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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE KAEHLER MANIFOLD WITH A QUARTER-SYMMETRIC METRIC CONNECTION

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Abstract. Jin studied lightlike hypersurfaces of an indefinite Kaehler manifold [6, 8] or indefinite trans-Sasakian manifold [7] with a quarter-symmetric metric connection. Jin also studied generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection [10]. We study generic lightlike submanifolds of an indefinite Kaehler manifold with a quarter-symmetric metric connection.

1. Introduction

A lightlike submanifold M of an indefinite almost complex manifold $\overline{M}$ is called *generic* if there exists a screen distribution $S(TM)$ of M such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.1}$$

where $S(TM)^{\perp}$ is the orthogonal complement of $S(TM)$ in the tangent bundle $T\overline{M}$ of $\overline{M}$. The generic lightlike submanifold was introduced by Jin-Lee [11] and later, studied by several authors [3–5,10,12]. The theory of generic lightlike submanifolds is an extension of that of lightlike hypersurfaces.

A linear connection $\overline{\nabla}$ on a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be a *quarter-symmetric connection* if its torsion tensor $\overline{T}$ satisfies

$$\overline{T}(\overline{X}, \overline{Y}) = \theta(\overline{Y})J\overline{X} - \theta(\overline{X})J\overline{Y}, \tag{1.2}$$

where J is a $(1,1)$-type tensor field and θ is a 1-form associated with a smooth unit vector field ζ by $\theta(X) = \overline{g}(X, \zeta)$. Throughout this paper, we denote by $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ the smooth vector fields on $\overline{M}$. Furthermore, if $\overline{\nabla}$ is a metric connection, then we say that $\overline{\nabla}$ is a *quarter-symmetric metric connection.* The notion of quarter-symmetric metric connection was introduced Yano-Imai [14]. Recently, Jin extended this notion to indefinite Kaehler manifold or indefinite trans-Sasakian manifold and then, studied the geometry of lightlike hypersurfaces of an indefinite Kaehler manifold [6, 8] or indefinite trans-Sasakian

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manifold [7] with a quarter-symmetric metric connection. Also, Jin studied generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection [10].

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of an indefinite Kaehler manifold $(\overline{M}, \overline{g}, J)$ with respect to $\overline{g}$. It is known [8] that *a linear connection $\overline{\nabla}$ on $\overline{M}$ is a quarter-symmetric metric connection if and only if $\overline{\nabla}$ sinisfies*

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \widetilde{\nabla}_{\overline{X}}\overline{Y} - \theta(\overline{X})J\overline{Y}. \tag{1.3}$$

The object of study of this paper is generic lightlike submanifolds of an indefinite Kaehler manifold $(\overline{M}, \overline{g}, J)$ with a quarter-symmetric metric connection, in which the tensor field J, defined by (1.2), is identical with the indefinite almost complex structure J on $(\overline{M}, \overline{g}, J)$.

In this paper, by the method of [10] but using indefinite Kaehler manifolds instead of indefinite trans-Sasakian manifolds and indefinite complex space forms instead of indefinite generalized Sasakian space forms, we study generic lightlike submanifolds of an indefinite Kaehler manifold and indefinite complex space form with a quarter-symmetric metric connection.

2. Preliminaries

Let $\overline{M} = (\overline{M}, \overline{g}, J)$ be an indefinite Kaeler manifold, where $\overline{g}$ is a semi-Riemannian metric and J is an indefinite almost complex structure such that

$$J^2\overline{X} = -\overline{X}, \qquad \overline{g}(J\overline{X}, J\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}), \qquad (\widetilde{\nabla}_{\overline{X}}J)\overline{Y} = 0. \tag{2.1}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the quarter-symmetric metric connection $\overline{\nabla}$ given by (1.3), the third equation of (2.1) is reduced to

$$(\overline{\nabla}_{\overline{X}}J)\overline{Y} = 0. \tag{2.2}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite Kaehler manifold $\overline{M}$ of dimension $(m+n)$. Then the radical distribution $Rad(TM)$ of M, defined by $Rad(TM) = TM \cap TM^{\perp}$, is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \le r \le \min\{m, n\})$. In case $1 < r < \min\{m, n\}$, we say that M is an *r-lightlike submanifold* [2] of $\overline{M}$. In this case, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which are called the *screen* and *co-screen* distributions of M [2], such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \ TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y, Z and W be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i, j, k, \ldots \ \in \{1, \ldots, r\}, \qquad a, b, c, \ldots \ \in \{r+1, \ldots, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\overline{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1, \dots, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0,$$

where $\{\xi_1, \dots, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. Then we have

$$\begin{aligned} T\overline{M} = &\ TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ = &\ \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

For the rest of this paper, we consider only r-lightlike submanifolds M, with following local quasi-orthonormal field of frames of $\overline{M}$:

$$\{\xi_1, \dots, \xi_r, N_1, \dots, N_r, F_{r+1}, \dots, F_m, E_{r+1}, \dots, E_n\},$$

where $\{F_{r+1}, \dots, F_m\}$ and $\{E_{r+1}, \dots, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \overline{g}(E_a, E_a)$. Then $\epsilon_a \delta_{ab} = \overline{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X,Y) N_i + \sum_{a=r+1}^{n} h_a^s(X,Y) E_a, \tag{2.3}$$

$$\overline{\nabla}_X N_i = -A_{N_i} X + \sum_{j=1}^{r} \tau_{ij}(X) N_j + \sum_{a=r+1}^{n} \rho_{ia}(X) E_a, \tag{2.4}$$

$$\overline{\nabla}_X E_a = -A_{E_a} X + \sum_{i=1}^{r} \phi_{ai}(X) N_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) E_b; \tag{2.5}$$

$$\nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h_i^*(X, PY) \xi_i, \tag{2.6}$$

$$\nabla_X \xi_i = -A^*_{\xi_i} X - \sum_{j=1}^{r} \tau_{ji}(X) \xi_j, \tag{2.7}$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h_i^{ℓ} and h_a^s are called the *local second fundamental forms* on M, h_i^* are called the *local screen second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are linear operators on M, and τ_{ij}, ρ_{ia}, ϕ_{ai} and σ_{ab} are 1-forms on M.

