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ON ROGERS–RAMANUJAN TYPE IDENTITIES FOR OVERPARTITIONS AND GENERALIZED LATTICE PATHS

Megha Goyal

Abstract. In this paper we introduce and study the lattice paths for which the horizontal step is allowed at height $h \geq 0$, $h \in \mathbb{Z}$. By doing so these paths generalize the heavily studied weighted lattice paths that consist of horizontal steps allowed at height zero only. Six q–series identities of Rogers–Ramanujan type are studied combinatorially using these generalized lattice paths. The results are further extended by using $(n+t)$–color overpartitions. Finally, we will establish that there are certain equinumerous families of $(n+t)$–color overpartitions and the generalized lattice paths.

1. Introduction and definitions

For n to be a natural number, the rising q–factorial of a with base q is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, where $|q| < 1$. Any series involving this rising q–factorial is called a q–series (or basic series or Eulerian series). In the literature, we see that several mathematicians have established connections between q–series, partition identities and different combinatorial parameters, see for instance [2, 9–11, 14]. Using weighted lattice paths as combinatorial tool, several basic series have been interpreted combinatorially [1, 4–6, 12]. But there are many q–series identities which cannot be interpreted combinatorially by using these classical weighted lattice paths. In this paper we will generalize these lattice paths by allowing the horizontal step to be at height $h \geq 0$. By doing so, the generalized lattice paths are quite helpful to interpret q–series identities combinatorially which have not been interpreted earlier in terms of weighted lattice paths. Our main objective in this paper is to use these generalized lattice paths as an elementary tool to study the following six q–series identities of Rogers–Ramanujan type combinatorially:

$$(1) \qquad \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda} = \frac{(-q^2;q^{10})_\infty(-q^5;q^{10})_\infty(-q^8;q^{10})_\infty(-q;q^2)_\infty}{(q^{10};q^{10})_\infty^{-1}(q^3;q^{10})_\infty^{-1}(q^7;q^{10})_\infty^{-1}(q^2;q^2)_\infty},$$

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$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda(q;q^2)_\lambda}=\frac{(-q^4;q^{10})_\infty(-q^5;q^{10})_\infty(-q^6;q^{10})_\infty(-q;q^2)_\infty}{(q^{10};q^{10})_\infty^{-1}(q;q^{10})_\infty^{-1}(q^9;q^{10})_\infty^{-1}(q^2;q^2)_\infty},\tag{2}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}}=\frac{(-q;q^2)_\infty}{(q^{20};q^{20})_\infty(q^5;q^{20})_\infty(q^{15};q^{20})_\infty(q^2;q^2)_\infty},\tag{3}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_\lambda}=\frac{(-q;q^2)_\infty(-q^2;q^6)_\infty(-q^4;q^6)_\infty(q^6;q^6)_\infty}{(q^2;q^2)_\infty},\tag{4}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_{\lambda+1}}=\frac{(q^4;q^{12})_\infty(q^8;q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty},\tag{5}$$

$$\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_{\lambda+1}}=\frac{(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}.\tag{6}$$

These six q–series identities are appearing in the Chu and Zhang compendium [7]. Note that the right hand sides of (1)–(6) can be easily interpreted as the generating functions for certain restricted ordinary partitions. Our main aim is to interpret the left hand side of (1)–(6) combinatorially by using generalized lattice paths and $(n+t)$–color overpartitions. This paper is organized as follows. In Section 2 we will use constructive approach to yield the combinatorial interpretation of the L.H.S. of q–series identities (1)–(6) in terms of generalized lattice paths. In Section 3 we will use q–functional equations and combinatorial arguments to further extend these results in terms of $(n+t)$–color overpartitions. Finally, in Section 4 we will establish bijections between certain classes of $(n+t)$–color overpartitions and the generalized lattice paths. Before we state our main results we first recall some definitions:

Definition ([3])**.** A *partition with "$(n+t)$ copies of n"*, $t \geq 0$, is a partition in which a part of size n, $n \geq 0$, can come in $(n+t)$ different colors denoted by subscripts: $n_1, n_2, \ldots, n_{n+t}$. Note that zeros are permitted if and only if t is greater than or equal to one. Also, zeros are not permitted to repeat in any partition. Furthermore, in an $(n+t)$–color partition, the parts are ordered in a nonincreasing sequence first by size and then by color, so $a_p > b_p$ if $a > b$ and $a_{p+k} > a_p$ if $k > 0$.

Remark 1.1. We note that if we take $t = 0$, then these are nothing but the n–color partitions.

Definition. The *weighted difference* of two parts a_p, b_q $(a \geq b)$ is defined by $a - b - p - q$ and is denoted by $((a_p - b_q))$.

