s) = L(s \wedge \tau).$$

So, if  $s < t$  and  $F \in \mathcal{F}_s \subseteq \mathcal{F}_t$ , we have

$$\begin{aligned} Q(s)(F) &= \int_F \frac{dQ(s)}{dP} dP = \int_F L(s \wedge \tau) dP = \int_F L(t \wedge \tau) dP \\ &= \int_F \frac{dQ(t)}{dP} dP = Q(t)(F). \end{aligned}$$



Then  $(ML)^\tau$  is a  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingale because

$$\begin{aligned}
 \int_F M(t \wedge \tau) L(t \wedge \tau) dP &= \int_F M(t \wedge \tau) \frac{dQ(t)}{dP} dP = \int_F M(t \wedge \tau) dQ(t) \\
 &= \int_F M(s \wedge \tau) dQ(s) = \int_F M(s \wedge \tau) \frac{dQ(s)}{dP} dP \\
 (12) \qquad &= \int_F M(s \wedge \tau) L(s \wedge \tau) dP.
 \end{aligned}$$

So,  $E_P(M^\tau(\infty)L^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = M_{t \wedge \tau} L_{t \wedge \tau}$ . On the other hand we have

$$\begin{aligned}
 M_{t \wedge \tau} L_{t \wedge \tau} &= E_P(M^\tau(\infty)L^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = E_P(M^\tau(\infty) \frac{dQ}{dP} \chi_{\{\tau < \infty\}} \mid \mathcal{F}_{t \wedge \tau}) \\
 &= E_Q(M^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = M_{t \wedge \tau}.
 \end{aligned}$$

Hence,  $L_{t \wedge \tau} = L_0 = 1$ . Follows  $P = Q$ . Similar, we prove that  $P = R$ .

2. On the other hand let  $\tau$  be a bounded stopping time  $\tau \leq t$ . Then, equality (11) becomes  $L(t \wedge \tau) = \frac{dQ(\tau)}{dP} = L(\tau)$ . Since  $Q \ll P$ , it follows  $Q \stackrel{\text{loc}}{\ll} P$ , so  $L(\tau)$  is Radon-Nykodim derivative  $\frac{dQ}{dP}$  on the filtration  $(\mathcal{F}_\tau)$ .

Also, if  $\tau$  is a bounded stopping time and  $\tau \leq t$ , then by the Optional Sampling Theorem, since  $L$  is a martingale, we have

$$L(\tau) = E_P(L(t) \mid \mathcal{F}_\tau).$$

That is, for  $F \in \mathcal{F}_\tau \subseteq \mathcal{F}_t$  it follows

$$\int_F L(\tau) dP = \int_F L(t) dP = Q(t)(F) = Q(F).$$

If  $F \in \mathcal{F}_{t \wedge \tau}$ , and  $r \geq t$ , then  $(ML)^\tau$  is  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingale. Indeed, due to (12) we have

$$\begin{aligned}
 \int_F M(t \wedge \tau) L(t \wedge \tau) dP &= \int_F M(t \wedge \tau) L(\tau) dP = \int_F M(t \wedge \tau) L(t) dP \\
 &= \int_F M(t \wedge \tau) \frac{dQ(t)}{dP} dP = \int_F M(t \wedge \tau) dQ \\
 &= \int_F M(r \wedge \tau) dQ = \int_F M(r \wedge \tau) L(r \wedge \tau) dP.
 \end{aligned}$$

Now, considering the fact that  $M^\tau$  is  $(\mathcal{F}_{t \wedge \tau}, P)$  martingale, we have

$$\begin{aligned}
 M_{t \wedge \tau} L_{t \wedge \tau} &= E_P(M^\tau(\infty)L^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = E_P(M^\tau(\infty) \frac{dQ}{dP} \mid \mathcal{F}_{t \wedge \tau}) \\
 &= E_Q(M^\tau(\infty) \mid \mathcal{F}_{t \wedge \tau}) = M_{t \wedge \tau}.
 \end{aligned}$$

Hence,  $L_{t \wedge \tau} = L_0 = 1$ . Follows  $P = Q$ . Similar, we prove that  $P = R$ . So, measure  $P$  is extremal measure in the set  $\mathcal{M}^\tau$ .  $\square$

Let consider a set

$$(13) \qquad K^\tau = \{X_t^\tau \mid X_t^\tau = X_{t \wedge \tau} = P(A \mid \mathcal{F}_{t \wedge \tau}^X), A \in \mathcal{F}_\tau^X\}$$

and let  $\mathcal{K}^\tau$  be the set of all measures  $Q$  where  $P = Q$  on  $(\mathcal{F}_0)$  and under which elements of the set  $K^\tau$  are  $(\mathcal{F}_{t \wedge \tau}, P)$ -martingales. As a consequence of Theorem 3.4, we have the following result to hold.

