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GLOBAL MAXIMAL ESTIMATE TO SOME OSCILLATORY INTEGRALS

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Abstract. Under the symbol Ω is a combination of ϕ_i $(i = 1, 2, 3, \dots, n)$ which has a suitable growth condition, for dimension $n = 2$ and $n \geq 3$, when the initial data f belongs to homogeneous Sobolev space, we obtain the global L^q estimate for maximal operators generated by operators family $\{S_{t,\Omega}\}_{t\in\mathbb{R}}$ associated with solution to dispersive equations, which extend some results in [27].

1. Introduction and main results

Assume that Ω is a continuous real-valued functions in $\mathbb{R}^n$. Let f be a Schwartz function in $\mathcal{S}(\mathbb{R}^n)$ and

$$S_{t,\Omega}f(x) = u(x,t) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi+it\Omega(\xi)}\hat{f}(\xi)d\xi, \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}.$$

Here $\hat{f}$ denotes Fourier transform of f defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}f(x)dx$. Define the global maximal operator associated with the family of operators $\{S_{t,\Omega}\}_{t\in\mathbb{R}}$ by

$$S_\Omega^{**}f(x) = \sup_{t\in\mathbb{R}}|S_{t,\Omega}f(x)|, \quad x\in\mathbb{R}^n.$$

We recall the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ $(s\in\mathbb{R})$ which is defined by

$$\dot{H}^s(\mathbb{R}^n) = \left\{f\in\mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^n}|\xi|^{2s}|\hat{f}(\xi)|^2d\xi\right)^{1/2} < \infty\right\},$$

and the inhomogeneous Sobolev space $H^s(\mathbb{R}^n)$ $(s\in\mathbb{R})$, which is defined by

$$H^s(\mathbb{R}^n) = \left\{f\in\mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^s} = \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^s|\hat{f}(\xi)|^2d\xi\right)^{1/2} < \infty\right\}.$$

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Here, $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.

In this paper, we will discuss the global estimate

$$\|S_{\Omega}^{**}f\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^n)}. \tag{1.1}$$

In case $\Omega(\xi) = |\xi|^a$, the maximal estimates (1.1) have been well studied associated with the following oscillatory integral:

$$S_{t,a}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{it|\xi|^a}\hat{f}(\xi)d\xi, \qquad t\in\mathbb{R} \text{ and } a>1,$$

which is the solution of the fractional Schrödinger equation:

$$\begin{cases} i\partial_t u + (-\Delta)^{a/2}u = 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.2}$$

Moreover, the global estimate (1.1) and related questions have been well studied in literature, see e.g. Carbery [3], Kenig and Ruiz [16], Kenig, Ponce and Vega [15], Rogers and Villarroya [23], Rogers [21], Sjölin [24–29], and so on.

In particular, if $\Omega(\xi) = |\xi|^2$, then u is the solution of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.3}$$

In 1979, Carleson [4] proposed a problem: if $f\in H^s(\mathbb{R}^n)$ for which the optimal s such that

$$\lim_{t\to 0} u(x,t) = f(x), \text{ a.e. } x\in\mathbb{R}^n. \tag{1.4}$$

When spatial dimension $n=1$, the pointwise convergence (1.4) is true if and only if $s\geq\frac{1}{4}$, (see [4], and [7]). In spatial dimension $n\geq 3$, Bourgain [1] showed that (1.4) holds for $s>\frac{1}{2}-\frac{1}{4n}$, and he also showed that the necessary condition of convergence (1.4) is $s\geq\frac{1}{2}-\frac{1}{n}$ when $n\geq 4$. Recently, when $n\geq 2$, Lucà, Rogers in [20] and Demeter, Guo in [8] improved above result and proved that (1.4) can fail if $s<\frac{n}{2(n+2)}$. Moreover, when $n\geq 2$, Bourgain in [2] showed that (1.4) fails if $s<\frac{n}{2(n+1)}$. Recently, in spatial dimension $n=2$, Du, Guth, Li [11] showed that (1.4) holds for data in $H^s(\mathbb{R}^2)$ with $s>\frac{1}{3}$, which is sharp up to the endpoint. For more results on the convergence (1.4) when $f\in H^s(\mathbb{R}^n)$. See [19, 24, 30–32], for example.

