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WEIGHTED COMPOSITION OPERATORS WHOSE RANGES CONTAIN THE DISK ALGEBRA II

Kei Ji Izuchi, Kou Hei Izuchi, and Yuko Izuchi

Abstract. Let $\{\varphi_n\}_{n\geq 1}$ be a sequence of analytic self-maps of $\mathbb{D}$. It is proved that if the union set of the ranges of the composition operators C_{φ_n} on the weighted Bergman spaces contains the disk algebra, then φ_k is an automorphism of $\mathbb{D}$ for some $k \geq 1$.

1. introduction

Let $\mathbb{D}$ be the open unit disk. We denote by $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. We denote by $\mathcal{S}$ the set of analytic self-maps of $\mathbb{D}$. For each $\varphi \in \mathcal{S}$ and $u \in H(\mathbb{D})$, we may defined the weighted composition operator $M_u C_\varphi$ on $H(\mathbb{D})$ by $(M_u C_\varphi) f = u(f \circ \varphi)$ for $f \in H(\mathbb{D})$. For a subset E of $H(\mathbb{D})$, write $(M_u C_\varphi)(E) = \{(M_u C_\varphi) f : f \in E\}$. There are a lot of studies of (weighted) composition operators on various space of analytic functions, see [1, 8].

We denote by $A(\overline{\mathbb{D}})$ the disk algebra, i.e., the space of functions in $H(\mathbb{D})$ which can be extended continuously on $\overline{\mathbb{D}}$ (see [4]). Let $\mathrm{Aut}\,(\mathbb{D})$ be the set of automorphisms of $\mathbb{D}$. For finitely many $\varphi_1, \varphi_2, \ldots, \varphi_\ell$ in $\mathcal{S}$ and $u_1, u_2, \ldots, u_\ell$ in $H(\mathbb{D})$, in the previous paper [5, Theorem 2.1] the authors proved that if $A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\ell} (M_{u_n} C_{\varphi_n})(H(\mathbb{D}))$, then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ and $Z(u_k) = \emptyset$ for some $1 \leq k \leq \ell$, where $Z(u_k)$ denotes the zero set of u_k in $\mathbb{D}$. We have a conjecture that for sequences $\{\varphi_n\}_{n\geq 1}$ in $\mathcal{S}$ and $\{u_n\}_{n\geq 1}$ in $H(\mathbb{D})$, if $A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H(\mathbb{D}))$, then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ and $Z(u_k) = \emptyset$ for some $k \geq 1$. At this moment, we can not prove this. In this paper, we shall study the same type of problems.

For $0 < p < \infty$, let A^p, the Bergman space, be the space of functions f in $H(\mathbb{D})$ satisfying that

$$\int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

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where dA is the normalized area measure on $\mathbb{D}$. It is well known that $C_\varphi(A^p) \subset A^p$. In Section 2, applying the Baire category theorem we shall prove that if

$$A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\infty} C_{\varphi_n}\Big(\bigcup_{0<p<\infty} A^p \Big),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ for some $k \geq 1$.

Let H^∞ be the space of bounded analytic functions on $\mathbb{D}$. We denote by $\partial\mathbb{D}$ the boundary of $\mathbb{D}$. For each $f \in H^\infty$, there exists $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ for almost all $e^{i\theta} \in \partial\mathbb{D}$. We identify f with f^*, so we may think of H^∞ the closed subalgebra of $L^\infty(\partial\mathbb{D})$, the space of bounded measurable functions on $\partial\mathbb{D}$. It is well known that $H^\infty + C(\partial\mathbb{D})$ is a closed subalgebra of $L^\infty(\partial\mathbb{D})$, where $C(\partial\mathbb{D})$ is the space of continuous functions on $\partial\mathbb{D}$ ([6]). We write

$$\overline{H^\infty + C(\partial\mathbb{D})} = \big\{ \overline{f} : f \in H^\infty + C(\partial\mathbb{D}) \big\}.$$

Let

$$QA = H^\infty \cap \overline{H^\infty + C(\partial\mathbb{D})}.$$

Then QA is the closed subalgebra of H^∞ and $A(\overline{\mathbb{D}}) \subset QA \subset H^\infty$. It is considered that QA is a fairly small and an interesting space in H^∞ (see [2, 7, 9, 10]). In [5, Theorem 4.1], the authors pointed out that if

$$H^\infty \subset \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H(\mathbb{D})),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ and $Z(u_k) = \emptyset$ for some $k \geq 1$. In Section 3, we shall show that the above result holds if we replace H^∞ by QA.

