

ON RADIAL OSCILLATION OF ENTIRE SOLUTIONS TO NONHOMOGENEOUS ALGEBRAIC DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we mainly investigate the properties of the solutions to a type of nonhomogeneous algebraic differential equation in an angular domain. It includes the Borel directions of the solutions, the width of angular domains in which the solutions take its order and the measure of radial distributions of Julia sets of the solutions.

1. Introduction

In this paper, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory; see [6, 7, 11, 20, 22]. Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [4, 25]. Let f be a meromorphic function in the whole complex plane. We use $\sigma(f)$ and $\mu(f)$ to denote the order and lower order of f respectively; see [20, p. 10] for the definitions.

Let $\Lambda = \{(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n)\}$, λ_j is a nonnegative integer and $0 \leq j \leq n < \infty$, be an index set with a finite cardinal number and let

$$Q_d(z, f) = \sum_{\lambda \in \Lambda} a_\lambda f^{\lambda_0} (f')^{\lambda_1} \dots (f^{(n)})^{\lambda_n}$$

be a polynomial of f and its derivatives with degree d and meromorphic function coefficients $a_\lambda(z)$, where $d := \deg(Q_d(z, f)) = \max_{\lambda \in \Lambda} \sum_{j=0}^n \lambda_j$. In the sequel, we simply call $Q_d(z, f)$ a differential polynomial of f with degree d .

Suppose that $0 \leq \alpha < \beta \leq 2\pi$, we set

$$\Omega(\alpha, \beta) = \{z \in \mathbb{C} : \arg z \in (\alpha, \beta)\}, \quad \Omega(\alpha, \beta, r) = \{z : z \in \Omega(\alpha, \beta), |z| < r\}$$

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and denote by $\overline{\Omega}(\alpha, \beta)$ the closure of $\Omega(\alpha, \beta)$. Let $g(z)$ be analytic on the angle $\overline{\Omega}(\alpha, \beta)$. We define the order of g on $\Omega(\alpha, \beta)$ by

$$\sigma_{\alpha, \beta}(g) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, \Omega(\alpha, \beta), g)}{\log r},$$

where $M(r, \Omega(\alpha, \beta), g) = \sup_{\alpha \leq \theta \leq \beta} |g(re^{i\theta})|$. If $g(z)$ is analytic on \mathbb{C} , the order $\sigma(g)$ of g satisfies $\sigma(g) \geq \sigma_{\alpha, \beta}(g)$. Moreover, the sectorial order $\sigma_{\theta, \varepsilon}(g)$ and the radial order $\sigma_{\theta}(g)$ are defined by

$$\sigma_{\theta, \varepsilon}(g) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ M(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), g)}{\log r}, \quad \sigma_{\theta}(g) = \lim_{\varepsilon \rightarrow 0} \sigma_{\theta, \varepsilon}(g).$$

Similarly, the sectorial, respectively radial, exponent of convergence for zeros of $g(z)$ are defined by

$$\lambda_{\theta, \varepsilon}(g) = \limsup_{r \rightarrow \infty} \frac{\log^+ \log^+ n(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), g = 0)}{\log r}, \quad \lambda_{\theta}(g) = \lim_{\varepsilon \rightarrow 0} \lambda_{\theta, \varepsilon}(g),$$

where $n(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), g = 0)$ stands for the number of zeros of $g(z)$ in $\Omega(\theta - \varepsilon, \theta + \varepsilon, r)$ counting multiplicity.

In 1919, Julia gave the concept of Julia direction which is an improvement of Picard's theorem, and started the study of singular directions for meromorphic functions. He showed that every transcendental entire function has at least one Julia direction. From Borel's theorem, which is another important theorem in the Nevanlinna theory, Valiron raised the notation of Borel direction as follows.

Definition 1.1. Let $f(z)$ be a transcendental meromorphic function of order σ . The ray $\arg z = \theta$ is called a Borel direction of f if for any $\varepsilon > 0$, $\lambda_{\theta, \varepsilon}(f - a) = \sigma$ with at most two exceptional value $a \in \mathbb{C} \cup \{\infty\}$.

In 2005, Wu [19] firstly studied the Borel directions of solutions of second order linear differential equation

$$(1) \quad f''(z) + A(z)f(z) = 0,$$

where $A(z)$ is a nonconstant polynomial or an transcendental entire function of finite order. For the case $A(z)$ is transcendental entire, set $E = f_1 f_2$ where f_1, f_2 are two linearly independent solutions of (1) and suppose that the exponent of the zero-sequence $\lambda(E)$ is infinite, he obtained that the ray $\arg z = \theta$ from the origin is a Borel direction for E if only if $\lambda_{\theta} = \infty$.

In 2015, Huang and Wang [10] considered the Borel directions of the solutions for nonhomogeneous second order linear differential equations

$$(2) \quad f''(z) + B(z)f'(z) + A(z)f(z) = F(z),$$

where $A(z), B(z)$ and $F(z)$ are entire functions. Indeed, they obtained the following two theorems.

Theorem A ([10]). *Let $A(z), B(z)$ be entire functions with finite order, let $F(z)$ be transcendental entire and $\max\{\sigma(A), \sigma(B)\} < \sigma(F) = \infty$. If $\arg z = \theta$*

is a Borel direction of F , then it is also a Borel direction of every non-trivial solution f of equation (2).