If $n=2$, $\xi=(\xi_1,\xi_2)$ and $\Omega(\xi)=\xi_2^2-\xi_1^2$, then u is the solution of the nonelliptic Schrödinger equation

$$\begin{cases} i\partial_t u = \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_1^2}, & (x,t)\in\mathbb{R}^2\times\mathbb{R}, \\ u(x,0) = f(x). \end{cases} \tag{1.5}$$

In 2006, to discuss the pointwise convergence problem on the solution of nonelliptic Schrödinger equation (1.5), Rogers, Vargas and Vega [22] obtained the following results of global estimate (1.1) for nonelliptic Schrödinger equation (1.5).

Theorem A ([22])**.** *Assume that* $n = 2$ *and* $\Omega(\xi) = \xi_1^2 - \xi_2^2$. *Then the global estimate* (1.1) *holds for* $s = \frac{1}{2}$ *and* $q = 4$.

In 2007, Sjölin [27] extended Theorem A and obtained the following results.

Theorem B ([27])**.**
(i) *Assume that* $n = 2$ *and* $\Omega(\xi) = |\xi_1|^a \pm |\xi_2|^a$, *where* $a > 1$. *Then the global estimate* (1.1) *holds for* $\frac{1}{2} \le s < 1$ *and* $q = \frac{2}{1-s}$;
(ii) *Assume that* $n \ge 3$ *and* $\Omega(\xi) = |\xi_1|^a \pm |\xi_2|^a \pm |\xi_3|^a \pm |\xi_4|^a \pm \cdots \pm |\xi_n|^a$, *where* $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ *and* $a > 1$. *Then the global estimate* (1.1) *holds for* $\frac{n}{4} \le s < \frac{n}{2}$ *and* $q = \frac{2n}{n-2s}$.

Assume $\phi : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following growth conditions:

(H1) There exists $m_1 > 1$, such that $|\phi'(r)| \sim r^{m_1-1}$ and $|\phi''(r)| \gtrsim r^{m_1-2}$ for all $0 < r < 1$;
(H2) There exists $m_2 > 1$, such that $|\phi'(r)| \sim r^{m_2-1}$ and $|\phi''(r)| \gtrsim r^{m_2-2}$ for all $r \ge 1$;
(H3) Either $\phi''(r) > 0$ or $\phi''(r) < 0$ for all $r > 0$.

In the present paper, we will consider the global maximal estimates for generalized oscillatory integral when symbol Ω is a combination of some ϕ. Now we state our main results as follows.

Theorem 1.1. *Assume that* $n = 2$ *and* $\Omega(\xi) = \phi_1(|\xi_1|) \pm \phi_2(|\xi_2|)$, *where* ϕ_i $(i = 1, 2)$ *satisfies* (H1)$\sim$(H3). *Then the global estimate* (1.1) *holds for* $\frac{1}{2} \le s < 1$ *and* $q = \frac{2}{1-s}$.

Theorem 1.2. *Assume that* $n \ge 3$ *and* $\Omega(\xi) = \phi_1(|\xi_1|) \pm \phi_2(|\xi_2|) \pm \phi_3(|\xi_3|) \pm \cdots \pm \phi_n(|\xi_n|)$, *where* ϕ_i $(i = 1, 2, 3, \ldots, n)$ *satisfies* (H1)$\sim$(H3). *Then the global estimate* (1.1) *holds for* $\frac{n}{4} \le s < \frac{n}{2}$ *and* $q = \frac{2n}{n-2s}$.

Remark 1.1. We recall that

$$S_{t,\Omega} f(x) = u(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi + it\Omega(\xi)} \hat{f}(\xi) d\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

As a consequence, when Ω satisfies conditions in Theorem 1.1 ($n = 2$) or Theorem 1.2 ($n \ge 3$), if $f \in H^s(\mathbb{R}^n)$ and $s \ge \frac{n}{4}$, we have

$$\lim_{t\to 0} u(x,t) = f(x), \ \text{ a.e. } x \in \mathbb{R}^n. \tag{1.6}$$

In fact, by a standard argument, for $f \in H^s(\mathbb{R}^n)$, the pointwise convergence (1.6) follows from the local estimate

$$\|S_\Omega^*\|_{L^q(\mathbb{B}^n)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \quad f \in H^s(\mathbb{R}^n) \tag{1.7}$$

for some $q \ge 1$ and $s \in \mathbb{R}$. Here $\mathbb{B}^n$ is the unit ball centered at the origin in $\mathbb{R}^n$ and the local maximal operator S_Ω^* associated with the family of operators $\{S_{t,\Omega}\}_{t\in\mathbb{R}}$ defined by

$$S_\Omega^* f(x) = \sup_{0<t<1} |S_{t,\Omega} f(x)|, \qquad x \in \mathbb{R}^n.$$