2. Weighted Bergman spaces

We study under more general setting. Let ω be a positive continuous function on $\mathbb{D}$ satisfying that

$$(\alpha) \qquad \omega(z) \to 0 \text{ as } |z| \to 1.$$

For $f \in H(\mathbb{D})$, let denote

$$\|f\|_\omega = \sup_{z\in\mathbb{D}} \omega(z)|f(z)|.$$

Write

$$H(\omega) = \{ f \in H(\mathbb{D}) : \|f\|_\omega < \infty \}.$$

Then $H(\omega)$ is a Banach space with the norm $\|\cdot\|_\omega$ and $A(\overline{\mathbb{D}}) \subset H(\omega)$. For each positive integer m, let

$$H_m(\omega) = \{ f \in H(\omega) : \|f\|_\omega \leq m \}.$$

We have $H(\omega) = \bigcup_{m=1}^{\infty} H_m(\omega)$.

Lemma 2.1. *$C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}})$ is closed in $A(\overline{\mathbb{D}})$.*

Proof. Let $\{f_j\}_{j\geq 1}$ be a sequence in $C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}})$ such that $f_j \to f_0$ in $A(\overline{\mathbb{D}})$ as $j \to \infty$. For each $j \geq 1$, there is $g_j \in H_m(\omega)$ such that $f_j = g_j \circ \varphi$. Since $\omega(z)|g_j(z)| \leq m$ on $\mathbb{D}$ for every $j \geq 1$, by the normal family argument we may assume that $g_j \to g_0 \in H(\mathbb{D})$ uniformly on any compact subset of $\mathbb{D}$ as $j \to \infty$. Then we have $g_0 \in H_m(\omega)$ and $f_0 = g_0 \circ \varphi$, so $f_0 \in C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}})$. Thus we get the assertion. □

Since $C_\varphi(H(\omega)) \cap A(\overline{\mathbb{D}})$ is a subspace of $A(\overline{\mathbb{D}})$ and $H(\omega) = \bigcup_{m\geq 1}^{\infty} H_m(\omega)$, we have the following.

Lemma 2.2. *If $C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}})$ contains a non-void open subset of $A(\overline{\mathbb{D}})$, then $A(\overline{\mathbb{D}}) \subset C_\varphi(H(\omega))$.*

Proof. Take a non-void open subset U of $A(\overline{\mathbb{D}})$ satisfying

$$U \subset C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}}).$$

Fix $f_0 \in U$. There is $g_0 \in H_m(\omega)$ such that $f_0 = g_0 \circ \varphi$. Take $f \in A(\overline{\mathbb{D}})$ arbitrary. We have

$$f_0 + \varepsilon f \in U \subset C_\varphi(H_m(\omega)) \cap A(\overline{\mathbb{D}})$$

for some $\varepsilon > 0$. Then there is $h \in H_m(\omega)$ such that $f_0 + \varepsilon f = h \circ \varphi$. Since $(h - g_0)/\varepsilon \in H(\omega)$, we have

$$f = \frac{(h - g_0) \circ \varphi}{\varepsilon} \in C_\varphi(H(\omega)).$$

Thus we get the assertion. □

By [5, Theorem 1.1], we have the following.

Lemma 2.3. *If $A(\overline{\mathbb{D}}) \subset C_\varphi(H(\omega))$, then $\varphi \in \mathrm{Aut}\,(\mathbb{D})$.*

Theorem 2.4. *Let $\{\varphi_n\}_{n\geq 1}$ be a sequence in $\mathcal{S}$ and $\{\omega_\ell\}_{\ell\geq 1}$ be a sequence of positive continuous functions on $\mathbb{D}$ satisfying condition (α). If*

$$A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\infty} C_{\varphi_n}\Big(\bigcup_{\ell=1}^{\infty} H(\omega_\ell)\Big),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ for some $k \geq 1$.