Theorem B ([10]). *Let $A(z), B(z)$ be entire functions with finite order, let $F(z)$ be transcendental entire and $\max\{\sigma(A), \sigma(B)\} < \sigma(F) = \sigma < \infty$. Suppose that f is a solution of equation (2). If $\arg z = \theta$ is a Borel direction of F , then for any angular domain $\Omega(\alpha, \beta)$ contained the ray $\arg z = \theta$ with $\beta - \alpha > \frac{\pi}{\sigma}$, there exists a Borel direction of f in $\Omega(\alpha, \beta)$.*

These two theorems gave the relations between Borel directions of $F(z)$ and that of solutions for equation (2). One of purposes of this paper is to study the Borel directions of solutions of nonhomogeneous algebraic differential equation

$$(3) \quad Q_d(z, f) = F(z),$$

where $Q_d(z, f)$ is a differential polynomial in f with degree d as defined at the beginning and $F(z)$ is a transcendental entire function. In fact, we get the following results.

Theorem 1.1. *Let $F(z)$ be an entire function of infinite order and $Q_d(z, f)$ be a differential polynomial in f of degree d with finite order entire coefficients $a_\lambda(z)$. If $\arg z = \theta$ is a Borel direction of F , then it is also a Borel direction of every non-trivial solution f of equation (3).*

If $\sigma(F)$ is of finite order in equation (3), it's easy to see for any non-trivial solution f , $\sigma(f) \geq \sigma(F)$, but one can not guarantee that $\sigma(f) = \sigma(F)$ in general. However, given some restrictions on the solution f , we obtain the result as follows.

Theorem 1.2. *Let $F(z)$ be an entire function of finite order, and $Q_d(z, f)$ be a differential polynomial in f of degree d with entire coefficients $a_\lambda(z)$ satisfying $\sigma(a_\lambda) < \sigma(F)$. Suppose that f is a non-trivial solution of equation (3) and has a finite Borel exceptional value, then $\sigma(f) = \sigma(F)$.*

Combining Theorem B and Theorem 1.2, we obtain the following result.

Theorem 1.3. *Let $F(z)$ be an entire function of finite order, and $Q_d(z, f)$ be a differential polynomial in f of degree d with entire coefficients $a_\lambda(z)$ satisfying $\sigma(a_\lambda) < \sigma(F)$. Suppose that f is a non-trivial solution of equation (3) and has a finite Borel exceptional value. If $\arg z = \theta$ is a Borel direction of F , then for any angular domain $\Omega(\alpha, \beta)$ contained the ray $\arg z = \theta$ with $\beta - \alpha > \frac{\pi}{\sigma}$, there exists a Borel direction of f in $\Omega(\alpha, \beta)$.*

Remark 1.1. By [12, Theorem 1.1], we know that for the special differential algebraic equation, $f^n f' + Q_d(z, f) = u(z)e^{v(z)}$, where $n \geq d + 1$, $u(z)$ is a nonzero polynomial, $v(z)$ is a nonconstant polynomial, and $Q_d(z, f)$ is a differential polynomial in f of degree d with polynomial coefficients, every non-trivial solution of this equation satisfies the conclusion of Theorem 1.3.

For a transcendental entire function $g(z)$, it is easy to see that there may exist some angular domain $\Omega(\alpha, \beta)$ such that $\sigma_{\alpha, \beta}(g) = \sigma(g)$, but this is not true for arbitrary angular domain. For example, $\sigma_{-\frac{\pi}{2}, \frac{\pi}{2}}(e^z) = \sigma(e^z) = 1$, however $\sigma_{\frac{\pi}{2}, \frac{3\pi}{2}}(e^z) = 0$. Thus, a natural question is that how wide are such $\Omega(\alpha, \beta)$ with $\sigma_{\alpha, \beta}(g) = \sigma(g)$? In [10], the authors showed that under certain conditions every non-trivial solution f of the following equation (4), the angular domain satisfying $\sigma_{\alpha, \beta} = \infty$ must have a definite range of measure. Indeed, they get

Theorem C ([10]). *Suppose that $A(z), B(z)$ are entire functions with $\mu(A) > \sigma(B)$. If $f(z)$ is a non-trivial solution of equation*

$$(4) \quad f''(z) + B(z)f'(z) + A(z)f(z) = 0,$$

then $\text{meas}I(f) \geq \min\{2\pi, \pi/\mu(A)\}$, where $I(f) = \{\theta \in [0, 2\pi) : \sigma_\theta(f) = \infty\}$.

Motivated by this theorem, we consider the measure of the angle $\Omega(\alpha, \beta)$ such that $\sigma_{\alpha, \beta}(f) = \sigma(f)$ for the non-trivial solutions for the nonhomogeneous algebraic differential equation (3).

Theorem 1.4. *Let $F(z)$ be an entire function of finite order, and $Q_d(z, f)$ be a differential polynomial in f of degree d with entire coefficients $a_\lambda(z)$ satisfying $\sigma(a_\lambda) < \sigma(F)$. Suppose that f is a non-trivial solution of equation (3) and has a finite Borel exceptional value, then $\text{meas}I(f) \geq \min\{2\pi, \pi/\mu(F)\}$, where $I(f) = \{\theta \in [0, 2\pi) : \sigma_\theta(f) = \sigma(f)\}$.*

Since entire function f and its derivatives $f^{(n)}$ have the same order, we also estimate the measure of the angle domain $\Omega(\alpha, \beta)$ satisfies $\sigma_{\alpha, \beta}(f^{(n)}) = \sigma(f^{(n)}) = \sigma(f)$ for any non-trivial solution f to equation (3).