For any generic lightlike submanifold M, from (1.1) we show that the distributions $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$\begin{aligned} S(TM) &= \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp})) \oplus_{orth} H_o, \\ H &= Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o. \end{aligned}$$

In this case, the tangent bundle TM of M is decomposed as follows:

$$TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp})). \tag{2.8}$$

Consider $2r$ local null vector fields U_i and V_i, $(n-r)$ local non-null unit vector fields W_a, and their 1-forms u_i, v_i and w_a defined by

$$U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a, \tag{2.9}$$

$$u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a). \tag{2.10}$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a. \tag{2.11}$$

Applying J to (2.11) and using $(2.1)_1$ and (2.9), we have

$$F^2X = -X + \sum_{i=1}^{r} u_i(X)U_i + \sum_{a=r+1}^{n} w_a(X)W_a. \tag{2.12}$$

We say that F is the *induced structure tensor field* of J on M.

3. Quarter-symmetric metric connection

Substituting (2.3) into $(\overline{\nabla}_X \overline{g})(Y, Z) = 0$, we obtain

$$(\nabla_X g)(Y, Z) = \sum_{i=1}^{r} \{h_i^{\ell}(X, Y)\eta_i(Z) + h_i^{\ell}(X, Z)\eta_i(Y)\}, \tag{3.1}$$

where η_i is a 1-form such that $\eta_i(X) = \overline{g}(X, N_i)$. Substituting (2.3) and (2.11) into (1.2) and then, comparing the tangent, lightlike transversal and co-screen components of the left and right terms, we have

$$T(X, Y) = \theta(Y)FX - \theta(X)FY, \tag{3.2}$$

$$h_i^{\ell}(X, Y) - h_i^{\ell}(Y, X) = \theta(Y)u_i(X) - \theta(X)u_i(Y), \tag{3.3}$$

$$h_a^{s}(X, Y) - h_a^{s}(Y, X) = \theta(Y)w_a(X) - \theta(X)w_a(Y), \tag{3.4}$$

where T is the torsion tensor with respect to the induced connection ∇.

From the facts that $h_i^{\ell}(X, Y) = \overline{g}(\overline{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^{s}(X, Y) = \overline{g}(\overline{\nabla}_X Y, E_a)$, we know that h_i^{ℓ} and h_a^{s} are independent of the choice of $S(TM)$. The local second fundamental forms are related to their shape operators by

$$h_i^{\ell}(X, Y) = g(A^*_{\xi_i}X, Y) - \sum_{k=1}^{r} h_k^{\ell}(X, \xi_i)\eta_k(Y), \tag{3.5}$$

$$\epsilon_a h_a^{s}(X, Y) = g(A_{E_a}X, Y) - \sum_{k=1}^{r} \phi_{ak}(X)\eta_k(Y), \tag{3.6}$$

$$h_i^{*}(X, PY) = g(A_{N_i}X, PY). \tag{3.7}$$

Applying $\overline{\nabla}_X$ to $g(\xi_i, \xi_j) = 0$, $\overline{g}(\xi_i, E_a) = 0$, $\overline{g}(N_i, N_j) = 0$, $\overline{g}(N_i, E_a) = 0$ and $\overline{g}(E_a, E_b) = \epsilon\delta_{ab}$, we obtain

$$
\begin{aligned}
&h_i^\ell(X, \xi_j) + h_j^\ell(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = -\epsilon_a \phi_{ai}(X),\\
&\eta_j(A_{N_i} X) + \eta_i(A_{N_j} X) = 0, \quad \overline{g}(A_{E_a} X, N_i) = \epsilon_a \rho_{ia}(X),\\
&\epsilon_b \sigma_{ab} + \epsilon_a \sigma_{ba} = 0 \quad \text{and} \quad h_i^\ell(X, \xi_i) = 0, \quad h_i^\ell(\xi_j, \xi_k) = 0.
\end{aligned} \tag{3.8}
$$

Applying $\overline{\nabla}_X$ to (2.9) and (2.11) by turns and using (2.2) ∼ (2.5), (2.7), (2.9) ∼ (2.11) and (3.5) ∼ $(3.8)_2$, we have

$$
\begin{aligned}
&h_j^\ell(X, U_i) = h_i^*(X, V_j), \qquad \epsilon_a h_i^*(X, W_a) = h_a^s(X, U_i),\\
&h_j^\ell(X, V_i) = h_i^\ell(X, V_j), \qquad \epsilon_a h_i^\ell(X, W_a) = h_a^s(X, V_i),\\
&\epsilon_b h_b^s(X, W_a) = \epsilon_a h_a^s(X, W_b),
\end{aligned} \tag{3.9}
$$

$$
\nabla_X U_i = F(A_{N_i} X) + \sum_{j=1}^{r} \tau_{ij}(X) U_j + \sum_{a=r+1}^{n} \rho_{ia}(X) W_a, \tag{3.10}
$$

$$
\begin{aligned}
\nabla_X V_i = {}& F(A^*_{\xi_i} X) - \sum_{j=1}^{r} \tau_{ji}(X) V_j + \sum_{j=1}^{r} h_j^\ell(X, \xi_i) U_j\\
&- \sum_{a=r+1}^{n} \epsilon_a \phi_{ai}(X) W_a,
\end{aligned} \tag{3.11}
$$

$$
\nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^{r} \phi_{ai}(X) U_i + \sum_{b=r+1}^{n} \sigma_{ab}(X) W_b, \tag{3.12}
$$

$$
\begin{aligned}
(\nabla_X F)(Y) = {}& \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X\\
&- \sum_{i=1}^{r} h_i^\ell(X, Y) U_i - \sum_{a=r+1}^{n} h_a^s(X, Y) W_a.
\end{aligned} \tag{3.13}
$$