Proof. We may assume that φ_n is non-constant for every $n \geq 1$. We have

$$\bigcup_{\ell=1}^{\infty} H(\omega_\ell) = \bigcup_{\ell,m=1}^{\infty} H_m(\omega_\ell).$$

By the assumption,

$$A(\overline{\mathbb{D}}) = \bigcup_{n,\ell,m=1}^{\infty} C_{\varphi_n}(H_m(\omega_\ell)) \cap A(\overline{\mathbb{D}}).$$

By Lemma 2.1, $C_{\varphi_n}(H_m(\omega_\ell)) \cap A(\overline{\mathbb{D}})$ is closed in $A(\overline{\mathbb{D}})$ for every $n, \ell, m \geq 1$. By the Baire category theorem, $C_{\varphi_n}(H_m(\omega_\ell)) \cap A(\overline{\mathbb{D}})$ contains a non-void open subset of $A(\overline{\mathbb{D}})$ for some n, ℓ and m. By Lemma 2.2, $A(\overline{\mathbb{D}}) \subset C_{\varphi_n}(H(\omega_\ell))$. By Lemma 2.3, we have $\varphi_n \in \mathrm{Aut}\,(\mathbb{D})$. □

For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space A^p_α is the space of $f \in H(\mathbb{D})$ satisfying that

$$\|f\|_{p,\alpha} := \Big(\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z) \Big)^{1/p},$$

where

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z)$$

(see [3, p. 2]). We have $A^p_0 = A^p$,

$$A^p_{\alpha_1} \subset A^p_{\alpha_2} \quad \text{if } -1 < \alpha_1 < \alpha_2 < \infty$$

and

$$A^{p_1}_\alpha \supset A^{p_2}_\alpha \quad \text{if } 0 < p_1 < p_2 < \infty.$$

Then

$$\bigcup_{0<p<\infty} A^p_\alpha = \bigcup_{\ell=1}^{\infty} A^{1/\ell}_\alpha \quad \text{and} \quad \bigcup_{-1<\alpha<\infty} A^p_\alpha = \bigcup_{k=1}^{\infty} A^p_k.$$

Hence

$$\bigcup_{0<p<\infty, -1<\alpha<\infty} A^p_\alpha = \bigcup_{\ell,k=1}^{\infty} A^{1/\ell}_k.$$

For $f \in A^p_\alpha$, we have

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{2+\alpha}{p}} |f(z)| \leq \|f\|_{p,\alpha} < \infty$$

(see [3, p. 53]). Set $\omega_{p,\alpha}(z) = (1 - |z|^2)^{(2+\alpha)/p}$. Then $A^p_\alpha \subset H(\omega_{p,\alpha})$ and

$$\bigcup_{0<p<\infty, -1<\alpha<\infty} A^p_\alpha \subset \bigcup_{\ell,k=1}^{\infty} H(\omega_{1/\ell,k}).$$

So by Theorem 2.4, we have the following.

Corollary 2.5. *Let $\{\varphi_n\}_{n\geq 1}$ be a sequence in $\mathcal{S}$. If*

$$A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\infty} C_{\varphi_n}\Big(\bigcup_{0<p<\infty, -1<\alpha<\infty} A^p_\alpha \Big),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ for some $k \geq 1$.

For $0 < p < \infty$, let H^p be the space of functions f in $H(\mathbb{D})$ satisfying that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, \frac{d\theta}{2\pi} < \infty.$$

The space H^p is call the Hardy space. Since $H^p \subset A^p$, we have the following.

Corollary 2.6. *Let* $\{\varphi_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{S}$. *If*

$$A(\overline{\mathbb{D}}) \subset \bigcup_{n=1}^{\infty} C_{\varphi_n}\Big(\bigcup_{0<p<\infty} H^p\Big),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ *for some* $k \geq 1$.

3. The QA space

In this section, we shall prove the following.

Theorem 3.1. *Let* $\{\varphi_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{S}$ *and let* $\{u_n\}_{n\geq 1}$ *be a sequence in* $H(\mathbb{D})$. *If*

$$QA \subset \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H(\mathbb{D})),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ *and* $Z(u_k) = \emptyset$ *for some* $k \geq 1$.

Proof. The proof is a modification of the one of [5, Theorem 1.1]. We may assume that $u_n \neq 0$ for every $n \geq 1$. To prove the assertion, suppose that either $\varphi_k \notin \mathrm{Aut}\,(\mathbb{D})$ or $Z(u_k) \neq \emptyset$ for every $k \geq 1$. We shall show the existence of $F \in QA$ such that

$$F \notin \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H(\mathbb{D})).$$

Let $\mathbb{N}$ be the set of positive integers,

$$\Sigma_1 = \big\{n \in \mathbb{N} : Z(u_n) \neq \emptyset\big\},$$

$$\Sigma_2 = \big\{n \in \mathbb{N} : n \notin \Sigma_1, \varphi_n \text{ is constant}\big\},$$

$$\Sigma_3 = \big\{n \in \mathbb{N} : n \notin \Sigma_1 \cup \Sigma_2, \varphi_n \text{ is not a finite Blaschke product}\big\},$$

and

$$\Sigma_4 = \big\{n \in \mathbb{N} : n \notin \Sigma_1, \varphi_n \text{ is a finite Blaschke product}\big\}.$$