Theorem 1.5. *Let $F(z)$ be a transcendental entire function of finite order and $Q_d(z, f)$ be a differential polynomial in f of degree d with entire coefficients $a_\lambda(z)$ satisfying $\sigma(a_\lambda) < \mu(F)$. Suppose that f is a non-trivial solution of equation (3), then $\text{meas}(I(f) \cap I(f^{(k)})) \geq \min\{2\pi, \pi/\mu(F)\}$, where $I(f^{(k)}) = \{\theta \in [0, 2\pi) : \sigma_\theta(f^{(k)}) = \sigma(f^{(k)})\}$ and $k \geq 0$ is an integer.*

Corollary 1.6. *Under the hypothesis of Theorem 1.5, we have $\text{meas}I(f^{(k)}) \geq \min\{2\pi, \pi/\mu(F)\}$.*

In the sequel, we shall study the radial distribution of the Julia sets of solutions to equation (3). At first we give some introduction of some related concepts. We define $f^{[n]}$, $n \in \mathbb{N}$ denote the n th iterate of f . The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane \mathbb{C} where the iterates $f^{[n]}$ of f form a normal family. The complement of $F(f)$ in \mathbb{C} is called the Julia set $J(f)$ of f . It is well known that $F(f)$ is open and completely invariant under f , $J(f)$ is closed and non-empty, for more information refer to [4].

Given $\theta \in [0, 2\pi)$, if $\Omega(\theta - \varepsilon, \theta + \varepsilon) \cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $\Delta(f)$ is the set

of $\theta \in [0, 2\pi)$ such that $\arg z = \theta$ is the radial distribution of $J(f)$. Obviously, $\Delta(f)$ is closed and so measurable. We use the $\text{meas}\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distribution of the Julia sets of transcendental meromorphic functions have been obtained, for example [3, 8, 9, 14, 15, 18, 23, 24, 26].

In [24], the authors gave a result which shows the connection between the radial order $\sigma_\theta(f)$ and the radial direction of Julia set for an entire function on the ray $\arg z = \theta$. It's stated as follows.

Theorem D ([24]). *Let $f(z)$ be a transcendental entire function. If $\sigma_\theta(f) = \sigma(f)$, then $\arg z = \theta$ is a radial distribution of the Julia set of f .*

Combining Corollary 1.6 and Theorem D, we have:

Corollary 1.7. *Under the hypothesis of Theorem 1.5, we have $\text{meas}\Delta(f) \geq \min\{2\pi, \pi/\mu(F)\}$ and $\text{meas}\Delta(f^{(k)}) \geq \min\{2\pi, \pi/\mu(F)\}$.*

2. Preliminary lemmas

At first, we recall the Nevanlinna characteristic in an angle. Following [6], we define

$$\begin{aligned} A_{\alpha,\beta}(r, g) &= \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \{ \log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})| \} \frac{dt}{t}; \\ B_{\alpha,\beta}(r, g) &= \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin \omega(\theta - \alpha) d\theta; \\ C_{\alpha,\beta}(r, g) &= 2 \sum_{1 < |b_n| < r} \left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}} \right) \sin \omega(\beta_n - \alpha), \end{aligned}$$

where $\omega = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r, g) = A_{\alpha,\beta}(r, g) + B_{\alpha,\beta}(r, g) + C_{\alpha,\beta}(r, g).$$

In particular, we denote the order of $S_{\alpha,\beta}(r, g)$ by

$$\rho_{\alpha,\beta}(g) = \limsup_{r \rightarrow \infty} \frac{\log S_{\alpha,\beta}(r, g)}{\log r}.$$

By [25, Corollary 2.2.2], we have $\sigma_{\alpha,\beta}(g) = \rho_{\alpha,\beta}(g) + \omega$ if g is analytic on $\Omega(\alpha, \beta)$. It's clear that if $\sigma_{\alpha,\beta}(g)$ is finite, then $\rho_{\alpha,\beta}(g)$ is finite. In 1928, Valiron [17] asked that is it true that a meromorphic function of finite positive order and its derivative always have a common Borel direction? This problem still open. But the following result given by Milloux [13] partially answered Valiron's problem.

Lemma 2.1 ([13]). *If g is an entire function with $0 < \sigma(g) = \sigma < \infty$, then a Borel direction of order σ for g' is also a Borel direction of order σ for g .*

For entire functions with infinite order, Sun [16] obtained the following lemma.

Lemma 2.2 ([16]). *Let g be an entire function of infinite order, then the ray $\arg z = \theta$ is a Borel direction of infinite order for g if and only if $\arg z = \theta$ is a Borel direction of infinite order for g' .*

The following lemma is a weaker version of Chuang's result.

Lemma 2.3 ([5]). *Let f be a meromorphic function of infinite order. Then the ray $\arg z = \theta$ is one Borel direction of infinite order of f if and only if f satisfies the equality*

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, f)}{\log r} = \infty$$

for any $\varepsilon \in (0, \pi/2)$.

Lemma 2.4 ([10]). *Suppose that f is a transcendental entire function with order $\sigma(f) = \sigma \in (0, \infty)$, and that $\Omega(\alpha, \beta)$ is an angular domain with $\beta - \alpha > \frac{\pi}{\sigma}$. If there is no Borel direction of order σ for f in $\Omega(\alpha, \beta)$, then $\sigma_{\alpha, \beta}(f) < \sigma$.*

Lemma 2.5 ([20, Borel Lemma]). *Let $f_j(z)$ ($j = 1, 2, \dots, n$) ($n \geq 2$) be meromorphic functions and $g_j(z)$ ($j = 1, 2, \dots, n$) be entire functions such that*

- (1) $\sum_{j=1}^n f_j(z) \exp\{g_j(z)\} \equiv 0$;
- (2) when $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ is not constant;
- (3) when $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}$, ($r \rightarrow \infty, r \notin E$), where $E \subset (1, \infty)$ is of finite linear measure or finite logarithmic measure. Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.6 ([20]). *Let $f(z)$ be a transcendental meromorphic function in complex plane such that $\sigma(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \sigma(f)$ and $\sigma(f)$ is a positive integer.*

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R -set; for reference, see [11]. Set $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \rightarrow \infty$, then $\cup_{n=1}^{\infty} B(z_n, r_n)$ is called an R -set. Clearly, the set $\{|z| : z \in \cup_{n=1}^{\infty} B(z_n, r_n)\}$ is of finite linear measure.