Remark 1.2. Notice that

$$u(x,t) = e^{it\phi(\sqrt{-\Delta})}f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi+it\phi(|\xi|)}\hat{f}(\xi)d\xi$$

is the formal solution of the following generalized dispersive equation:

$$\begin{cases} i\partial_t u + \phi(\sqrt{-\Delta})u = 0, & (x,t)\in\mathbb{R}^n\times\mathbb{R}, \\ u(x,0) = f(x), & f\in\mathcal{S}(\mathbb{R}^n), \end{cases} \tag{1.8}$$

where $\phi(\sqrt{-\Delta})$ is a pseudo-differential operator with symbol $\phi(|\xi|)$. Many dispersive equation can be reduced this type. For instance, the half-wave equation ($\phi(r) = r$), the fractional Schrödinger equation ($\phi(r) = r^a$ $(0 < a, a \neq 1)$), the Beam equation ($\phi(r) = \sqrt{1+r^4}$), Klein-Gordon or semirelativistic equation ($\phi(r) = \sqrt{1+r^2}$), iBq ($\phi(r) = r\sqrt{1+r^2}$), imBq ($\phi(r) = \frac{r}{\sqrt{1+r^2}}$) and the fourth-order Schrödinger equation ($\phi(r) = r^2 + r^4$) (see [5, 6, 9, 12–14, 17, 18] and references therein).

Remark 1.3. There are many elements ϕ satisfying the conditions (H1)$\sim$(H3), for instance, r^a $(a \geq 1)$, $(1+r^2)^{\frac{a}{2}}$ $(a \geq 1)$, $\sqrt{1+r^4}$, r^2+r^4, $r\sqrt{1+r^2}$ and so on. Moreover, the results of Theorem 1.1 and Theorem 1.2 can be applied to symbol Ω is a combination of ϕ_i $(i = 1,2)$ or $(i = 1,2,3,\ldots,n)$, where $\phi_i(|\xi_i|) = |\xi_i|^a$, $a > 1$, $\phi_i(|\xi_i|) = |\xi_i|^2 + |\xi_i|^4$, $\phi_i(|\xi_i|) = \sqrt{1+|\xi_i|^4}$, $\phi_i(|\xi_i|) = |\xi_i|^2 + |\xi_i|^4$, or $\phi_i(|\xi_i|) = |\xi_i|\sqrt{1+|\xi_i|^2}$, and so on. Hence, Theorem 1.1 and Theorem 1.2 are an extension of Theorem A and Theorem B, respectively.

This paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 2 and Section 3, respectively.

2. The proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To do this, we need an important lemma (i.e., Lemma 2.1 below), which plays a key role in proving Theorem 1.1.

Lemma 2.1. *Assume ϕ satisfies* (H1)$\sim$(H3) *with $m_1 > 1$, $m_2 > 1$, $\frac{1}{2} \leq s < 1$, and $\mu \in C_0^\infty(\mathbb{R})$. Then*

$$\left|\int_{\mathbb{R}} e^{ix\xi+it\phi(|\xi|)}|\xi|^{-s}\mu\left(\frac{\xi}{N}\right)d\xi\right| \leq C\frac{1}{|x|^{1-s}}$$

for $x \in \mathbb{R}\setminus\{0\}$, $t \in \mathbb{R}$ and $N = 1,2,3,\ldots$. Here the constant C may depend on s, m_1, m_2 and μ but not on x, t or N.

Proof. The proof of Lemma 2.1 is similar to that of Lemma 2.1 in [10]. Here, we omit the proof of Lemma 2.1. □

Proof of Theorem 1.1. Let $t(x)$ be a measurable function on $\mathbb{R}^2$ with $t(x) \in \mathbb{R}$. Assume that $n = 2$, $\Omega(\xi) = \phi(|\xi_1|) \pm \phi(|\xi_2|)$, where ϕ_i $(i = 1,2)$ satisfies (H1)$\sim$(H3). We set

$$Sf(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}e^{it(x)\Omega(\xi)}\hat{f}(\xi)d\xi, \quad x\in\mathbb{R}^2 \quad f\in\mathcal{S}(\mathbb{R}^2).$$