4. Recurrent and Lie recurrent submanifolds

Recently, Jin [10] studied recurrent and Lie recurrent generic lightlike submanifolds of an indefinite trans-Sasakian manifold with a quarter-symmetric metric connection. We now follow the general scheme in [10]. Using the method of [10], we study recurrent and Lie recurrent generic lightlike submanifolds of an indefinite Kaehler manifold with a quarter-symmetric metric connection.

Definition. We say that a lightlike submanifold M of $\overline{M}$ is called
(1) *irrotational* [13] if $\overline{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \ldots, r\}$,
(2) *solenoidal* [12] if A_{E_a} and A_{N_i} are $S(TM)$-valued,
(3) *statical* [12] if M is both irrotational and solenoidal.

Remark 4.1. From (2.3) and $(3.8)_2$, the item (1) is equivalent to

$$h^\ell_j(X,\xi_i)=0, \qquad h^s_a(X,\xi_i)=\phi_{ai}(X)=0. \tag{4.1}$$

By using $(3.8)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i}X)=0, \qquad \rho_{ia}(X)=\eta_i(A_{E_a}X)=0. \tag{4.2}$$

Denote by λ_{ij}, μ_{ia}, ν_{ia}, κ_{ab} and χ_{ij} the 1-forms on M such that

$$\begin{aligned} &\lambda_{ij}(X)=h^\ell_i(X,U_j)=h^*_j(X,V_i), && \kappa_{ab}(X)=\epsilon_a h^s_a(X,W_b),\\ &\mu_{ia}(X)=h^\ell_i(X,W_a)=\epsilon_a h^s_a(X,V_i), && \chi_{ij}(X)=h^\ell_i(X,V_j),\\ &\nu_{ai}(X)=h^*_i(X,W_a)=\epsilon_a h^s_a(X,U_i). \end{aligned} \tag{4.3}$$

Definition. The structure tensor field F of the generic lightlike submanifold M is said to be *recurrent* [9,10] if there exists a 1-form ϖ on M such that

$$(\nabla_X F)Y=\varpi(X)FY.$$

Theorem 4.2. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold with a quarter-symmetric metric connection. If F is recurrent, then*

(1) *F is parallel with respect to the induced connection ∇ on M,*
(2) *M is statical,*
(3) *$J(ltr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions on M, and*
(4) *M is locally a product manifold $M_r\times M_{n-r}\times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H respectively.*

Proof. (1) From the above definition and (3.13), we obtain

$$\begin{aligned} \varpi(X)FY \;=\; &\sum_{i=1}^r u_i(Y)A_{N_i}X+\sum_{a=r+1}^n w_a(Y)A_{E_a}X\\ &-\sum_{i=1}^r h^\ell_i(X,Y)U_i-\sum_{a=r+1}^n h^s_a(X,Y)W_a. \end{aligned} \tag{4.4}$$

Replacing Y by ξ_j to this equation and using the fact that $F\xi_j=-V_j$, we get

$$\varpi(X)V_j=\sum_{k=1}^r h^\ell_k(X,\xi_j)U_k+\sum_{b=r+1}^n h^s_b(X,\xi_j)W_b. \tag{4.5}$$

Taking the scalar product with U_j, V_i and W_a by turns, we obtain

$$\varpi=0, \qquad h^\ell_i(X,\xi_j)=0, \qquad h^s_a(X,\xi_j)=\phi_{aj}(X)=0,$$

respectively. As $\varpi=0$, F is parallel with respect to the connection ∇.

(2) As $h^\ell_i(X,\xi_j)=0$ and $h^s_a(X,\xi_j)=0$, by (4.1) M is irrotational. Also, M is solenoidal. In fact, taking the scalar product with N_j to (4.4), we have

$$\sum_{i=1}^r u_i(Y)\bar g(A_{N_i}X,N_j)+\sum_{a=r+1}^n w_a(Y)\bar g(A_{E_a}X,N_j)=0.$$

Taking $Y = U_i$ and $Y = W_a$ by turns, we get (4.2). Thus M is statical.

(3) Taking the scalar product with U_j to (4.4), we get

$$\sum_{i=1}^{r} u_i(Y)g(A_{N_i}X, U_j) + \sum_{a=r+1}^{n} w_a(Y)g(A_{E_a}X, U_j) = 0.$$

Taking $Y = U_k$ and $Y = W_b$ to this equation by turns, we obtain

$$h_i^*(X, U_j) = \bar{g}(A_{N_i}X, U_j) = 0, \qquad \nu_{ai}(X) = \bar{g}(A_{E_a}X, U_i) = 0. \tag{4.6}$$

Taking the scalar product with V_j and W_b to (4.4) by turns, we have

$$h_i^\ell(X, Y) = \sum_{j=1}^{r} \lambda_{ij}(X)u_j(Y) + \sum_{a=r+1}^{n} \mu_{ia}(X)w_a(Y), \tag{4.7}$$

$$\epsilon_a h_a^s(X, Y) = \sum_{b=r+1}^{n} \kappa_{ba}(X)w_b(Y),$$

by (3.6), (3.7), (4.3) and $(4.6)_2$. Replacing Y by V_j to $(4.7)_{1,2}$, we have

$$\chi_{ij}(X) = h_i^\ell(X, V_j) = 0, \qquad \mu_{ia}(X) = h_a^s(X, V_i) = 0. \tag{4.8}$$