**Lemma 3.5.** *Let  $\tau$  be a bounded  $(\mathcal{F}_t^X)$ -stopping time. Then, the measure  $P \in \mathcal{K}^\tau$  is extremal measure in the set  $\mathcal{K}^\tau$  if and only if  $\mathbf{F}^{X^\tau}$  is its own cause within  $\mathbf{F}^\tau = \{\mathcal{F}_{t \wedge \tau}\}$ , i.e.,  $\mathbf{F}^{X^\tau} \llcorner \mathbf{F}^{X^\tau}; \mathbf{F}^\tau; P$  holds.*

#### 4. Example

In this section we give an example about the connections between the given concept of statistical causality and the equivalent martingale measure.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space,  $T < \infty$  be a fixed time horizon, and let  $\mathbf{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}$  be a filtration that satisfies the usual conditions with  $(\mathcal{F}_0)$  containing only  $\Omega$  and the null sets of  $P$  and  $\mathcal{F}_T = \mathcal{F}$  containing all subsets of  $\Omega$ . The filtration describes how information is revealed to the investors; they have access to past and current price information only.

Let  $S = \{S_t; 0 \leq t \leq T\}$  be a vector valued stochastic process whose components  $S^0, S^1, \dots, S^k$  are adapted, right continuous with left limits and strictly positive.  $S_t^k$  represents the time  $t$  value of the  $k$ -th security. The discounted price process (the price of security “corrected for inflation”) is  $\tilde{S}_t = \frac{S_t^k}{S_t^0}$ .

A probability measure  $Q \sim P$  is an equivalent martingale measure (EMM) for  $S$  if the discounted price processes  $\tilde{S}^k, 1 \leq k \leq d$  are  $(\mathbf{F}, Q)$ -martingales (see [5]). According to Theorem 3.1 in [17]

$$(14) \quad \mathbf{F}^{\tilde{S}} \llcorner \mathbf{F}^{\tilde{S}}; \mathbf{F}; Q$$

holds, or the discounted price process  $\tilde{S}_t$  is its own cause within  $\mathbf{F}$  relative to  $Q$ . Let  $\mathcal{M}(\tilde{S})$  denote the set of all probability measures  $Q$  on  $(\Omega, \mathcal{F})$  (not necessarily equivalent to  $P$ ) that makes the given stochastic process  $\tilde{S}$  a  $(\mathcal{F}_t, Q)$ -martingale, and let  $\mathcal{M}_e(\tilde{S})$  denote the set consisting of all extreme points of  $\mathcal{M}(\tilde{S})$ .

A trading strategy is defined to be adapted, predictable stochastic process  $\Phi = \{\Phi_t; 0 \leq t \leq T\}$  and represents the number of units of security in the portfolio at time  $t$ . The value of the portfolio at time  $t$  equals  $V_t = V_t^\Phi$ . The strategy is called self-financing, if for all  $t \in [0, T]$  we have  $V_t^\Phi = V_0^\Phi + G_t^\Phi$ , where  $G_t^\Phi$  is the discounted gains process and represent the discounted net profit or loss due to the transactions by the investor. A self-financing trading strategy is called  $Q$ -admissible if the discounted gains process  $G_t^\Phi$  is a  $Q$ -martingale.

A contingent claim  $X$  is an  $(\mathcal{F}_T)$ -measurable random variable, to be interpreted as the payoff of some financial claims. Such a claim is said to be attainable if there exists an admissible trading strategy  $\Phi$  such that  $V_T^\Phi = X$ , where  $V_T^\Phi$  is the value of the portfolio.

For the discounted price process  $\tilde{S}$ , in [3] is established the condition of no free lunch with vanishing risk (NFLVR). For  $\tilde{S}$ -integrable, predictable process

$H = (H_t)_{0 \leq t \leq T}$  there exists a constant  $M > 0$  such that  $\int_0^t H_u d\tilde{S}_u \geq -M$ . Then

$$N = \left\{ \int_0^T H_t dS_t, H \text{ is admissible} \right\}$$

define a set of random variables uniformly bounded from below. The set  $C$  contains the random variables

$$C = [N - L_+^0(\Omega, \mathcal{F}, P)] \cap L^\infty(\Omega, \mathcal{F}, P),$$

where  $L_+^0(\Omega, \mathcal{F}, P)$  represents the set of non-negative measurable functions.