For $\frac{1}{2} \leq s < 1$ and $q = \frac{2}{1-s}$, by linearising the maximal operator to prove the global estimate (1.1) it suffices to prove that

$$\|Sf\|_{L^q(\mathbb{R}^2)} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^2)} = C\bigg(\int_{\mathbb{R}^2} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi\bigg)^{1/2}. \tag{2.1}$$

For $f \in \mathcal{S}(\mathbb{R}^2)$, notice that

$$\bigg(\int_{\mathbb{R}^2} |\xi_1|^s|\xi_2|^s|\hat{f}(\xi)|^2 d\xi\bigg)^{1/2} \leq \bigg(\int_{\mathbb{R}^2} |\xi|^{2s}|\hat{f}(\xi)|^2 d\xi\bigg)^{1/2}.$$

Thus to prove (2.1) it suffices to prove that

$$\|Sf\|_{L^q(\mathbb{R}^2)} \leq C\bigg(\int_{\mathbb{R}^2} |\xi_1|^s|\xi_2|^s|\hat{f}(\xi)|^2 d\xi\bigg)^{1/2}. \tag{2.2}$$

Let $g(\xi) = |\xi_1|^{\frac{s}{2}}|\xi_2|^{\frac{s}{2}}\hat{f}(\xi)$, and then we have

$$Sf(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}e^{it(x)\Omega(\xi)}|\xi_1|^{-\frac{s}{2}}|\xi_2|^{-\frac{s}{2}}g(\xi)d\xi =: \ Rg(x), \tag{2.3}$$

where

$$Rg(x) = \int_{\mathbb{R}^2} e^{ix\cdot\xi}e^{it(x)\Omega(\xi)}|\xi_1|^{-\frac{s}{2}}|\xi_2|^{-\frac{s}{2}}g(\xi)d\xi.$$

Thus, by (2.3), to prove (2.2) it suffices to prove that

$$\|Rg\|_{L^q(\mathbb{R}^2)} \leq C\|g\|_{L^2(\mathbb{R}^2)} \tag{2.4}$$

for g which is a continuous and rapidly decreasing at infinity function.

We take a real-valued function $\rho \in C_0^\infty(\mathbb{R}^2)$ such that $\rho(x) = 1$ if $|x| \leq 1$, and $\rho(x) = 0$ if $|x| \geq 2$. And we choose a real-valued function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(x) = 1$ if $|x| \leq 1$, and $\psi(x) = 0$ if $|x| \geq 2$, and set $\sigma(\xi) = \psi(\xi_1)\psi(\xi_2)$. For $\xi \in \mathbb{R}^2$ and for $N = 1, 2, 3, \ldots$, we set $\rho_N(x) = \rho(\frac{x}{N})$ and $\sigma_N(\xi) = \sigma(\frac{\xi}{N})$. For $x \in \mathbb{R}^2$, $g \in L^2(\mathbb{R}^2)$, and for $N = 1, 2, 3, \ldots$, the operator R_N is defined by

$$R_N g(x) = \rho_N(x)\int_{\mathbb{R}^2} e^{ix\cdot\xi}e^{it(x)\Omega(\xi)}|\xi_1|^{-\frac{s}{2}}|\xi_2|^{-\frac{s}{2}}\sigma_N(\xi)g(\xi)d\xi.$$

The adjoint of R_N is given by

$$R_N' g(\xi) = \sigma_N(\xi)|\xi_1|^{-\frac{s}{2}}|\xi_2|^{-\frac{s}{2}}\int_{\mathbb{R}^2} e^{-ix\cdot\xi}e^{-it(x)\Omega(\xi)}\rho_N(x)h(x)dx,$$
$$\xi \in \mathbb{R}^2, \quad h \in L^2(\mathbb{R}^2).$$