Taking $Y = U_j$ and $Y = W_b$ to (4.4) and using (4.3), $(4.6)_2$ and $(4.8)_2$, we get

$$A_{N_i}X = \sum_{j=1}^{r} \lambda_{ji}(X)U_j, \qquad A_{E_a}X = \sum_{b=r+1}^{n} \epsilon_b \kappa_{ba}(X)W_b. \tag{4.9}$$

Using (3.5), (4.1), $(4.8)_2$ and the non-degenerateness of $S(TM)$, $(4.7)_1$ reduces

$$A^*_{\xi_i}X = \sum_{j=1}^{r} \lambda_{ij}(X)V_j. \tag{4.10}$$

Applying F to $(4.9)_{1,2}$, we have $F(A_{N_i}X) = 0$ and $F(A_{E_a}X) = 0$. Substituting these results into (3.10) and (3.12), we obtain

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \qquad \nabla_X W_a = \sum_{b=r+1}^{n} \sigma_{ab}(X)W_b. \tag{4.11}$$

It follow that $J(ltr(TM))$ and $J(S(TM^\perp))$ are parallel distributions on M with respect to the induced connection ∇ on M, that is,

$$\nabla_X U_i \in \Gamma(J(ltr(TM))), \qquad \nabla_X W_a \in \Gamma(J(S(TM^\perp))).$$

Applying F to (4.10), we get $F(A^*_{\xi_i}X) = \sum_{j=1}^{r} \lambda_{ij}(X)\xi_j$. Thus we have

$$\nabla_X V_i = \sum_{j=1}^{r} \{\lambda_{ij}(X)\xi_j - \tau_{ji}(X)V_j\}. \tag{4.12}$$

By directed calculations from (2.3), (4.8), $(4.11)_2$ and (4.12), we see that $g(\nabla_X Y, V_i) = 0$ and $g(\nabla_X Y, W_a) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Thus

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \;\; \forall\, Y \in \Gamma(H).$$

Thus H is also a parallel distribution on M with respect to ∇.

(4) As $J(ltr(TM))$, $J(S(TM^{\perp}))$ and H are parallel distributions and satisfy the decomposition form (2.8), by the decomposition theorem of de Rham [1], M is locally a product manifold $M_r \times M_{n-r} \times M^{\sharp}$, where M_r, M_{n-r} and $M^{\sharp}$ are leaves of $J(ltr(TM)), J(S(TM^{\perp}))$ and H respectively. $\square$

Definition. The structure tensor field F of M is said to be *Lie recurrent* or *L-recurrent* [9] if there exists a 1-form ϑ on M such that

$$(\mathcal{L}_X F)Y = \vartheta(X)FY.$$

In particular, if $\vartheta = 0$, that is, $\mathcal{L}_X F = 0$, then F is called *Lie parallel*, where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X, that is,

$$(\mathcal{L}_X F)Y = [X, FY] - F[X, Y].$$

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite Kaehler manifold with a quarter-symmetric metric connection. If F is L-recurrent, then*

(1) *F is Lie parallel,*

(2) *τ_{ij} and ρ_{ia} are satisfied $\tau_{ij} \circ F = 0$ and $\rho_{ia} \circ F = 0$. Moreover,*

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X) g(A_{N_k} V_j, N_i).$$

Proof. (1) Using (2.12), (3.2) and (3.13), we get

$$\begin{aligned}
\vartheta(X)FY \;=\; & -\nabla_{FY}X + F\nabla_Y X - \theta(Y)X - \theta(FY)FX \\
& + \sum_{i=1}^{r} u_i(Y)A_{N_i}X + \sum_{a=r+1}^{n} w_a(Y)A_{E_a}X \\
& - \sum_{i=1}^{r}\{h_i^{\ell}(X,Y) - \theta(Y)u_i(X)\}U_i \\
& - \sum_{a=r+1}^{n}\{h_a^{s}(X,Y) - \theta(Y)w_a(X)\}W_a.
\end{aligned} \tag{4.13}$$

Let $\alpha_i = \theta(\xi_i)$. Taking $Y = \xi_j$ and $Y = V_j$ to (4.13) respectively, we have

$$\begin{aligned}
-\vartheta(X)V_j \;=\; & \nabla_{V_j}X + F\nabla_{\xi_j}X - \alpha_j X + \theta(V_j)FX \\
& - \sum_{i=1}^{r}\{h_i^{\ell}(X,\xi_j) - \alpha_j u_i(X)\}U_i \\
& - \sum_{a=r+1}^{n}\{h_a^{s}(X,\xi_j) - \alpha_j w_a(X)\}W_a,
\end{aligned} \tag{4.14}$$

$$\vartheta(X)\xi_j = -\nabla_{\xi_j}X + F\nabla_{V_j}X - \theta(V_j)X - \alpha_j FX \tag{4.15}$$

$$- \sum_{i=1}^{r}\{h_i^\ell(X, V_j) - \theta(V_j)u_i(X)\}U_i$$
$$- \sum_{a=r+1}^{n}\{h_a^s(X, V_j) - \theta(V_j)w_a(X)\}W_a.$$

Taking the scalar product with U_i to (4.14) and then, taking the scalar product with N_i to (4.15), we obtain respectively

$$-\vartheta(X)\delta_{ij} = g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i) - \alpha_j v_i(X) - \theta(V_j)\eta_i(X),$$
$$\vartheta(X)\delta_{ij} = g(\nabla_{V_j}X, U_i) - \bar{g}(\nabla_{\xi_j}X, N_i) - \alpha_j v_i(X) - \theta(V_j)\eta_i(X).$$

Comparing these two equations, we get $\vartheta = 0$. Thus F is Lie parallel.