Example 1.2. There are twelve $(n+1)$–color partitions of 2:

$$\begin{aligned}&2_3,\ 2_3+0_1,\ 1_2+1_2,\ 1_2+1_2+0_1,\\&2_2,\ 2_2+0_1,\ 1_2+1_1,\ 1_2+1_1+0_1,\\&2_1,\ 2_1+0_1,\ 1_1+1_1,\ 1_1+1_1+0_1.\end{aligned}$$

In [13] the $(n+t)$–*color overpartitions* are defined as:

Definition. An $(n+t)$–color overpartition is an $(n+t)$–color partition in which the final occurrence of a part a_p may be overlined.

Example 1.3. There are sixteen n–color overpartitions of 3:

$$\begin{aligned}
&3_3,\ 3_2,\ 3_1,\ \overline{3}_3,\ \overline{3}_2,\ \overline{3}_1,\\
&2_2+1_1,\ \overline{2}_2+1_1,\ 2_2+\overline{1}_1,\ \overline{2}_2+\overline{1}_1,\\
&2_1+1_1,\ \overline{2}_1+1_1,\ 2_1+\overline{1}_1,\ \overline{2}_1+\overline{1}_1,\\
&1_1+1_1+1_1,\ 1_1+1_1+\overline{1}_1.
\end{aligned}$$

We describe *Generalized Lattice Paths* as follows:

Definition. All paths will be of finite length lying in the first quadrant. They will begin on the Y–axis and terminate on the X–axis. Following four unitary steps are allowed at each step:

North–East *NE*: from (i,j) to $(i+1,j+1)$.
South–East *SE*: from (i,j) to $(i+1,j-1)$, only allowed if $j>0$.
South *S*: from (i,j) to $(i,j-1)$, only allowed if $j\geq 1$.
Horizontal (East) *H*: from (i,j) to $(i+1,j)$.

All our lattice paths are either empty or terminate with a southeast step: from $(i,1)$ to $(i+1,0)$.

In describing generalized lattice paths, the following terminology is used:

Peak: Either a vertex on the Y–axis which is followed by a *S* step or *SE* step or a vertex preceded by a North–East step and followed by a South step (in which case it is called a *NES* peak) or by a South–East step (in which case it is called *NESE* peak).

Valley: A vertex preceded by a *S* step or *SE* step and followed by a *NE* step. Note that a *S* step or *SE* step followed by *H* step followed by a *NE* step does not constitute a valley.

Mountain: A section of the path which starts on either the X– or Y–axis, which ends on the X–axis and which does not touch the X–axis anywhere in between the end points. Every mountain has at least one peak and may have more than one.

Plain: A section of the path consisting of only *H* steps which starts at a vertex preceded by a *SE* step and ends at a vertex followed by a *NE* step. Note that a sequence of consecutive horizontal steps which is immediately preceded by a *S* step and immediately followed by a *SE* step or a *NE* step does not constitute a plain.

Height of a vertex is its Y–coordinate.

Weight of a vertex is its X–coordinate.

Weight of a Generalized Lattice Path is the sum of the weights of its peaks.

Length of a sequence of Horizontal steps is the number of *H* steps which belong to it.

Depth of a sequence of South steps is the number of S steps which belong to it.

Remark 1.4. When in the lattice paths H steps are allowed only at height zero, then this definition reduces to the definition of the lattice paths as introduced and studied in [8]. And when the lattice paths have no S steps and H steps are allowed only at height zero, then this is the definition of the lattice paths given in [4]. Thus by allowing the H steps at height $h \geq 0$ makes it possible to have plains above the horizontal axis also.

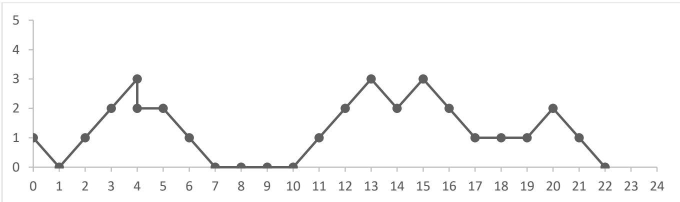


FIGURE 1

Example 1.5. In this example, there is one peak of height 1 followed by a valley at height zero, one peak of height 3 followed by a sequence of S steps of depth 1 and a sequence of H steps of length 1 at height 2. These are further followed by a plain of length 3 at height 0, two peaks of height 3 along with a valley at height 2, further followed by a plain of length 2 at height 1 which is followed by a peak of height 2. The weight of this path is $0+4+13+15+20 = 52$.

2. Generalized lattice paths and combinatorial identities

In this section we shall prove that the basic series identities (1)–(6) have their combinatorial counterparts for generalized lattice paths in the form of the following theorems respectively.