Lemma 2.7 ([9, Lemma 2.2]). *Let $z = re^{i\psi}$, $r_0 + 1 < r$ and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that $n(\geq 2)$ is an integer, and that $g(z)$ is analytic in $\Omega(r_0, \alpha, \beta)$ with $\rho_{\alpha, \beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0, (\beta_j - \alpha_j)/2)$ ($j = 1, 2, \dots, n-1$) outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2, 3, \dots, n-1,$$

there exist $K > 0$ and $M > 0$ only depending on $g, \varepsilon_1, \dots, \varepsilon_{n-1}$ and $\Omega(\alpha_{n-1}, \beta_{n-1})$, and not depending on z , such that

$$\left| \frac{g'(z)}{g(z)} \right| \leq Kr^M (\sin k(\psi - \alpha))^{-2}$$

and

$$\left| \frac{g^{(n)}(z)}{g(z)} \right| \leq Kr^M \left(\sin k(\psi - \alpha) \prod_{j=1}^{n-1} \sin k_{\varepsilon_j}(\psi - \alpha_j) \right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1}, \beta_{n-1})$ outside an R -set D , where $k = \pi/(\beta - \alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j - \alpha_j)$ ($j = 1, 2, \dots, n-1$).

Lemma 2.8 ([21, 25]). *Let $f(z)$ be a transcendental meromorphic function with lower order $\mu(f) < \infty$ and order $0 < \sigma(f) \leq \infty$. Then, for any positive number λ with $\mu(f) \leq \lambda \leq \sigma(f)$ and any set H of finite measure, there exists a sequence $\{r_n\}$ satisfies*

- (1) $r_n \notin H, \lim_{n \rightarrow \infty} r_n/n = \infty$;
- (2) $\liminf_{n \rightarrow \infty} \log T(r_n, f)/\log r_n \geq \lambda$;
- (3) $T(r, f) < (1 + o(1))(2t/r_n)^\lambda T(r_n/2, f), t \in [r_n/n, nr_n]$;
- (4) $t^{-\lambda-\varepsilon_n} T(t, f) \leq 2^{\lambda+1} r_n^{-\lambda-\varepsilon_n} T(r_n, f), 1 \leq t \leq nr_n, \varepsilon_n = (\log n)^{-2}$.

Such $\{r_n\}$ is called a sequence of Pólya peaks of order λ outside H . The following lemma, which related to Pólya peaks, is called the spread relation; see [2].

Lemma 2.9 ([2]). *Let $f(z)$ be a transcendental meromorphic function with positive order and finite lower order, and has a deficient value $a \in \overline{\mathbb{C}}$. Then, for any sequence of Pólya peaks $\{r_n\}$ of order $\lambda > 0$, $\mu(f) \leq \lambda \leq \sigma(f)$, and any positive function $\Upsilon(r) \rightarrow 0$ as $r_n \rightarrow \infty$, we have*

$$\liminf_{r_n \rightarrow \infty} \text{meas} D_{\Upsilon}(r_n, a) \geq \min \left\{ 2\pi, \frac{4}{\lambda} \arcsin \sqrt{\frac{\delta(a, f)}{2}} \right\},$$

where

$$D_{\Upsilon}(r, a) = \left\{ \theta \in [0, 2\pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Upsilon(r) T(r, f) \right\}, \quad a \in \mathbb{C}$$

and

$$D_{\Upsilon}(r, \infty) = \left\{ \theta \in [0, 2\pi) : \log^+ |f(re^{i\theta})| > \Upsilon(r) T(r, f) \right\}.$$

Lemma 2.10 ([22]). *Let $f(z)$ be meromorphic and of order $\sigma(0 < \sigma < \infty)$ in the finite plane. If $\arg z = \theta_0 (0 \leq \theta_0 < 2\pi)$ is a Borel direction of $f(z)$, then there exists a sequence of Borel filling discs $\Gamma_j = \{z : |z - z_j| < \varepsilon_j |z_j|\}, z_j = |z_j|e^{i\theta_0}, (j = 1, 2, \dots)$ with $\lim_{j \rightarrow \infty} |z_j| = \infty, \lim_{j \rightarrow \infty} \varepsilon_j = 0$ such that $f(z)$ takes every complex number at least $n_j = |z_j|^{\sigma-\delta_j}$ times in Γ_j , except possibly for those numbers contained in two spherical disks with radius e^{-n_j} , where $\lim_{j \rightarrow \infty} \delta_j = 0$.*

3. Proof of theorems

Proof of Theorem 1.1. By the assumption that $F(z)$ is of infinite order, $a_\lambda(z)$ is of finite order and the fact $f^{(k)}$ has the same order as f , it's easy to see that every solution f of (3) must be entire function with infinite order. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of $F(z)$. By Lemma 2.3, for any sufficiently small ε ,

$$(5) \quad \limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, F)}{\log r} = \infty.$$