To prove (2.4) it suffices to prove that

$$\|R_N g\|_{L^q(\mathbb{R}^2)} \leq C\|g\|_{L^2(\mathbb{R}^2)}. \tag{2.5}$$

By duality, show (2.5) it suffices to show that

$$\|R_N' h\|_{L^2(\mathbb{R}^2)} \leq C\|h\|_{L^{q'}(\mathbb{R}^2)}, \quad N = 1, 2, 3, \ldots, \tag{2.6}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Since

$$\begin{aligned}
\|R'_N h\|^2_{L^2(\mathbb{R}^2))} &= \int_{\mathbb{R}^2} |R'_N h(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^2} R'_N h(\xi)\overline{R'_N h(\xi)} d\xi \\
&= \int_{\mathbb{R}^2} \sigma_N(\xi)^2 |\xi_1|^{-s}|\xi_2|^{-s} \left(\int_{\mathbb{R}^2} e^{-ix\cdot\xi} e^{-it(x)\Omega(\xi)} \rho_N(x) h(x) dx \right) \\
&\quad \times \left(\int_{\mathbb{R}^2} e^{iy\cdot\xi} e^{it(y)\phi(\xi)} \rho_N(y)\overline{h(y)} dy \right) d\xi \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left(\int |\xi_1|^{-s}|\xi_2|^{-s} e^{i(y-x)\cdot\xi} e^{i(t(y)-(t(x))\Omega(\xi)} \sigma_N(\xi)^2 d\xi \right) \\
&\quad \times \rho_N(x)\rho_N(y) h(x)\overline{h(y)} dxdy \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} K_N(x,y)\rho_N(x)\rho_N(y)h(x)\overline{h(y)} dxdy, \qquad (2.7)
\end{aligned}$$

where

$$\begin{aligned}
K_N(x,y) = &\int_{\mathbb{R}^2} |\xi_1|^{-s}|\xi_2|^{-s} e^{i((y_1-x_1)\xi_1+(y_2-x_2)\xi_2)} e^{i(t(y)-t(x))\phi(|\xi_1|)} e^{\pm i(t(y)-t(x))\phi(|\xi_2|)} \\
&\psi(\frac{\xi_1}{N})^2 \psi(\frac{\xi_2}{N})^2 d\xi \\
= &\left(\int_{\mathbb{R}} |\xi_1|^{-s} e^{i(y_1-x_1)\xi_1} e^{i(t(y)-t(x))\phi(|\xi_1|)} \psi(\frac{\xi_1}{N})^2 d\xi_1 \right) \qquad (2.8) \\
&\times \left(\int_{\mathbb{R}} |\xi_2|^{-s} e^{i(y_2-x_2)\xi_1} e^{\pm i(t(y)-t(x))\phi(|\xi_2|)} \psi(\frac{\xi_2}{N})^2 d\xi_2 \right).
\end{aligned}$$

Since $\frac{1}{2}\le s<1$, using Lemma 2.1, we obtain

$$|K_N(x,y)| \le C\frac{1}{|x_1-y_1|^{1-s}}\frac{1}{|x_2-y_2|^{1-s}}. \qquad (2.9)$$

We set

$$P_1 f(x_1,x_2) = \int_{\mathbb{R}} \frac{1}{|x_1-y_1|^{1-s}} f(y_1,x_2) dy_1,$$

and

$$P_2 f(x_1,x_2) = \int_{\mathbb{R}} \frac{1}{|x_2-y_2|^{1-s}} f(x_1,y_2) dy_2.$$

Thus, by (2.7) and (2.9), we obtain

$$\begin{aligned}
&\int |R'_N h(x)|^2 dx \\
\le\; & C\int\int \frac{1}{|x_1-y_1|^{1-s}}\frac{1}{|x_2-y_2|^{1-s}} |h(x)||h(y)| dxdy \\
=\; & C\int\int \frac{1}{|x_2-y_2|^{1-s}} \left(\int \frac{1}{|x_1-y_1|^{1-s}} |h(y_1,y_2)| dy_1 \right) dy_2 |h(x)| dx
\end{aligned}$$

$$= C \int_{\mathbb{R}^2} P_2 P_1 |h|(x) |h(x)| dx. \tag{2.10}$$

By (2.10) and invoking Hölder's inequality, we get

$$\int_{\mathbb{R}^2} |R'_N h(x)|^2 dx \le C \|P_2 P_1 |h|\|_{L^q(\mathbb{R}^2)} \|h\|_{L^{q'}(\mathbb{R}^2)}, \tag{2.11}$$

where $q = \frac{2}{1-s}$, $q' = \frac{2}{1+s}$ and $\frac{1}{2} \le s < 1$. Denote I_σ the Riesz potential of order σ, which is defined by

$$I_\sigma(f)(u) = \int_{\mathbb{R}} \frac{f(v)}{|u-v|^{1-\sigma}} dv.$$

Applying the fact I_s is bounded from $L^{q'}(\mathbb{R})$ to $L^q(\mathbb{R})$, we have