Theorem 2.1. *Let $P_1(\mu)$ denote the number of generalized lattice paths of weight μ which start at $(0,0)$, such that* (i) *they have no valley above height 0,* (ii) *the length of plains, if any, is $\equiv 0 \pmod 4$ between any two consecutive mountains,* (iii) *there is no sequence of S steps with depth > 1 and* (iv) *any sequence of H steps at height $h > 0$ cannot have length > 1. Let*

$$Q_1(\mu) = \sum_{l=0}^{\mu} U_1(\mu - l)V_1(l),$$

where $U_1(\mu)$ is the number of ordinary partitions of μ into parts $\equiv \pm 2, \pm 4, \pm 8 \pmod{20}$ and $V_1(\mu)$ denotes the number of ordinary partitions of μ into distinct parts $\equiv \pm 1, \pm 2, 5 \pmod{10}$, where parts $\equiv 5 \pmod{10}$ are counted twice. Then

$$\sum_{\mu=0}^{\infty} P_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} Q_1(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_{\lambda}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda}} \text{ for all } \mu.$$

Theorem 2.2. *Let $P_2(\mu)$ denote the number of generalized lattice paths of weight μ which start at $(0,0)$, such that* (i) *they have no valley above height* 0*,* (ii) *there is a plain of length* 2 *in the beginning of the path and other plains, if any, are of length* $\equiv 0 \pmod 4$ *between any two consecutive mountains,* (iii) *there is no sequence of S steps with depth > 1,* (iv) *any sequence of H steps at height $h > 0$ cannot have length > 1 and* (v) *the weight of the first peak is ≥ 3. Let*

$$Q_2(\mu) = \sum_{l=0}^{\mu} U_2(\mu - l)V_2(l),$$

where $U_2(\mu)$ is the number of partitions of μ into parts $\equiv \pm 4, \pm 6, \pm 8 \pmod{20}$ and $V_2(\mu)$ denotes the number of partitions of μ into distinct parts $\equiv \pm 3, \pm 4, 5 \pmod{10}$, where parts $\equiv 5 \pmod{10}$ are counted twice. Then

$$\sum_{\mu=0}^{\infty} P_2(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} Q_2(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_{\lambda}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda}} \text{ for all } \mu.$$

Theorem 2.3. *Let $P_3(\mu)$ denote the number of generalized lattice paths of weight μ which start at $(0,2)$, such that* (i) *they have no valley above height* 0*,* (ii) *the length of plains, if any, is $\equiv 0 \pmod 4$ between any two consecutive mountains,* (iii) *there is no sequence of S steps with depth > 1 and* (iv) *any sequence of H steps at height $h > 0$ cannot have length > 1. Let $Q_3(\mu)$ denote the number of partitions of μ into parts $\equiv \pm 1, \pm 3, \pm 4 \pm 7, \pm 8 \pm 9 \pmod{20}$. Then*

$$\sum_{\mu=0}^{\infty} P_3(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} Q_3(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_{\lambda}}{(q^4;q^4)_{\lambda}(q;q^2)_{\lambda+1}} \text{ for all } \mu.$$

Theorem 2.4. *Let $P_4(\mu)$ denote the number of generalized lattice paths of weight μ which start at $(0,0)$, such that* (i) *they have no valley above height* 0*,* (ii) *there is no plain with odd length between any two consecutive mountains,* (iii) *there is no sequence of S steps with depth > 1 and* (iv) *any sequence of H steps at height $h > 0$ cannot have length > 1. Let $Q_4(\mu)$ denote the number of partitions of μ such that odd parts are distinct, even parts are $\equiv \pm 2, \pm 4 \pmod{12}$ and the parts which are $\equiv \pm 2 \pmod{12}$ are counted twice. Then*

$$\sum_{\mu=0}^{\infty} P_4(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} Q_4(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_{\lambda}}{(q^2;q^2)_{\lambda}(q;q^2)_{\lambda}} \text{ for all } \mu.$$

Theorem 2.5. *Let $P_5(\mu)$ denote the number of generalized lattice paths of weight μ which start at $(0,1)$, such that* (i) *they have no valley above height* 0*,* (ii) *there is no plain with odd length between any two consecutive mountains,* (iii) *there is no sequence of S steps with depth > 1 and* (iv) *any sequence of H steps at height $h > 0$ cannot have length > 1. Let $Q_5(\mu)$ denote the number of* *partitions of* μ *into parts* $\not\equiv 0, \pm 4 \pmod{12}$. *Then*

$$\sum_{\mu=0}^{\infty} P_5(\mu) q^{\mu} = \sum_{\mu=0}^{\infty} Q_5(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda (q;q^2)_{\lambda+1}} \text{ for all } \mu.$$

Theorem 2.6. *Let* $P_6(\mu)$ *denote the number of generalized lattice paths of weight* μ *which start at* $(0,2)$, *such that* (i) *they have no valley above height* 0, (ii) *there is no plain with odd length between any two consecutive mountains,* (iii) *there is no sequence of* S *steps with depth* > 1 *and* (iv) *any sequence of* H *steps at height* $h > 0$ *cannot have length* > 1. *Let* $Q_6(\mu)$ *denote the number of partitions of* μ *into parts* $\not\equiv 0, \pm 2 \pmod{12}$. *Then*

$$\sum_{\mu=0}^{\infty} P_6(\mu) q^{\mu} = \sum_{\mu=0}^{\infty} Q_6(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda (q;q^2)_{\lambda+1}} \text{ for all } \mu.$$

As Theorems 2.1–2.6 have similar proofs, so we will discuss the detailed proof of Theorem 2.1 and provide an outline of the proofs of the remaining theorems.