(2) Taking the scalar product with N_i to (4.14) such that $X = W_a$ and using (3.4), (3.6), $(3.8)_4$ and (3.12), we get $h_a^s(U_i, V_j) = \rho_{ia}(\xi_j)$. On the other hand, taking the scalar product with W_a to (4.15) such that $X = U_i$ and using (3.10), we have $h_a^s(U_i, V_j) = -\rho_{ia}(\xi_j)$. Thus $\rho_{ia}(\xi_j) = 0$ and $h_a^s(U_i, V_j) = 0$.

Taking the scalar product with U_i to (4.14) such that $X = W_a$ and using (3.4), (3.6), $(3.8)_{2,4}$ and (3.12), we get $\epsilon_a\rho_{ia}(V_j) = \phi_{aj}(U_i)$. On the other hand, taking the scalar product with W_a to (4.14) such that $X = U_i$ and using $(3.8)_2$ and (3.10), we get $\epsilon_a\rho_{ia}(V_j) = -\phi_{aj}(U_i)$. Thus $\rho_{ia}(V_j) = 0$ and $\phi_{aj}(U_i) = 0$.

Taking the scalar product with V_i to (4.14) such that $X = W_a$ and using (3.3), (3.4), $(3.8)_2$, $(3.9)_4$ and (3.12), we get $\phi_{ai}(V_j) = -\phi_{aj}(V_i)$. On the other hand, taking the scalar product with W_a to (4.14) such that $X = V_i$ and using $(3.8)_2$ and (3.11), we have $\phi_{ai}(V_j) = \phi_{aj}(V_i)$. Thus $\phi_{ai}(V_j) = 0$.

Taking the scalar product with W_a to (4.14) such that $X = \xi_i$ and using (2.7), (3.5) and $(3.8)_2$, we get $h_i^\ell(V_j, W_a) = \phi_{ai}(\xi_j)$. On the other hand, taking the scalar product with V_i to (4.15) such that $X = W_a$ and using (3.3) and (3.12), we have $h_i^\ell(V_j, W_a) = -\phi_{ai}(\xi_j)$. Thus $\phi_{ai}(\xi_j) = 0$ and $h_i^\ell(V_j, W_a) = 0$. Summarizing the above results, we obtain

$$\begin{aligned}(4.16)\quad & \rho_{ia}(\xi_j) = 0,\ \rho_{ia}(V_j) = 0,\ \phi_{ai}(U_j) = 0,\ \phi_{ai}(V_j) = 0,\ \phi_{ai}(\xi_j) = 0,\\ & h_a^s(U_i, V_j) = h_j^\ell(U_i, W_a) = 0, \quad h_i^\ell(V_j, W_a) = h_a^s(V_j, V_i) = 0.\end{aligned}$$

Taking the scalar product with N_i to (4.13) and using $(3.8)_4$, we have

$$\begin{aligned}(4.17)\qquad & -\bar{g}(\nabla_{FY}X, N_i) + \bar{g}(\nabla_Y X, U_i) - \theta(Y)\eta_i(X) - \theta(FY)v_i(X)\\ & + \sum_{k=1}^{r} u_k(Y)\bar{g}(A_{N_k}X, N_i) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\rho_{ia}(X) = 0.\end{aligned}$$

Replacing X by V_j to (4.17) and using (3.5), (3.11) and $(4.16)_2$, we have

$$(4.18)\qquad h_j^\ell(FX, U_i) + \tau_{ij}(X) + \delta_{ij}\theta(FX) = \sum_{k=1}^{r} u_k(X)\bar{g}(A_{N_k}V_j, N_i).$$

Replacing X by ξ_j to (4.17) and using (2.7), (3.5) and $(4.16)_1$, we have

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = \sum_{k=1}^{r} u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) + \tau_{ij}(FX). \tag{4.19}$$

Taking $X = U_k$ to (4.19), we have

$$h_i^*(U_k, V_j) = h_j^\ell(U_k, U_i) = \overline{g}(A_{N_k}\xi_j, N_i) - \delta_{ij}\theta(U_k). \tag{4.20}$$

Replacing X by U_i to (4.13) and using (2.12), (3.3), (3.4), (3.7), $(3.9)_{1,2}$ and (3.10), we obtain

$$\begin{aligned}\sum_{k=1}^{r} u_k(Y)A_{N_k}U_i &+ \sum_{a=r+1}^{n} w_a(Y)A_{E_a}U_i - \theta(Y)U_i \\ &+ \theta(U_i)\{\sum_{i=1}^{r} u_j(Y)U_j + \sum_{a=r+1}^{n} w_a(Y)W_a\} - A_{N_i}Y \\ &- F(A_{N_i}FY) - \sum_{j=1}^{r}\tau_{ij}(FY)U_j - \sum_{a=r+1}^{n}\rho_{ia}(FY)W_a = 0.\end{aligned} \tag{4.21}$$

Taking the scalar product with V_j to (4.21) and using (3.7), $(3.8)_3$, $(3.9)_1$, $(4.16)_6$ and (4.20), we get

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = -\sum_{k=1}^{r} u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) - \tau_{ij}(FX).$$

Comparing this equation with (4.19), we obtain

$$h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = 0, \quad \tau_{ij}(FX) + \sum_{k=1}^{r} u_k(X)\overline{g}(A_{N_k}\xi_j, N_i) = 0.$$

Replacing X by U_h to the second equation, we get $\overline{g}(A_{N_k}\xi_j, N_i) = 0$. Thus,

$$\tau_{ij}(FX) = 0, \qquad h_j^\ell(X, U_i) + \delta_{ij}\theta(X) = 0. \tag{4.22}$$