$$\left(\int_{\mathbb{R}} |P_j h(x)|^q dx_j \right)^{1/q} \le C \left(\int_{\mathbb{R}} |h(x)|^{q'} dx_j \right)^{1/q'}, \tag{2.12}$$

where $j = 1, 2$. By (2.12) and Minkowski's inequality, we have

$$\|P_2 P_1 |h|\|_{L^q(\mathbb{R}^2)} \le C \|h\|_{L^{q'}(\mathbb{R}^2)}, \tag{2.13}$$

where using the fact $q' = \frac{2}{1+s}$ and $\frac{1}{q} = \frac{1}{q'} - s$. Therefore, (2.6) follows from (2.11) and (2.13). Now we complete the proof of Theorem 1.1.

3. The proof of Theorem 1.2

Let $t(x)$ be a measurable function on $\mathbb{R}^n$ with $t(x) \in \mathbb{R}$. Assume that $n \ge 3$, $\Omega(\xi) = \phi_1(|\xi_1|) \pm \phi_2(|\xi_2|) \pm \phi_3(|\xi_3|) \pm \cdots \pm \phi_n(|\xi_n|)$, where ϕ_i $(i = 1, 2, 3, \ldots, n)$ satisfies the conditions (H1)$\sim$(H3). We will show that the global estimate (1.1) holds for $\frac{n}{4} \le s < \frac{n}{2}$ and $q = \frac{2n}{n-2s}$. We set

$$Sf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it(x)\Omega(\xi)} \hat{f}(\xi) d\xi, \quad x \in \mathbb{R}^n \quad f \in \mathcal{S}(\mathbb{R}^n).$$

For $\frac{n}{4} \le s < \frac{n}{2}$ and $q = \frac{2n}{n-2s}$, by linearising the maximal operator to prove the global estimate (1.1) it suffices to prove that

$$\|Sf\|_{L^q(\mathbb{R}^n)} \le C \left(\int_{\mathbb{R}^n} |\xi_1|^{\frac{2s}{n}} |\xi_2|^{\frac{2s}{n}} |\cdots |\xi_n|^{\frac{2s}{n}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}. \tag{3.1}$$

Let $g(\xi) = |\xi_1|^{\frac{s}{n}} |\xi_2|^{\frac{s}{n}} \cdots |\xi_n|^{\frac{s}{n}} \hat{f}(\xi)$, then we have

$$Sf(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it(x)\Omega(\xi)} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} g(\xi) d\xi =: \ Rg(x), \tag{3.2}$$

where

$$Rg(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it(x)\Omega(\xi)} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} f(\xi) d\xi.$$

To prove (3.1) it suffices to prove that

$$\|Rg\|_{L^q(\mathbb{R}^n)} \le C \|g\|_{L^2(\mathbb{R}^n)} \tag{3.3}$$

for g is a function of continuous and rapidly decreasing at infinity.

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a real-valued function such that $\rho(x) = 1$ if $|x| \leq 1$ and $\rho(x) = 0$ if $|x| \geq 2$. Also let $\psi \in C_0^\infty(\mathbb{R})$ be a real-valued function such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| \geq 2$. For $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, we set $\sigma(\xi) = \psi(\xi_1)\psi(\xi_2)\cdots\psi(\xi_n)$. Thus, for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $N = 1, 2, 3, \ldots$, we set $\rho_N(x) = \rho(\frac{x}{N})$ and $\sigma_N(\xi) = \sigma(\frac{\xi}{N})$. For $x \in \mathbb{R}^n$, $g \in L^2(\mathbb{R}^n)$, and $N = 1, 2, 3, \ldots$, the operator R_N is defined by

$$R_N g(x) = \rho_N(x) \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{it(x)\phi(\xi)} |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} \sigma_N(\xi) g(\xi) d\xi.$$

The adjoint of R_N is given by

$$\begin{aligned} R_N' g(\xi) &= \sigma_N(\xi) |\xi_1|^{-\frac{s}{n}} |\xi_2|^{-\frac{s}{n}} \cdots |\xi_n|^{-\frac{s}{n}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{-it(x)\Omega(\xi)} \rho_N(x) h(x) dx, \\ &\quad \xi \in \mathbb{R}^n, \quad h \in L^2(\mathbb{R}^n). \end{aligned}$$