Since $\varphi_n \notin \mathrm{Aut}\,(\mathbb{D})$ for every $n \geq 1$ with $n \notin \Sigma_1$, $\varphi_j \notin \mathrm{Aut}\,(\mathbb{D})$ for every $j \in \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$. We note also that $\mathbb{N} = \bigcup_{i=1}^{4} \Sigma_i$ and $\Sigma_i \cap \Sigma_\ell = \emptyset$ for $i \neq \ell$. For each $k \in \Sigma_1$, take $\zeta_k \in \mathbb{D}$ satisfying $u_k(\zeta_k) = 0$.

Associated with the set Σ_3, we shall define a function in QA satisfying some additional conditions. First, suppose that $\Sigma_3 \neq \emptyset$. Let $j \in \Sigma_3$. Then there is a sequence $\{z_{j,n}\}_{n\geq 1}$ in $\mathbb{D}$ such that $\{z_{j,n}\}_{n\geq 1} \cap \{\zeta_k\}_{k\in\Sigma_1} = \emptyset$, $|z_{j,n}| \to 1$ as $n \to \infty$ and $\varphi_j(z_{j,n}) \to \alpha_j$ as $n \to \infty$ for some $\alpha_j \in \mathbb{D}$. Since φ_j is non-constant, we may take $\{z_{j,n}\}_{n\geq 1}$ as $\varphi_j(z_{j,n}) \neq \varphi_j(z_{j,m})$ for every $n \neq m$. Considering a subsequence of $\{z_{j,n}\}_{n\geq 1}$, further we may assume that

$$\sum_{j\in\Sigma_3} \sum_{n=1}^{\infty} (1 - |z_{j,n}|) < \infty.$$

Let b_{Σ_3} be the Blaschke product with zeros $\{z_{j,n} : j \in \Sigma_3, n \geq 1\}$, that is,

$$b_{\Sigma_3}(z) = \prod_{j \in \Sigma_3, 1 \leq n < \infty} \frac{-\overline{z}_{j,n}}{|z_{j,n}|} \frac{z - z_{j,n}}{1 - \overline{z}_{j,n} z}, \quad z \in \mathbb{D}.$$

Then $b_{\Sigma_3}(\zeta_k) \neq 0$ for every $k \in \Sigma_1$. By the Wolff theorem [10], there is a function q_{Σ_3} in QA such that $b_{\Sigma_3} q_{\Sigma_3} \in QA$ and $|q_{\Sigma_3}| > 0$ on $\mathbb{D}$. When $\Sigma_3 = \emptyset$, we set $b_{\Sigma_3} = q_{\Sigma_3} = 1$.

Associated with the set Σ_4, we shall define a function in QA satisfying some additional conditions. First, suppose that $\Sigma_4 \neq \emptyset$. Let $j \in \Sigma_4$. Note that $\varphi_j \notin \mathrm{Aut}\,(\mathbb{D})$. Since φ_j is a finite Blaschke product, there are $\gamma_j, \xi_j \in \mathbb{D}$ such that $\gamma_j \neq \xi_j$ and $\varphi_j(\gamma_j) = \varphi_j(\xi_j)$. Moreover, we may assume that $b_{\Sigma_3}(\xi_j) \neq 0$ and $\{\gamma_j, \xi_j : j \in \Sigma_4\}$ is a set of distinct points. Further, we may assume that $\{\gamma_j\}_{j \in \Sigma_4} \cap \{\zeta_k\}_{k \in \Sigma_1} = \emptyset$ and $\sum_{j \in \Sigma_4}(1 - |\gamma_j|) < \infty$. Let b_{Σ_4} be the Blaschke product with zeros $\{\gamma_j\}_{j \in \Sigma_4}$. Then $b_{\Sigma_4}(\xi_j) \neq 0$ for every $j \in \Sigma_4$ and $b_{\Sigma_4}(\zeta_k) \neq 0$ for every $k \in \Sigma_1$. By the Wolff theorem [10] again, there is a function q_{Σ_4} in QA such that $b_{\Sigma_4} q_{\Sigma_4} \in QA$ and $|q_{\Sigma_4}| > 0$ on $\mathbb{D}$. When $\Sigma_4 = \emptyset$, we set $b_{\Sigma_4} = q_{\Sigma_4} = 1$.