By (3), there exist positive constants M_i , ($i = 0, 1, \dots, n$) such that

$$\begin{aligned} S_{\theta-\varepsilon, \theta+\varepsilon}(r, F) &= S_{\theta-\varepsilon, \theta+\varepsilon}(r, Q_d(z, f)) \\ &\leq \sum_{a_\lambda \in \Lambda} S_{\theta-\varepsilon, \theta+\varepsilon}(r, a_\lambda) + \sum_{i=0}^n M_i S_{\theta-\varepsilon, \theta+\varepsilon}(r, f^{(i)}), \end{aligned}$$

which implies

$$(1 - o(1)) S_{\theta-\varepsilon, \theta+\varepsilon}(r, F) \leq \sum_{i=0}^n M_i S_{\theta-\varepsilon, \theta+\varepsilon}(r, f^{(i)}).$$

This means there exists at least one i_0 , ($0 \leq i_0 \leq n$) satisfies

$$(6) \quad \limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, f^{(i_0)})}{\log r} = \infty.$$

Using Lemma 2.3 again, we know that the ray $\arg z = \theta$ is also a Borel direction of $f^{(i_0)}$. Thus, by Lemma 2.2, the ray $\arg z = \theta$ is a Borel direction of f . \square

Proof of Theorem 1.2. Without loss of generality, assume the finite Borel exceptional value of f is 0. Let $z_1, z_2, \dots, z_n, \dots$ be non-null zeros of f , and each zero is repeated as many times as its multiplicity. Then by Lemma 2.6 and Weierstrass factorization theorem [11, Theorem 1.2.4], we know that

$$(7) \quad f(z) = e^{g(z)} z^{m_0} \prod_{n=1}^{\infty} E_{m_n} \left(\frac{z}{z_n} \right) := h(z) e^{g(z)},$$

where m_0, m_n , ($n = 1, 2, \dots$) are some integers, $g(z)$ is an entire function, $E_{m_n} \left(\frac{z}{z_n} \right)$ is the canonical product of $f(z)$ and $h(z) = z^{m_0} \prod_{n=1}^{\infty} E_{m_n} \left(\frac{z}{z_n} \right)$. From Ash [1, Theorem 4.3.6] we have $\lambda(f) = \lambda(h) = \sigma(h)$. From the definition of Borel exceptional value [20, p. 104], we get

$$\lambda(f) = \limsup_{r \rightarrow \infty} \frac{\log^+ N(r, 1/f)}{\log r} < \sigma(f).$$

Thus, we have $\sigma(h) < \sigma(f)$. Calculating the derivatives of f , it is clear that we can set $f^{(i)} := h_i(z) e^{g(z)}$ for $i = 1, 2, \dots, n$, where $h_i(z)$ are entire functions

satisfying $\sigma(h_i) < \sigma(f)$. Substituting the expressions of $f, f', \dots, f^{(n)}$ into (3) and noting $\sigma(a_\lambda) < \sigma(F) \leq \sigma(f)$, we can obtain

$$(8) \quad H_d(z)e^{dg(z)} + H_{d-1}(z)e^{(d-1)g(z)} + \dots + H_1(z)e^{g(z)} + H_0(z) = F(z),$$

where $H_i(z)$ ($i = 1, 2, \dots, d$) are entire functions satisfying $\sigma(H_i) < \sigma(f)$ and $H_0(z) = a_{\lambda_0}$ ($\lambda_0 = (0, 0, \dots, 0) \in \Lambda$) satisfying $\sigma(H_0) < \sigma(F)$. Applying Lemma 2.5 to (8), we can deduce that there exists integer $i_0, 1 \leq i_0 \leq d$ such that $F(z) - H_0(z) = H_{i_0}e^{i_0g(z)}$. Therefore, we have $\sigma(f) = \sigma(F) < +\infty$ and $g(z)$ is a polynomial. \square

Proof of Theorem 1.3. Suppose that $\arg z = \theta$ ($0 \leq \theta < 2\pi$) is a Borel direction of order σ for $F(z)$. By Lemma 2.10, there exists a sequence of disks $\Gamma_j = \{z : |z - z_j| < \varepsilon_j |z_j|\}$ with $\arg z_j = \theta, \lim_{j \rightarrow \infty} |z_j| = \infty, \lim_{j \rightarrow \infty} \varepsilon_j = 0$. Note that for entire functions, ∞ is always a Picard value, ∞ is located in one of the two spherical discs in Lemma 2.10. Denote the spherical distance of z_1, z_2 by $|z_1, z_2|$. Therefore, for all sufficiently large j , we can find a point $b_j \in \Gamma_j$ such that

$$|F(b_j), \infty| = \frac{1}{(1 + |F(b_j)|^2)^{\frac{1}{2}}} = 2e^{-|z_j|^{\sigma-\delta_j}}$$

with $\lim_{j \rightarrow \infty} \delta_j = 0$. Then we can find a constant C independent of j such that for all sufficiently large j

$$|F(b_j)| > Ce^{|z_j|^{\sigma-\delta_j}}.$$

Since $|b_j| = (1 + o(1))|z_j|$, we can conclude that for any given $\varepsilon > 0$

$$(9) \quad \limsup_{r \rightarrow \infty} \frac{\log \log M(r, \Omega(\theta - \varepsilon, \theta + \varepsilon), F)}{\log r} \geq \sigma.$$

Suppose that there exists no Borel direction of f in $\Omega(\alpha, \beta)$. By Theorem 1.2 we know that $0 < \sigma(f) = \sigma < \infty$. By Lemma 2.1, there also exists no Borel direction of $f^{(k)}$, ($k = 1, 2, \dots, n$) in $\Omega(\alpha, \beta)$. Then by Lemma 2.4, we have

$$(10) \quad \limsup_{r \rightarrow \infty} \frac{\log \log M(r, \Omega(\alpha, \beta), f^{(k)})}{\log r} < \sigma, (k = 0, 1, 2, \dots, n).$$