2.1. Proof of Theorem 2.1

We will prove this theorem by using the constructive approach.

Proof. In $\frac{q^{m^2}(-q;q^2)_m}{(q;q^2)_m(q^4;q^4)_m}$ the factor q^{m^2} generates the lattice path of m peaks each of height 1 starting at (0,0) and terminating at $(2m, 0)$.
If m=4, the path begins as: In the above figure we consider two successive

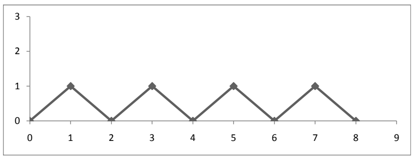


FIGURE 2

peaks, say, i^{th} and $(i+1)^{st}$. Their corresponding coordinates are $(2i-1, 1)$ and $(2i+1, 1)$ respectively. The factor $\frac{1}{(q^4;q^4)_m}$ generates m nonnegative multiples of

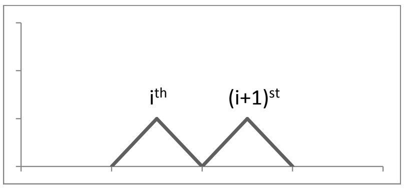
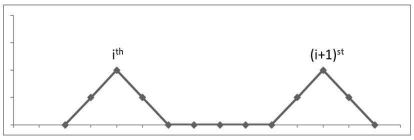


FIGURE 3. i^{th} and $(i+1)^{st}$ peaks

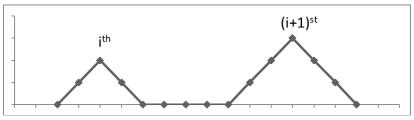
4, say $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \geq 0$, which are encoded by inserting α_m horizontal steps in front of the first mountain and $\alpha_i - \alpha_{i+1}$ horizontal steps in front of the $(m - i + 1)st$ mountain, $1 \leq i \leq m$. Fig. 3 now becomes Fig. 4. The

FIGURE 4. i^{th} and $(i+1)^{st}$ peaks

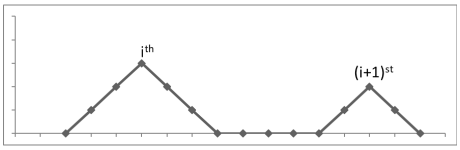
factor $1/(q;q^2)_m$ generates m nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $\beta_1 \times 1,\ \beta_2 \times 3, \ldots, \beta_m \times (2m-1)$. This is encoded by having the i^{th} peak grow to height $\beta_{m-i+1} + 1$ by inserting NESE steps. Each increase by one in the height of a given peak increases its weight by one and the weight of each subsequent peak by two. Fig. 4 changes to Fig. 5a or 5b or 5c depending upon whether $\beta_{m-i} = \beta_{m-i+1}$ or $\beta_{m-i} > \beta_{m-i+1}$ or $\beta_{m-i} < \beta_{m-i+1}$. The



(A)



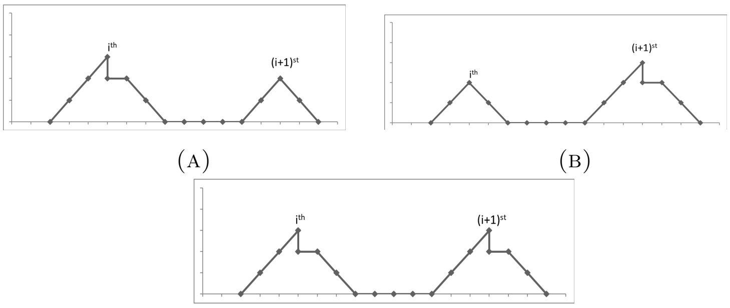
(B)



(C)