Taking $X = FY$ to $(4.22)_2$, we obtain

$$h_j^\ell(FX, U_i) + \delta_{ij}\theta(FY) = 0.$$

From this equation and (4.18), we see that

$$\tau_{ij}(X) = \sum_{k=1}^{r} u_k(X)\overline{g}(A_{N_k}V_j, N_i). \tag{4.23}$$

Replacing Y by W_b to (4.21), we have

$$A_{N_i}W_a + \theta(W_a)U_i = A_{E_a}U_i + \theta(U_i)W_a.$$

Taking the product with U_j and using (3.4), (3.6), (3.7) and $(3.9)_2$, we get

$$h_i^*(W_a, U_j) = \epsilon_a h_a^s(U_i, U_j) = \epsilon_a h_a^s(U_j, U_i) = h_i^*(U_j, W_a). \tag{4.24}$$

Taking the scalar product with W_a to (4.21), we have

$$\epsilon_a\rho_{ia}(FY) = -h_i^*(Y, W_a) + \epsilon_a\theta(U_i)w_a(Y) + \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) + \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Taking the scalar product with U_i to (4.13) and then, taking $X = W_a$ and using (3.4), (3.6), (3.7), $(3.8)_4$, $(3.9)_2$, (3.12) and (4.24), we obtain

$$\epsilon_a\rho_{ia}(FY) = h_i^*(Y, W_a) - \epsilon_a\theta(U_i)w_a(Y) - \sum_{k=1}^{r} u_k(Y)h_k^*(U_i, W_a) - \sum_{b=r+1}^{n} \epsilon_b w_b(Y)h_b^s(U_i, W_a).$$

Comparing the last two equations, we obtain $\rho_{ia}(FY) = 0$. □

5. Indefinite complex space forms

Denote by $\overline{R}$, R and R^* the curvature tensor of the quarter-symmetric metric connection $\overline{\nabla}$ on $\overline{M}$, and the curvature tensors of the induced connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain the Gauss equations for M and $S(TM)$, respectively:

$$\begin{aligned}\overline{R}(X,Y)Z = R(X,Y)Z &+ \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\ &+ \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\ &+ \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ &+ \sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)] \\ &+ \sum_{a=r+1}^{n}[\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)] \\ &- \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z)\}N_i \\ &+ \sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\ &+ \sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_a^s(X,Z)]\end{aligned} \tag{5.1}$$

$$+ \sum_{b=r+1}^{n} [\sigma_{ba}(X)h_b^s(Y,Z) - \sigma_{ba}(Y)h_b^s(X,Z)] - \theta(X)h_a^s(FY,Z) + \theta(Y)h_a^s(FX,Z)\}E_a,$$

$$\begin{aligned}(5.2)\qquad R(X,\,Y)PZ \;=\; & R^*(X,\,Y)PZ \\ & + \sum_{i=1}^{r}\{h_i^*(X,\,PZ)A^*_{\xi_i}Y - h_i^*(Y,\,PZ)A^*_{\xi_i}X\}, \\ & + \sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,\,PZ) - (\nabla_Y h_i^*)(X,\,PZ) \\ & + \sum_{j=1}^{r}[h_j^*(X,\,PZ)\tau_{ij}(Y) - h_j^*(Y,\,PZ)\tau_{ij}(X)] \\ & - \theta(X)h_i^*(FY,PZ) + \theta(Y)h_i^*(FX,PZ)\}\xi_i.\end{aligned}$$

Definition. An indefinite complex space form $\overline{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\begin{aligned}(5.3)\qquad \widetilde{R}(\overline{X},\overline{Y})\overline{Z} \;=\; & \frac{c}{4}\{\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y} \\ & + \overline{g}(J\overline{Y},\overline{Z})J\overline{X} - \overline{g}(J\overline{X},\overline{Z})J\overline{Y} + 2\overline{g}(\overline{X},J\overline{Y})J\overline{Z}\},\end{aligned}$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\widetilde{\nabla}$ on $\overline{M}$.

By directed calculations from (1.2) and (1.3), we see that

$$\begin{aligned}(5.4)\qquad \overline{R}(\overline{X},\overline{Y})\overline{Z} = \widetilde{R}(\overline{X},\overline{Y})\overline{Z} - \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) \\ + \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}JZ.\end{aligned}$$

Taking the scalar product with ξ_i and N_i to (5.4) by turns and substituting (5.1) and (5.3) into the left-right terms and using $(3.8)_4$ and (5.2), we get

$$\begin{aligned}(5.5)\qquad & (\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z) \\ & + \sum_{k=1}^{r}\{\tau_{ki}(X)h_k^\ell(Y,Z) - \tau_{ki}(Y)h_k^\ell(X,Z)\} \\ & + \sum_{a=r+1}^{n}\{\phi_{ai}(X)h_a^s(Y,Z) - \phi_{ai}(Y)h_a^s(X,Z)\} \\ & - \theta(X)h_i^\ell(FY,Z) + \theta(Y)h_i^\ell(FX,Z) \\ & + \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) + \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}u_i(Z) \\ = \; & \frac{c}{4}\{u_i(X)\overline{g}(JY,Z) - u_i(Y)\overline{g}(JX,Z) + 2u_i(Z)\overline{g}(X,JY)\},\end{aligned}$$