Since $\sigma(a_\lambda) < \sigma(F) = \sigma$, substituting (10) and (9) into (3) yields a contradiction. Therefore, there must have at least one Borel direction in $\Omega(\alpha, \beta)$. \square

Proof of Theorem 1.4. Suppose that f is a non-trivial solution of equation (3) under the hypothesis of this theorem. By Theorem 1.2 we know that $\sigma(f) = \sigma(F) := \sigma < +\infty$. We assume that $\text{meas} I(f) < \nu := \min\{2\pi, \pi/\mu(F)\}$, so $\zeta := \nu - \text{meas} I(f) > 0$. Clearly $S = (0, 2\pi) \setminus \overline{I(f)}$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i := (\alpha_i, \beta_i), i = 1, 2, \dots, m$ satisfying $[\alpha_i, \beta_i] \subset S$ and $\text{meas}(S \setminus \cup_{i=1}^m I_i) < \frac{\zeta}{4}$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see

$$\Omega(\alpha_i, \beta_i) \cap I(f) = \emptyset.$$

This implies that for each $i = 1, 2, \dots, m$, we have $\sigma_{\alpha_i, \beta_i}(f) < \sigma$, that is, for sufficiently large r , $M(r, \Omega(\alpha_i, \beta_i), f) < \exp\{r^{\sigma-\varepsilon}\}$, where ε is a sufficiently small positive constant. Moreover, it's clear that $\rho_{\alpha_i, \beta_i}(f)$ is finite. Therefore, by Lemma 2.7, for sufficiently small $\varepsilon > 0$, there exist two constants $M > 0$ and $K > 0$ such that

$$(11) \quad \left| \frac{f^{(s)}(z)}{f(z)} \right| \leq Kr^M, s = 1, 2, \dots, n,$$

for all $z \in \cup_{i=1}^k \Omega(\alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside an R -set H . Applying Lemma 2.9 to $F(z)$, there exists a sequence of Pólya peak $\{r_n\}$ of order $\mu(F)$ such that $r_n \notin \{|z|, z \in H\}$, and for sufficiently large n ,

$$(12) \quad \text{meas}\{D_{\Upsilon}(r_n, \infty)\} \geq \nu - \frac{\zeta}{4},$$

where we take the function $\Upsilon(r)$ as

$$\Upsilon(r) = \sqrt{\frac{r^{\sigma-\varepsilon}}{T(r, F)}}$$

for given sufficiently small positive constant ε . Without loss of generality, we assume that (12) holds for all n , and simplified denote $D(r_n) = D_{\Upsilon}(r_n, \infty)$. Obviously,

$$(13) \quad \begin{aligned} \text{meas}(D(r_n) \cap S) &= \text{meas}(D(r_n) \setminus (I(f) \cap D(r_n))) \\ &\geq \text{meas} D(r_n) - \text{meas} I(f) > \frac{3\zeta}{4}. \end{aligned}$$

Then, for each n we have

$$(14) \quad \begin{aligned} \text{meas}((\cup_{i=1}^m I_i) \cap D(r_n)) &= \text{meas}(S \cap D(r_n)) - \text{meas}((S \setminus \cup_{i=1}^m I_i) \cap D(r_n)) \\ &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2}. \end{aligned}$$

This means there exists at least one open interval $I_{i_0} = (\alpha, \beta)$ of $I_i (i = 1, 2, \dots, m)$ such that for infinitely many j ,

$$(15) \quad \text{meas}(D(r_j) \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0.$$

Set $G_j = D(r_j) \cap (\alpha + 2\varepsilon, \beta - 2\varepsilon)$, it follows from the definition of $D(r_j)$ in Lemma 2.9, $T(r, F) = m(r, F)$ and (15) that

$$(16) \quad \begin{aligned} \int_{G_j} \log^+ |F(r_j e^{i\theta})| d\theta &\geq \text{meas}(G_j) \Upsilon(r_j) m(r_j, F) \\ &\geq \frac{\zeta}{4m} \Upsilon(r_j) m(r_j, F). \end{aligned}$$

Note $\sigma(a_\lambda) < \sigma(F) = \sigma(f)$ and set $E_1 := \{\theta \in [0, 2\pi) : |f(re^{i\theta})| < 1\}$, $E_2 := [0, 2\pi) \setminus E_1$. By (3), (11), we can get

(17)

$$\begin{aligned}
 & \int_{G_j} \log^+ |F(r_j e^{i\theta})| d\theta \\
 &= \int_{G_j \cap E_1} \log^+ \left| \sum_{\lambda \in \Lambda} a_\lambda f^{\lambda_0} (f')^{\lambda_1} \dots (f^{(n)})^{\lambda_n} \right| d\theta \\
 & \quad + \int_{G_j \cap E_2} \log^+ \left| f^d \sum_{\lambda \in \Lambda} \frac{a_\lambda f^{\lambda_0} (f')^{\lambda_1} \dots (f^{(n)})^{\lambda_n}}{f^d} \right| d\theta \\
 &\leq \int_{G_j \cap E_1} \sum_{\lambda \in \Lambda} \left(\log^+ |a_\lambda| + \lambda_0 \log^+ |f| + \sum_{i=1}^n \log^+ \left| \frac{f^{(i)}}{f} \right|^{\lambda_i} \right) d\theta \\
 & \quad + \int_{G_j \cap E_2} \sum_{\lambda \in \Lambda} \left(d \log^+ |f| + \log^+ |a_\lambda| + \log^+ \left| \frac{1}{f^{d - \sum_{i=0}^n \lambda_i}} \right| + \sum_{i=1}^n \log^+ \left| \frac{f^{(i)}}{f} \right|^{\lambda_i} \right) d\theta \\
 & \quad + O(1) \\
 &\leq \int_{G_j} \sum_{\lambda \in \Lambda} \left(\sum_{i=1}^n \log^+ \left| \frac{f^{(i)}}{f} \right| + d \log^+ |f| + \log^+ |a_\lambda| \right) d\theta + O(1) \\
 &\leq \text{meas}(G_j) c_0 r^{\sigma - \varepsilon} = \text{meas}(G_j) c_0 \Upsilon^2(r_j) m(r_j, F),
 \end{aligned}$$