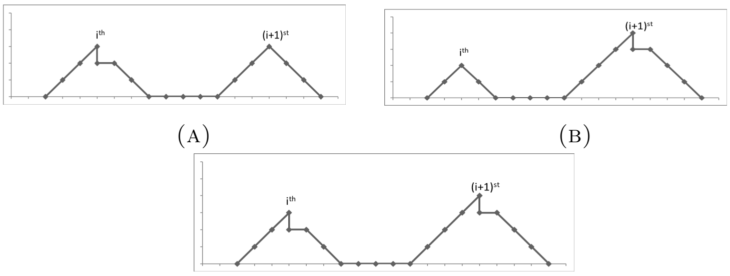
FIGURE 5

factor $(-q;q^2)_m$ generates m nonnegative multiples of $(2i-1)$, $1 \leq i \leq m$, say, $\gamma_1 \times 1,\ \gamma_2 \times 3, \ldots,\ \gamma_m \times (2m-1)$, where each γ_i $(1 \leq i \leq m)$ is 0 or 1. This is encoded by having the i^{th} peak grow to height $\gamma_{m-i+1} + \beta_{m-i+1} + 1$ by inserting NE steps followed by S steps with depth γ_{m-i+1} and then followed by H steps with length γ_{m-i+1}. The depth of this S step and the length of this H step cannot exceed one as $\gamma_i = 0$ or 1; $1 \leq i \leq m$. This causes an increase of γ_{m-i+1} in the height of i^{th} peak. Fig. 5 now changes to Fig. 6 or Fig. 7 or Fig. 8 depending upon $\gamma_{m-i+1} = 1,\ \gamma_{m-i} = 0$ or $\gamma_{m-i+1} = 0,\ \gamma_{m-i} = 1$ or $\gamma_{m-i+1} = \gamma_{m-i} = 1$. In the case when $\gamma_{m-i+1} = \gamma_{m-i} = 0$, the new figure looks like Fig. 5. Every lattice path enumerated by $P_1(\mu)$ is uniquely generated in



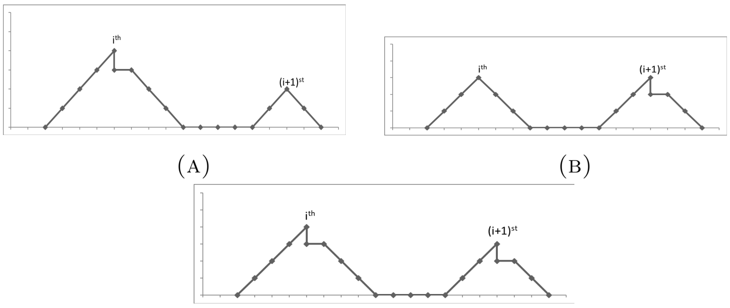
(A) (B) (C)

Figure 6



(A) (B) (C)

Figure 7



(A) (B) (C)

Figure 8

this manner. Hence

$$\sum_{\mu=0}^{\infty} P_1(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda}.$$

Furthermore, we have

$$\frac{(-q^2;q^{10})_\infty(-q^5;q^{10})_\infty(-q^8;q^{10})_\infty(-q;q^2)_\infty}{(q^{10};q^{10})_\infty^{-1}(q^3;q^{10})_\infty^{-1}(q^7;q^{10})_\infty^{-1}(q^2;q^2)_\infty}$$
$$= \prod_{n=1}^{\infty} \tfrac{(1+q^{10n-5})(1+q^{10n-2})(1+q^{10n-8})(1+q^{2n-1})(1-q^{10n})(1-q^{10n-7})(1-q^{10n-3})}{(1-q^{2n})}$$
$$= \prod_{n=1}^{\infty}(1+q^{10n-5})(1+q^{10n-2})(1+q^{10n-8})(1-q^{10n-3})(1-q^{10n-7})$$
$$\left(\tfrac{(1+q^{10n-1})(1+q^{10n-3})(1+q^{10n-5})(1+q^{10n-7})(1+q^{10n-9})}{(1-q^{10n-2})(1-q^{10n-4})(1-q^{10n-6})(1-q^{10n-8})}\right)$$
$$= \prod_{n=1}^{\infty} \tfrac{(1+q^{10n-5})^2(1+q^{10n-2})(1+q^{10n-8})(1+q^{10n-1})(1+q^{10n-9})}{(1-q^{10n-2})(1-q^{10n-8})}$$
$$\left(\tfrac{(1-q^{20n-14})(1-q^{20n-6})}{(1-q^{20n-4})(1-q^{20n-14})(1-q^{20n-6})(1-q^{20n-16})}\right)$$
$$= \prod_{n=1}^{\infty} \tfrac{(1+q^{10n-5})^2(1+q^{10n-2})(1+q^{10n-8})(1+q^{10n-1})(1+q^{10n-9})}{(1-q^{20n-2})(1-q^{20n-12})(1-q^{20n-4})(1-q^{20n-16})(1-q^{20n-8})(1-q^{20n-18})}$$
$$= \sum_{\mu=0}^{\infty} Q_1(\mu)q^\mu.$$

Thus, by q–series identity given in (1), we get

$$\sum_{\mu=0}^{\infty} P_1(\mu)q^\mu = \sum_{\mu=0}^{\infty} Q_1(\mu)q^\mu \text{ for all } \mu.$$

This completes the proof of Theorem 2.1 □

2.2. Outline of the proofs of Theorems 2.2–2.6

Theorem 2.2: An appeal to Theorem 2.1, the extra factor q^{2m} puts a plain of length 2 in front of the first peak. This causes a total increase of $2m$ in the weight of the path and makes the weight of first peak ≥ 3.
Theorem 2.3: An appeal to Theorem 2.1, the extra factor q^{2m} puts two *SE* steps (0,2) to (1,1) and (1,1) to (2,0). So in this case the path begins with $(m+1)$ peaks starting from (0,2) and ending at $(2m+2, 0)$.