A locally bounded semimartingale  $\tilde{S}$  satisfies the no free lunch with vanishing risk (NFLVR) condition if

$$\bar{C} \cap L^\infty(\Omega, \mathcal{F}, P) = \{0\},$$

where  $\bar{C}$  denotes the closure of  $C$ . Lets mention that NFLVR guarantees the existence of an equivalent martingale measure and thus opens the way for applications from martingale theory. An important limitation is that the price process is locally bounded. But we deals with continuous price processes and this condition is satisfied (see [4]). If  $\tilde{S}$  is a bounded real valued semi-martingale and there exists a measure  $Q$  equivalent to  $P$  under which  $\tilde{S}$  is a martingale. According to condition (14), Lemma 3.3 and Theorem 6 in [4]  $\tilde{S}$  satisfy the condition of no free lunch with vanishing risk if and only if the discounted price process  $\tilde{S}_t$  is its own cause within  $\mathbf{F}$  relative to  $Q$ .

As another example of application of causality to financial markets we consider its connection to completeness of the market. Due to Proposition 3.4 in [12], Lemma 3.3 and condition (14) the continuous financial market is complete if and only if the discounted price process  $\tilde{S}$  is its own cause within filtration of the market  $(\mathcal{F}_t)$ .

## References

- [1] P. Bremaud and M. Yor, *Changes of filtrations and of probability measures*, Z. Wahrsch. Verw. Gebiete **45** (1978), no. 4, 269–295.
- [2] F. Comte and E. Renault, *Noncausality in continuous time models*, Econometric Theory **12** (1996), no. 2, 215–256.
- [3] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann. **300** (1994), no. 3, 463–520.
- [4] A. Downarowicz, *The first fundamental theorem of asset pricing*, Revista de Economia Financiera **21** (2010), 23–35.
- [5] R. J. Elliott and P. E. Kopp, *Mathematics of Financial Markets*, Second edition, Springer Finance, Springer-Verlag, New York, 2005.
- [6] J.-P. Florens and D. Fougère, *Noncausality in continuous time*, Econometrica **64** (1996), no. 5, 1195–1212.
- [7] J. B. Gill and L. Petrović, *Causality and stochastic dynamic systems*, SIAM J. Appl. Math. **47** (1987), no. 6, 1361–1366.
- [8] C. W. J. Granger, *Investigating causal relations by econometric models and cross spectral methods*, Econometrica **37** (1969), no. 3, 424–438.

- [9] J. M. Harrison and S. R. Pliska, *A stochastic calculus model of continuous trading: complete markets*, Stochastic Process. Appl. **15** (1983), no. 3, 313–316.
- [10] S. W. He, J. G. Wang, and J. A. Yan, *Semimartingale theory and stochastic calculus*, Kexue Chubanshe (Science Press), Beijing, 1992.
- [11] J. Jacod, *Calcul stochastique et problèmes de martingales*, Lecture Notes in Mathematics, 714, Springer, Berlin, 1979.
- [12] A. Lyasoff, *The two fundamental theorems of asset pricing for a class of continuous-time financial markets*, Math. Finance **24** (2014), no. 3, 485–504.
- [13] P. Medvegyev, *Stochastic Integration Theory*, Oxford Graduate Texts in Mathematics, **14**, Oxford University Press, Oxford, 2007.
- [14] P. A. Mykland, *Statistical Causality*, Report No.14. University of Bergen, (1986), 1–26.
- [15] Lj. Petrović, *Causality and Markovian representations*, Statist. Probab. Lett. **29** (1996), no. 3, 223–227.
- [16] Lj. Petrović, S. Dimitrijević, and D. Valjarević, *Granger causality and stopping times*, Lith. Math. J. **56** (2016), 410–416.
- [17] Lj. Petrović and D. Stanojević, *Some Models of Causality and Weak Solutions of Stochastic Differential Equations with Driving Semimartingales*, Facta Univ. Ser. Math. Inform. **5**, (2005), 103–112.
- [18] ———, *Statistical Causality, Extremal Measures and Weak Solutions of Stochastic Differential Equations with Driving Semimartingales*, J. Math. Model. Algor. **9** (2010), 113–128.
- [19] Lj. Petrović and D. Valjarević, *Statistical Causality and stable subspaces of  $H^p$* , Bull. Aust. Math. Soc. **88** (2013), 17–25.
- [20] ———, *Statistical Causality and Local Solutions of the Stochastic Differential Equations Driven with Semimartingales*, Lecture Notes in Computer Science, Springer, 2015.
- [21] Y. A. Rozanov, *Innovation Processes*, translated from the Russian, preface by translation editor A. V. Balakrishnan, V. H. Winston & Sons, Washington, DC, 1977.
- [22] A. N. Shiryaev, *Absolute continuity and singularity of probability measures in functional spaces*, in Proceedings of the International Congress of Mathematicians (Helsinki, 1978), 209–225, Acad. Sci. Fennica, Helsinki.
- [23] D. Valjarević and Lj. Petrović, *Statistical causality and orthogonality of local martingales*, Statist. Probab. Lett. **82** (2012), no. 7, 1326–1330.

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