where c_0 is a positive constant. From (16) and (17) we get a contradiction as the fact $\Upsilon(r_j) \rightarrow 0$ when $j \rightarrow \infty$. Thus, we complete the proof. \square

Proof of Theorem 1.5. We know that every non-trivial solution f of the equation is an entire function with $\sigma(f) = \sigma(F) := \sigma$. We obtain the assertion by reduction to contradiction. Assume that

$$(18) \quad \text{meas}(I(f) \cap I(f^{(k)})) < \nu = \min\{2\pi, \pi/\mu(F)\}$$

and so

$$(19) \quad \xi := \nu - \text{meas}(I(f) \cap I(f^{(k)})) > 0.$$

Applying Lemma 2.8 to F , we have a Pólya peak $\{r_j\}$ of order $\mu(F)$ with all $r_j \notin H$. Since F is transcendental entire function, it follows the Nevanlinna deficient $\delta(\infty, F) = 1$. By Lemma 2.9, for the Pólya peak $\{r_j\}$, we have

$$(20) \quad \liminf_{r_j \rightarrow \infty} \text{meas}(D_{\Upsilon}(r_j, \infty)) \geq \pi/\mu(F),$$

where the function $\Upsilon(r)$ is defined by

$$(21) \quad \Upsilon(r) = \sqrt{\frac{r^{\sigma - \varepsilon}}{m(r, F)}},$$

ε is a sufficiently small positive constant and $m(r, F)$ is the proximation function of F . Obviously, $\Upsilon(r)$ is positive and $\lim_{r \rightarrow \infty} \Upsilon(r) = 0$. For the sake of simplicity, we denote $D_{\Upsilon}(r_j, \infty)$ by $D(r_j)$ in the following. We shall show that there must exist an open interval

$$(22) \quad E = (\alpha, \beta) \subset I(f^{(k)})^c$$

such that

$$(23) \quad \lim_{j \rightarrow \infty} \text{meas}(I(f) \cap D(r_j) \cap E) > 0,$$

where $I(f^{(k)})^c := [0, 2\pi) \setminus I(f^{(k)})$. In order to achieve this goal, we shall prove the following firstly.

$$(24) \quad \lim_{j \rightarrow \infty} \text{meas}(D(r_j) \setminus I(f)) = 0.$$

Otherwise, suppose that there is a subseries $\{r_{j_k}\}$ such that

$$(25) \quad \lim_{k \rightarrow \infty} \text{meas}(D(r_{j_k}) \setminus I(f)) > 0,$$

then there exist $\theta_0 \in I^c$ and $\eta > 0$ satisfying

$$(26) \quad \lim_{k \rightarrow \infty} \text{meas}((\theta_0 - \eta, \theta_0 + \eta) \cap (D(r_{j_k}) \setminus I(f))) > 0.$$

Since $\arg z = \theta_0$ is not in $I(f)$, it follows $\sigma_{\theta_0 - \eta, \theta_0 + \eta}(f) < \sigma$, that is, for sufficiently large r ,

$$(27) \quad M(r, \Omega(\theta_0 - \eta, \theta_0 + \eta), f) < \exp\{r^{\sigma - \varepsilon}\},$$

where ε is a sufficiently small positive constant. Moreover, by Lemma 2.7, there exist constants $M > 0$ and $K > 0$ such that

$$(28) \quad \left| \frac{f^{(s)}(z)}{f(z)} \right| \leq Kr^M, \quad (s = 1, 2, \dots, n),$$

for all $z \in \Omega(r_0, \theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta)$, outside an R -set H .

Since ζ can be chosen sufficiently small, from (26) we have

$$(29) \quad \lim_{k \rightarrow \infty} \text{meas}((\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap D(r_{j_k})) > 0.$$

Thus, by Lemma 2.9 we can find an infinite series $\{r_{j_k} e^{i\theta_{j_k}}\}$ such that for all sufficiently large k ,

$$(30) \quad \log^+ |F(r_{j_k} e^{i\theta_{j_k}})| > \Upsilon(r_{j_k})T(r_{j_k}, F) = \Upsilon(r_{j_k})m(r_{j_k}, F),$$

where $\theta_{j_k} \in G_{j_k} := (\theta_0 - \eta + \zeta, \theta_0 + \eta - \zeta) \cap D(r_{j_k})$. Then, for sufficiently large k , we have

$$(31) \quad \int_{G_{j_k}} \log^+ |F(r_{j_k} e^{i\theta_{j_k}})| d\theta \geq \text{meas}(G_{j_k}) \Upsilon(r_{j_k})m(r_{j_k}, F).$$

By the same arguments used in (17) in the proof of Theorem 1.4, we obtain a contradiction, which implies (24) is valid.