If m=4, the path begins as shown in Figure 9.

Again an appeal to Theorem 2.1, due to the factor $\frac{1}{(q^4;q^4)_m}$, the length of the plains, if any, is $\equiv 0 \pmod 4$ between any two mountains.

The factor $1/(q;q^2)_{m+1}$ generates $(m+1)$ nonnegative multiples of $(2i-1)$, $1 \leq i \leq m+1$, say, $\beta_1 \times 1,\ \beta_2 \times 3, \ldots, \beta_{m+1} \times (2m+1)$. This is encoded by increasing the height of i^{th} peak by β_{m-i+2} by inserting NESE steps. This will cause the i^{th} peak to grow to height $\beta_{m-i+2} + 1\ \forall\ 2 \leq i \leq m+1$. And the first peak which is already at height 2 will grow to height $\beta_{m+1} + 2$.

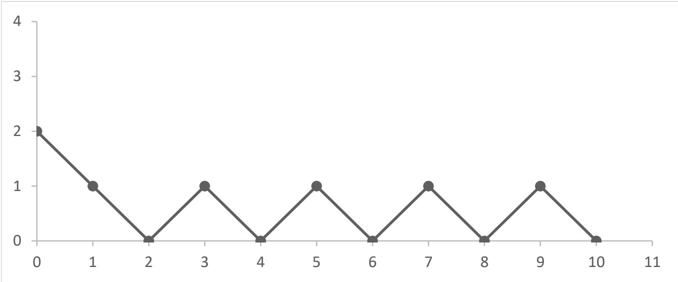


FIGURE 9

Consider the first three peaks of Figure 9 with two plains each of length 4 in between two mountains and take $\beta_5 = 2,\ \beta_4 = 0 = \beta_3$.

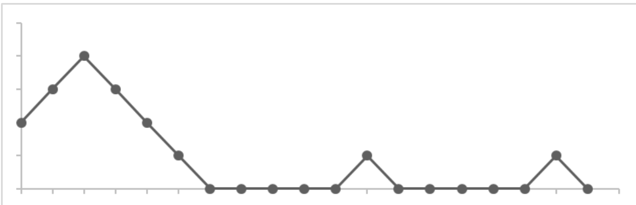


FIGURE 10

Again an appeal to Theorem 2.1, the factor $(-q;q^2)_m$ generates m nonnegative multiples of $(2j-1)$, $1 \le j \le m$, say, $\gamma_1 \times 1,\ \gamma_2 \times 3, \dots,\ \gamma_m \times (2m-1)$, where each γ_j $(1 \le j \le m)$ is 0 or 1. This is encoded by having the i^{th} peak grow to height $\gamma_{m-i+2} + \beta_{m-i+2} + 1$ by inserting *NE* steps followed by S steps with depth γ_{m-i+2} and then followed by H steps with length $\gamma_{m-i+2}\ \forall\ 2 \le i \le m+1$. The depth of this S step and the length of this H step cannot exceed one as $\gamma_j = 0$ or 1; $1 \le j \le m$. Thus the height of first peak remains unaffected. This is illustrated in Figure 11 with $\gamma_4 = 0,\ \gamma_3 = 1$.

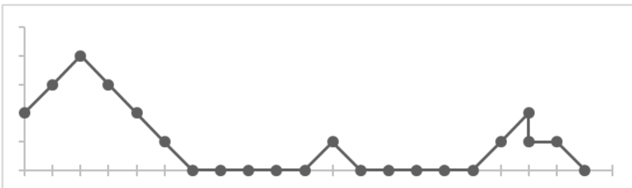


FIGURE 11

Theorem 2.4: This is treated in the same manner as Theorem 2.1. The only difference is the change in condition which states the length of the plains, if

any, is even because of the presence of the factor $\frac{1}{(q^2;q^2)_m}$.
Theorem 2.5: An appeal to Theorem 2.4, the extra factor q^m puts one *SE* step (0,1) to (1,0). So in this case the path begins with $(m+1)$ peaks starting from (0,1) and ending at $(2m+1, 0)$ and length of the plains, if any, is $\equiv 0 \pmod 2$ between any two mountains. Also, the factor $\frac{1}{(1-q^{2m+1})}$ introduces a nonnegative multiple of $2m+1$, say $\beta_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $\beta_{m+1}+1$ in the northeast direction.
Theorem 2.6: An appeal to Theorem 2.4, the extra factor q^{2m} puts two *SE* steps (0,2) to (1,1) and (1,1) to (2,0). So in this case the path begins with $(m+1)$ peaks starting from (0,2) and ending at $(2m+2, 0)$ and length of the plains, if any, is $\equiv 0 \pmod 2$ between any two mountains. Also, the factor $\frac{1}{(1-q^{2m+1})}$ introduces a nonnegative multiple of $2m+1$, say $\beta_{m+1} \times (2m+1)$. This is encoded by having the first peak grow to height $\beta_{m+1}+2$ in the northeast direction.