We have

$$F := b_{\Sigma_3} q_{\Sigma_3} b_{\Sigma_4} q_{\Sigma_4} \in QA.$$

Note that $F \neq 0$,

$$Z(F) = Z(b_{\Sigma_3}) \cup Z(b_{\Sigma_4}) = \{z_{j,n} : j \in \Sigma_3, n \geq 1\} \cup \{\gamma_j\}_{j \in \Sigma_4}$$

and

$$Z(F) \cap \big(\{\zeta_k\}_{k \in \Sigma_1} \cup \{\xi_j\}_{j \in \Sigma_4}\big) = \emptyset.$$

To show that $F \notin \bigcup_{n=1}^{\infty}(M_{u_n} C_{\varphi_n})(H(\mathbb{D}))$, suppose that there is a non-zero function G in $H(\mathbb{D})$ such that

$$F = b_{\Sigma_3} q_{\Sigma_3} b_{\Sigma_4} q_{\Sigma_4} = u_j (G \circ \varphi_j)$$

for some $j \geq 1$. To lead a contradiction, we divide the proof into four cases.

Case 1. Suppose that $j \in \Sigma_1$. Since $u_j(\zeta_j) = 0$, we have

$$F(\zeta_j) = u_j(\zeta_j)(G \circ \varphi_j)(\zeta_j) = 0.$$

This contradicts $Z(F) \cap \{\zeta_k\}_{k \in \Sigma_1} = \emptyset$.

Case 2. Suppose that $j \in \Sigma_2$. Then $\varphi_j(z) \equiv a$ for some $a \in \mathbb{D}$. We have $F(z) = u_j(z) G(a)$. If $G(a) = 0$, then $F = 0$. This is a contradiction. Suppose that $G(a) \neq 0$. Since $Z(u_j) = \emptyset$, $Z(F) = \emptyset$. This is also a contradiction.

Case 3. Suppose that $j \in \Sigma_3$. Since $F(z_{j,n}) = 0$, we have

$$u_j(z_{j,n}) G(\varphi_j(z_{j,n})) = 0$$

for every $n \geq 1$. Since $Z(u_j) = \emptyset$, $G(\varphi_j(z_{j,n})) = 0$ for $n \geq 1$. We have $z_{j,n} \to \alpha_j \in \mathbb{D}$ as $n \to \infty$ and $\varphi_j(z_{j,n}) \neq \varphi_j(z_{j,m})$ for every $n \neq m$. By the uniqueness theorem, we have $G = 0$. This is a contradiction.

Case 4. Suppose that $j \in \Sigma_4$. Since $F(\gamma_j) = 0$, we have

$$u_j(\gamma_j)G(\varphi_j(\gamma_j)) = 0.$$

Since $Z(u_j) = \emptyset$, $G(\varphi_j(\gamma_j)) = 0$. Since $\varphi_j(\gamma_j) = \varphi_j(\xi_j)$, we have $G(\varphi_j(\xi_j)) = 0$. This shows that $F(\xi_j) = 0$. But this contradicts $Z(F) \cap \{\xi_j\}_{j \in \Sigma_4} = \emptyset$. Therefore we get the assertion. □

Remark 3.2. Let $\{\varphi_n\}_{n \geq 1}$ be a sequence in $\mathcal{S}$ and let $\{u_n\}_{n \geq 1}$ be in H^∞. It is known that $(M_{u_n} C_{\varphi_n})(H^2) \subset H^2$ for every $n \geq 1$. If

$$H^2 \subset \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H^2),$$

then $\varphi_k \in \mathrm{Aut}\,(\mathbb{D})$ and $Z(u_k) = \emptyset$ for some $k \geq 1$. For, we have

$$QA \subset H^2 \subset \bigcup_{n=1}^{\infty} (M_{u_n} C_{\varphi_n})(H(\mathbb{D})),$$

so applying Theorem 3.1 we have the assertion.

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Kei Ji Izuchi
Department of Mathematics
Niigata University
Niigata 950-2181, Japan
Email address: izuchi@math.sc.niigata-u.ac.jp

Kou Hei Izuchi
Department of Mathematics
Faculty of Education
Yamaguchi University
Yamaguchi 753-8511, Japan
Email address: izuchi@yamaguchi-u.ac.jp

Yuko Izuchi
Asahidori 2-2-23, Yamaguchi 753-0051, Japan
Email address: yfd10198@nifty.com