$$(5.6)\qquad (\nabla_X h_i^*)(Y,PZ) - (\nabla_Y h_i^*)(X,PZ)$$

$$
\begin{aligned}
&+ \sum_{k=1}^{r}\{\tau_{ik}(Y)h_k^*(X,PZ) - \tau_{ik}(X)h_k^*(Y,PZ)\} \\
&+ \sum_{k=1}^{r}\{h_k^\ell(X,PZ)\eta_i(A_{N_k}Y) - h_k^\ell(Y,PZ)\eta_i(A_{N_k}X)\} \\
&+ \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(Y)h_a^s(X,\,PZ) - \rho_{ia}(X)h_a^s(Y,\,PZ)\} \\
&- \theta(X)h_i^*(FY,PZ) + \theta(Y)h_i^*(FX,PZ) \\
&+ \{(\overline{\nabla}_X\theta)(Y) - (\overline{\nabla}_Y\theta)(X) + \theta(Y)\theta(JX) - \theta(X)\theta(JY)\}v_i(PZ) \\
= \;& \frac{c}{4}\{g(Y,PZ)\eta_i(X) - g(X,PZ)\eta_i(Y) \\
&\quad + v_i(X)\overline{g}(JY,PZ) - v_i(Y)\overline{g}(JX,PZ) + 2v_i(PZ)\overline{g}(X,JY)\}.
\end{aligned}
$$

Theorem 5.1. *Let M be a generic lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ with a quarter-symmetric metric connection. If one of the following four conditions is satisfied;*

(1) *F is recurrent,*

(2) *F is Lie recurrent,*

(3) *U_is are parallel with respect to the induced connection ∇, or*

(4) *W_as are parallel with respect to ∇ and $A_{N_i}\xi_i$ belong to $S(TM)$,*

then $c = 0$ and $\overline{M}(c)$ is flat. Furthermore, in case (3) *M is solenoidal.*

Proof. (1) By Theorem 4.1, we show that M is solenoidal, *i.e.*, $\eta_j(A_{N_i}X) = 0$ and $\rho_{ia} = 0$, and the equations (4.9) and (4.11) are satisfied. Taking the scalar product with U_j to $(4.9)_1$ and using (3.7), we obtain

$$h_i^*(X,U_j) = 0.$$

Applying ∇_X to $h_i^*(Y,U_j) = 0$ and using $(4.11)_1$, we obtain

$$(\nabla_X h_i^*)(Y,U_j) = 0.$$

Taking $PZ = U_j$ to (5.6) and using the last two equations, we have

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(2) Replacing Y by U_j to (3.3) and using $(4.22)_2$, we have

$$h_i^\ell(U_j,X) = -\theta(U_j)u_i(X).$$

From this equation and (2.10), we obtain

$$h_i^\ell(U_i,F(A_{N_j}\xi_j)) = -\theta(U_i)u_i(F(A_{N_j}\xi_j)) = 0.$$

Replacing X by ξ_j and Y by $F(A_{N_j}U_i)$ to (3.3), we obtain

$$h_i^\ell(\xi_j,F(A_{N_j}U_i)) = h_i^\ell(F(A_{N_j}U_i),\xi_j).$$

Taking $Y = U_j$ to (4.21), we have

$$A_{N_j} U_i + \theta(U_i)U_j = A_{N_i} U_j + \theta(U_j)U_i.$$

Applying F to this equation, we get $F(A_{N_j} U_i) = F(A_{N_i} U_j)$. Thus $F(A_{N_j} U_i)$ is symmetric with respect to i and j. From this result and $(3.8)_1$, we obtain

$$h_i^\ell(\xi_j, F(A_{N_j} U_i)) = h_i^\ell(F(A_{N_j} U_i), \xi_j) = 0.$$

By Theorem 4.2, the equations (4.16), (4.21) and (4.22) are satisfied. Applying ∇_X to $h_i^\ell(Y, U_j) = -\delta_{ij}\theta(Y)$ and using (2.3) and (3.10), we have

$$\begin{aligned}
&(\nabla_X h_i^\ell)(Y, U_j)\\
&= -\delta_{ij}\{(\overline{\nabla}_X\theta)(Y) + \sum_{k=1}^{r} \beta_k h_k^\ell(X,Y) + \sum_{a=r+1}^{n} \gamma_a h_a^s(X,Y)\}\\
&\quad - h_i^\ell(Y, F(A_{N_j} X)) + \theta(Y)\tau_{ji}(X) - \sum_{a=r+1}^{n} \rho_{ja}(X) h_i^\ell(Y, W_a).
\end{aligned}$$

Substituting this equation and $(4.22)_2$ into (5.5) such that $Z = U_j$ and using (2.11), (3.2)∼(3.4), $(3.9)_4$ and (3.10), we get

$$\begin{aligned}
&h_i^\ell(X, F(A_{N_j} Y)) - h_i^\ell(Y, F(A_{N_j} X))\\
&+ \sum_{a=r+1}^{n} \epsilon_a\{\rho_{ja}(Y)h_a^s(X, V_i) - \rho_{ja}(X)h_a^s(Y, V_i)\}\\
&+ \sum_{a=r+1}^{n} \{\phi_{ai}(X)h_a^s(Y, U_j) - \phi_{ai}(Y)h_a^s(X, U_j)\}\\
&= \frac{c}{4}\{u_i(Y)\eta_j(X) - u_i(X)\eta_j(Y) + 2\delta_{ij}\overline{g}(X, JY)\}.
\end{aligned}$$

Taking $Y = U_i$ and $X = \xi_j$ and using (3.4), $(3.8)_2$ and $(4.16)_{1,3,4,5}$, we get

$$h_i^\ell(U_i, F(A_{N_j}\xi_j)) - h_i^\ell(\xi_j, F(A_{N_j} U_i)) = \frac{3}{4}c. \tag{5.7}$$

From the above results and (5.7), we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(3) If U_i is parallel with respect to ∇, then, taking the scalar product with U_j, W_a and N_j to (3.10) with $\nabla_X U_i = 0$ by turns, we get respectively

$$\eta_j(A_{N_i} X) = 0, \qquad \rho_{ia} = 0, \qquad h_i^*(X, U_j) = 0. \tag{5.8}$$

From (4.2) and $(5.8)_{1,2}$, we see that M is solenoidal.