3. Combinatorial interpretation using $(n+t)$–color overpartitions

Theorem 3.1. *Let $R_1(\mu)$ denote the number of n–color overpartitions of μ such that* (i) *if a_p is the smallest or the only part in the partition, then $a \equiv p \pmod 4$,* (ii) *a part a_p with $p = 1$ is never overlined,* (iii) *the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Then*

$$\sum_{\mu=0}^{\infty} R_1(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda}.$$

Example 3.2. $R_1(7) = 8$, the relevant n–color overpartitions are: 7_7, $\overline{7}_7$, 7_3, $\overline{7}_3$, 6_4+1_1, $\overline{6}_4+1_1$, 5_1+2_2, $5_1+\overline{2}_2$.

Theorem 3.3. *Let $R_2(\mu)$ denote the number of n–color overpartitions of μ such that* (i) *if a_p is the smallest or the only part in the partition, then $a \equiv p+2 \pmod 4$,* (ii) *a part a_p with $p = 1$ is never overlined,* (iii) *all parts are greater than or equal to* 3 *and* (iv) *the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Then*

$$\sum_{\mu=0}^{\infty} R_2(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_\lambda}.$$

Theorem 3.4. *Let $R_3(\mu)$ denote the number of $(n+2)$–color overpartitions of μ such that* (i) *the smallest part or the only part is of the form a_{a+2},* (ii) *the smallest part and a part a_p with $p = 1$ are never overlined,* (iii) *the weighted difference between any two consecutive parts is nonnegative and is $\equiv 0 \pmod 4$. Then*

$$\sum_{\mu=0}^{\infty} R_3(\mu) q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_{\lambda+1}}.$$

Theorem 3.5. *Let $R_4(\mu)$ denote the number of n–color overpartitions of μ such that* (i) *the parts and their subscripts have same parity,* (ii) *a part a_p with $p = 1$ is never overlined, and* (v) *the weighted difference between any two consecutive parts is nonnegative and is even. Then*

$$\sum_{\mu=0}^{\infty} R_4(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_\lambda}.$$

Example 3.6. $R_4(7) = 13$, the relevant n–color overpartitions are: 7_1, 7_3, 7_5, 7_7, $\overline{7}_3$, $\overline{7}_5$, $\overline{7}_7$, $6_4 + 1_1$, $\overline{6}_4 + 1_1$, $6_2 + 1_1$, $\overline{6}_2 + 1_1$, $5_1 + 2_2$, $5_1 + \overline{2}_2$.

Theorem 3.7. *Let $R_5(\mu)$ denote the number of $(n+1)$–color overpartitions of μ such that* (i) *the parts and their subscripts have opposite parity,* (ii) *the least part is of the form a_{a+1},* (iii) *the smallest part and a part a_p with $p = 1$ are never overlined and* (iv) *the weighted difference between any two consecutive parts is nonnegative and is even. Then*

$$\sum_{\mu=0}^{\infty} R_5(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_{\lambda+1}}.$$

Theorem 3.8. *Let $R_6(\mu)$ denote the number of $(n+2)$–color overpartitions of μ such that* (i) *the parts and their subscripts have same parity,* (ii) *the least part is of the form a_{a+2},* (iii) *the smallest part and a part a_p with $p = 1$ are never overlined and* (iv) *the weighted difference between any two consecutive parts is nonnegative and is even. Then*

$$\sum_{\mu=0}^{\infty} R_6(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^2;q^2)_\lambda(q;q^2)_{\lambda+1}}.$$

To prove the above theorems, we study a more general $(n+t)$–color overpartition function $R_k(m,\mu)$ $(1 \le k \le 6)$ which counts the $(n+t)$–color overpartitions of μ of the kind as described in Theorem $3.i$ $(i = 1, 3, 4, 5, 7, 8)$ with the added restriction that there be exactly m parts. For $1 \le k \le 6$, $\phi_k(z;q)$ will denote the 2–variable generating function given by

$$\phi_k(z;q) = \sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty} R_k(m,\mu)z^m q^{\mu}, \tag{7}$$

where $|q| < 1$ and $|z| < |q|^{-1}$.