By Theorem 1.4, we know that

$$(32) \quad \text{meas} I(f) \geq \nu.$$

From Lemma 2.9, we have, for all sufficiently large j and any positive ε ,

$$(33) \quad \text{meas} D(r_j) > \nu - \varepsilon.$$

Combining (24), (32) and (33) follows that, for all sufficiently large j ,

$$(34) \quad \text{meas}(I(f) \cap D(r_j)) \geq \nu - \xi/4,$$

where ξ is defined in (19). Since the interior of $\Delta(f^{(k)})^c$ is open, so it consists of at most countably open intervals. We can choose finitely many open intervals $E_j, (j = 1, 2, \dots, m)$ satisfying

$$(35) \quad E_j \subset I(f^{(k)})^c, \quad \text{meas}(I(f^{(k)})^c \setminus \cup_{i=1}^m E_i) < \xi/4.$$

Since, for sufficiently large j ,

$$(36) \quad \begin{aligned} & \text{meas}(I(f) \cap D(r_j) \cap (\cup_{i=1}^m E_i)) + \text{meas}(I(f) \cap D(r_j) \cap I(f^{(k)})) \\ &= \text{meas}(I(f) \cap D(r_j) \cap (I(f^{(k)}) \cup (\cup_{i=1}^m E_i))) \geq \nu - \xi/2, \end{aligned}$$

we have

$$(37) \quad \begin{aligned} & \text{meas}(I(f) \cap D(r_j) \cap (\cup_{i=1}^m E_i)) \geq \nu - \xi/2 - \text{meas}(I \cap D(r_j) \cap I(f^{(k)})) \\ & \geq \nu - \xi/2 - \text{meas}(I(f) \cap I(f^{(k)})) = \xi/2. \end{aligned}$$

Thus, there exists an open interval $E_{i_0} = (\alpha, \beta) \subset \cup_{i=1}^m E_i \subset I(f^{(k)})^c$ such that, for infinitely many sufficiently large j ,

$$(38) \quad \text{meas}(I(f) \cap D(r_j) \cap E_{i_0}) \geq \frac{\xi}{2m} > 0.$$

Then, we prove (23) holds.

From (23), we know that there are $\tilde{\theta}_0$ and $\tilde{\eta} > 0$ such that

$$(39) \quad (\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}) \subset E$$

and

$$(40) \quad \lim_{j \rightarrow \infty} \text{meas}(I(f) \cap D(r_j) \cap (\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta})) > 0.$$

By (39) it follows $\sigma_{\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}}(f^{(k)}) < \sigma$, that is, for sufficiently large r ,

$$(41) \quad M(r, \Omega(\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}), f^{(k)}) < \exp\{r^{\sigma - \varepsilon}\},$$

where ε is a sufficiently small positive constant.

By (40) and Lemma 2.9 we can choose an unbounded series $\{r_j e^{i\theta_j}\}$, for all sufficiently large j such that

$$(42) \quad \log^+ |F(r_j e^{i\theta_j})| > \Upsilon(r_j) m(r_j, F),$$

where $\theta_j \in I(f) \cap D(r_j) \cap (\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta})$. Fixed $r_j e^{i\theta_j}$, and take a $r_j e^{i\theta_j} \in \{r_j e^{i\theta_j}\}$. Take a simple Jordan arc γ in $\Omega(\tilde{r}_0, \tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta})$ which connecting

$r_J e^{i\theta_J}$ to $r_J e^{i\theta_j}$ along $|z| = r_J$, and connecting $r_J e^{i\theta_j}$ to $r_j e^{i\theta_j}$ along $\arg z = \theta_j$. For any $z \in \gamma$, γ_z denotes a part of γ , which connecting $r_J e^{i\theta_J}$ to z . Let $L(\gamma)$ be the length of γ . Clearly, $L(\gamma) = O(r_j)$, $j \rightarrow \infty$. By (41), it follows

$$\begin{aligned} |f^{(k-1)}(z)| &\leq \int_{\gamma_z} |f^{(k)}(z)| |dz| + c_k \\ &\leq O(\exp\{r_j^{\sigma-\varepsilon}\} L(\gamma)) + c_k \leq O(r_j \exp\{r_j^{\sigma-\varepsilon}\}), \quad j \rightarrow \infty. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |f^{(k-2)}(z)| &\leq \int_{\gamma_z} |f^{(k-1)}(z)| |dz| + c_{k-1} \leq O(r_j^2 \exp\{r_j^{\sigma-\varepsilon}\}), \quad j \rightarrow \infty, \\ &\vdots \\ |f(z)| &\leq \int_{\gamma_z} |f'(z)| |dz| + c_1 \leq O(r_j^k \exp\{r_j^{\sigma-\varepsilon}\}) \\ (43) \quad &\leq O(2 \exp\{r_j^{\sigma-\varepsilon}\}), \quad j \rightarrow \infty, \end{aligned}$$

where c_1, c_2, \dots, c_k are constants, which are independent of j . Therefore,

$$(44) \quad M(r, \Omega(\tilde{\theta}_0 - \tilde{\eta}, \tilde{\theta}_0 + \tilde{\eta}), f) < O(2 \exp\{r^{\sigma-\varepsilon}\}),$$

where ε is a sufficiently small positive constant.

By Lemma 2.7, we know (28) also holds for all $z \in \Omega(\tilde{r}_0, \tilde{\theta}_0 - \tilde{\eta} + \tilde{\zeta}, \tilde{\theta}_0 + \tilde{\eta} - \tilde{\zeta})$, outside an R -set H . Combining (28), (42) and (44), and applying the similar argument as (16) and (17) in the proof of Theorem 1.4, we can deduce a contradiction. Therefore, it follows

$$(45) \quad \text{meas}(I(f) \cap I(f^{(k)})) \geq \min\{2\pi, \pi/\mu(F)\}.$$

Then we complete the proof. \square

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