Applying ∇_X to $h_i^*(Y, U_j) = 0$ and using the fact that $\nabla_X U_j = 0$, we have

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting (5.8) and the last equation into (5.6) with $PZ = U_j$, we obtain

$$\frac{c}{4}\{v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y) + v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)\} = 0.$$

Taking $X = \xi_i$ and $Y = V_j$ to this equation, we have $c = 0$. Thus $\overline{M}(c)$ is flat.

(4) If W_a is parallel with respect to ∇, then, taking the scalar product with V_i, U_i and N_i to (3.12) with $\nabla_X W_a = 0$ by turns, we get respectively

$$\phi_{ai} = 0, \qquad \rho_{ia} = 0, \qquad h_i^*(X, W_a) = 0. \tag{5.9}$$

Applying ∇_X to $h_i^*(Y, W_a) = 0$ and using the fact that $\nabla_X W_a = 0$, we have

$$(\nabla_X h_i^*)(Y, W_a) = 0.$$

Substituting this equation and (5.9) into (5.6) with $PZ = W_a$, we get

$$\begin{aligned}&\sum_{k=1}^{r}\{h_a^s(X, V_k)\eta_i(A_{N_k}Y) - h_a^s(Y, V_k)\eta_i(A_{N_k}X)\}\\ &= \frac{c}{4}\{w_a(Y)\eta_i(X) - w_a(X)\eta_i(Y)\}.\end{aligned}$$

Taking $X = \xi_i$ and $Y = W_a$ and using (3.4) and $(3.8)_{2,3}$, we obtain

$$h_a^s(W_a, V_k)\eta_k(A_{N_i}\xi_i) = \frac{c}{4}.$$

Assume that $A_{N_i}\xi_i$ belong to $S(TM)$. Then we have $c = 0$. □

Definition. The lightlike submanifold M is called *screen totally umbilical* [2] if there exist smooth functions γ_i on a coordinate neighborhood $\mathcal{U}$ such that

$$h_i^*(X, PY) = \gamma_i g(X, Y). \tag{5.10}$$

Theorem 5.2. *Let M be a screen totally umbilical generic lightlike submanifold of an indefinite complex space form $\overline{M}(c)$ with a quarter-symmetric metric connection. If M is irrotational or solenoidal, then $\overline{M}(c)$ is flat.*

Proof. From $(3.9)_{1,2}$ and (5.10), we have

$$h_j^\ell(X, U_i) = \gamma_i u_j(X), \qquad h_a^s(X, U_i) = \gamma_i w_a(X).$$

Taking $X = \xi_k$ to these two equations, we obtain

$$h_j^\ell(\xi_k, U_i) = 0, \qquad h_a^s(\xi_k, U_i) = 0. \tag{5.11}$$

Taking $X = \xi_k$ and $Y = U_j$ to (3.3) and (3.4) and using $(5.11)_{1,2}$, we have

$$h_j^\ell(U_i, \xi_k) = \delta_{ij}\alpha_k, \qquad h_a^s(U_i, \xi_k) = 0.$$

Taking $j = k$ to the first equation and using $(3.8)_6$, we get $\alpha_i = 0$ for all i.

Applying ∇_Z to (5.10) and using (3.1), we obtain

$$(\nabla_X h_i^*)(Y, PZ) = (X\gamma_i)g(Y, PZ) + \sum_{k=1}^{r}\gamma_i h_k^\ell(X, PZ)\eta_k(Y).$$

Substituting the last equation into (5.6), we have

$$\{X\gamma_i - \sum_{k=1}^{r}\gamma_k\tau_{ik}(X)\}g(Y, PZ) - \{Y\gamma_i - \sum_{k=1}^{r}\gamma_k\tau_{ik}(Y)\}g(X, PZ)$$

$$
\begin{aligned}
&+\gamma_i\{\sum_{k=1}^{r}[h_k^{\ell}(X,PZ)\eta_k(Y)-h_k^{\ell}(Y,PZ)\eta_k(X)]\\
&\qquad\qquad+g(FX,PZ)\theta(Y)-g(FY,PZ)\theta(X)\}\\
&+\sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(Y)h_a^s(X,\,PZ)-\rho_{ia}(X)h_a^s(Y,\,PZ)\}\\
&+\sum_{k=1}^{r}\{h_k^{\ell}(X,PZ)\eta_i(A_{N_k}Y)-h_k^{\ell}(Y,PZ)\eta_i(A_{N_k}X)\}\\
&+\{(\overline{\nabla}_X\theta)(Y)-(\overline{\nabla}_Y\theta)(X)+\theta(Y)\theta(JX)-\theta(X)\theta(JY)\}v_i(PZ)\\
=\;&\frac{c}{4}\{g(Y,PZ)\eta_i(X)-g(X,PZ)\eta_i(Y)\\
&+v_i(X)\overline{g}(JY,PZ)-v_i(Y)\overline{g}(JX,PZ)+2v_i(PZ)\overline{g}(X,JY)\}.
\end{aligned}
$$

Taking $X=V_i$, $Y=U_j$ and $PZ=\xi_j$ and using (5.11), we obtain

$$
\sum_{k=1}^{r}\{h_k^{\ell}(V_i,\xi_j)\eta_i(A_{N_k}U_j)-\sum_{a=r+1}^{n}\rho_{ia}(U_j)\phi_{aj}(V_i)\}=\frac{c}{4}.
$$

Therefore, if M is irrotational or solenoidal, then $\overline{M}(c)$ is flat. □

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