To prove (3.3) it is sufficient to prove that

$$\|R_N g\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^2(\mathbb{R}^n)}. \tag{3.4}$$

By duality, prove (3.4) it suffices to prove that

$$\|R_N' h\|_{L^2(\mathbb{R}^n)} \leq C\|h\|_{L^{q'}(\mathbb{R}^n)}, \tag{3.5}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. A similar calculation as (2.7) in proof of Theorem 1.1, we have

$$\begin{aligned} \|R_N' h\|^2_{L^2(\mathbb{R}^n))} &= \int_{\mathbb{R}^n} |R_N' h(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_N(x,y) \rho_N(x) \rho_N(y) h(x) \overline{h(y)} dx dy, \end{aligned} \tag{3.6}$$

where

$$K_N(x,y) = \int_{\mathbb{R}^n} |\xi_1|^{-\frac{2s}{n}} |\xi_2|^{-\frac{2s}{n}} \cdots |\xi_n|^{-\frac{2s}{n}} e^{i(y-x)\cdot\xi} e^{i(t(y)-(t(x))\Omega(\xi)} \sigma_N(\xi)^2 d\xi. \tag{3.7}$$

Since $\frac{n}{4} \leq s < \frac{n}{2}$, it follows that $\frac{1}{2} \leq \frac{2s}{n} < 1$, thus, by Lemma 2.1 , we obtain

$$|K_N(x,y)| \leq C \frac{1}{|x_1 - y_1|^{1-\frac{2s}{n}}} \frac{1}{|x_2 - y_2|^{1-\frac{2s}{n}}} \cdots \frac{1}{|x_n - y_n|^{1-\frac{2s}{n}}}. \tag{3.8}$$

We set

$$P_i f(x_1, x_2, \ldots, x_n) = \int_{\mathbb{R}} \frac{1}{|x_i - y_i|^{1-\frac{2s}{n}}} f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) dy_i,$$

$i = 1, 2, \ldots, n$. Thus, by (3.6) and (3.8), we obtain

$$\begin{aligned} &\int |R_N' h(x)|^2 dx \\ &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{1}{|x_1 - y_1|^{1-\frac{2s}{n}}} \frac{1}{|x_2 - y_2|^{1-\frac{2s}{n}}} \cdots \frac{1}{|x_n - y_n|^{1-\frac{2s}{n}}} |h(x)||h(y)| dx dy \end{aligned} \tag{3.9}$$

$$= C \int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{1}{|x_n - y_n|^{1-\frac{2s}{n}}} \frac{1}{|x_{n-1} - y_{n-1}|^{1-\frac{2s}{n}}} \cdots \frac{1}{|x_2 - y_2|^{1-\frac{2s}{n}}}$$
$$\times \bigg(\int_{\mathbb{R}} \frac{1}{|x_1 - y_1|^{1-\frac{2s}{n}}} |h(y_1, y_2, \dots, y_n)| dy_1 \bigg) dy_2 \cdots dy_{n-1} dy_n \bigg) |h(x)| dx$$
$$= C \int_{\mathbb{R}^n} P_n P_{n-1} \cdots P_2 P_1 |h|(x) |h(x)| dx.$$

Invoking Hölder's inequality, we get

$$\int_{\mathbb{R}^n} |R'_N h(\xi)|^2 d\xi \leq C \|P_n P_{n-1} \cdots P_2 P_1 |h|\|_{L^q(\mathbb{R}^n)} \|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{3.10}$$

Since $q = \frac{2n}{n-2s}$, it follows that $q' = \frac{2n}{n+2s}$ and the fact $\frac{1}{q} = \frac{1}{q'} - \frac{2s}{n}$. Similar to estimate (2.12), we have

$$\bigg(\int_{\mathbb{R}} |P_j h(x)|^q dx_j \bigg)^{1/q} \leq C \bigg(\int_{\mathbb{R}} |h(x)|^{q'} dx_j \bigg)^{1/q'}, \tag{3.11}$$

where $j = 1, 2, \dots, n$. By (3.11) and Minkowski's inequality, we have

$$\|P_n P_{n-1} \cdots P_2 P_1 |h|\|_{L^q(\mathbb{R}^n)} \leq C \|h\|_{L^{q'}(\mathbb{R}^n)}. \tag{3.12}$$

Therefore, (3.5) follows from (3.10) and (3.12). Now we complete the proof of Theorem 1.2.

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