3.1. Proof of Theorem 3.1

Proof. We shall prove that

$$\sum_{\mu=0}^{\infty} R_1(\mu)q^{\mu} = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q;q^2)_\lambda(q^2;q^2)_\lambda}.$$

For this, we shall first prove the identity,

(8) $\quad R_1(m,\mu) = R_1(m,\mu-4m) + R_1(m-1,\mu-2m+1) + R_1(m-1,$

$$\mu - 4m + 2) + R_1(m, \mu - 2m + 1) - R_1(m, \mu - 6m + 1).$$

To prove Theorem 3.1, we split the overpartitions enumerated by $R_1(m, \mu)$ into four classes:

(i) those that do not contain a_a or $\overline{a}_a$ as a part,
(ii) those that contain 1_1 as a part,
(iii) those that contain $\overline{2}_2$ as a part and
(iv) those that contain $a_a, (a \geq 2)$ or $\overline{a}_a, (a \geq 3)$ as a part.

We now transform the overpartitions in class (i) by subtracting 4 from each part ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the parts and so the transformed overpartition will be of the type enumerated by $R_1(m, \mu - 4m)$.

Next we transform the overpartitions in class (ii) by deleting the part 1_1 and then subtracting 2 from all the remaining parts ignoring the subscripts. Obviously, this transformation will not disturb the inequalities between the parts and so the transformed overpartition will be of the type enumerated by $R_1(m - 1, \mu - 2m + 1)$.

Next, we transform the overpartitions in class (iii) by deleting the part $\overline{2}_2$ and then subtracting 4 from all the remaining parts ignoring the subscripts. The transformed overpartition will be of the type enumerated by $R_1(m-1, \mu - 4m + 2)$.

Finally, we transform the overpartitions in class (iv) by replacing a_a by $(a-1)_{a-1}$ or $\overline{a}_a$ by $\overline{(a-1)}_{a-1}$ as the case may be and then subtracting 2 from all the remaining parts. This will produce an overpartition of $\mu - 2m + 1$ into m parts. It is important to note here that by this transformation we get only those overpartitions of $\mu - 2m + 1$ into m parts which contain a part of the form a_a or $\overline{a}_a$. Therefore, the actual number of overpartitions which belong to class (iv) is $R_1(m, \mu - 2m + 1) - R_1(m, \mu - 6m + 1)$, where $R_1(m, \mu - 6m + 1)$ is the number of overpartitions of $\mu - 2m + 1$ into m parts which are free from the parts like a_a or $\overline{a}_a$.

The above transformations are clearly reversible and this leads to the identity (8).

Now substituting for $R_1(m, \mu)$ from (8) into (7) and then simplifying, we get

$$\phi_1(z;q) = \phi_1(zq^4;q) + zq\phi_1(zq^2;q) + zq^2\phi_1(zq^4;q) + q^{-1}\phi_1(zq^2;q) - q^{-1}\phi_1(zq^6;q)$$

taking

$$\phi_1(z;q) = \sum_{\lambda=0}^{\infty} \alpha_\lambda(q) z^\lambda$$

and then comparing the coefficients of z^λ, we get

$$\alpha_\lambda(q) = \frac{q^{2\lambda-1}(1 + q^{2\lambda-1})}{(1 - q^{2\lambda-1})(1 - q^{4\lambda})} \alpha_{\lambda-1}(q). \tag{9}$$

Iterating (9) λ–times and taking $\alpha_0(q) = 1$, we may easily get

$$\alpha_\lambda(q) = \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q;q^2)_\lambda(q^4;q^4)_\lambda}. \tag{10}$$

Thus

$$\phi_1(z;q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q;q^2)_\lambda(q^4;q^4)_\lambda} z^\lambda.$$

Now

$$\sum_{\mu=0}^{\infty} R_1(\mu)q^\mu = \sum_{\mu=0}^{\infty}\left(\sum_{m=0}^{\infty} R_1(m,\mu)\right)q^\mu = \phi_1(1;q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2}(-q;q^2)_\lambda}{(q;q^2)_\lambda(q^4;q^4)_\lambda}.$$

This completes the proof of Theorem 3.1. □

3.2. Proof of Theorem 3.3

Proof. To prove Theorem 3.3, we split the overpartitions enumerated by $R_2(m,\mu)$ into four classes:

(i) those that do not contain a_{a-2} or $\overline{a}_{(a-2)}$ as a part,

(ii) those that contain 3_1 as a part,

(iii) those that contain $\overline{4}_2$ as a part and

(iv) those that contain $a_{a-2},\ (a \geq 4)$ or $\overline{a}_{a-2},\ (a \geq 5)$ as a part.

Now by performing some elementary reversible transformations, we get the following recurrence relation

$$\begin{aligned} R_2(m,\mu) = {} & R_2(m,\mu-4m) + R_2(m-1,\mu-2m-1) + R_2(m-1,\mu-4m) \\ & + R_2(m,\mu-2m+1) - R_2(m,\mu-6m+1). \end{aligned}$$

Now using (7) for $k = 2$ and after simplification, we get the following q–functional equation

$$\phi_2(z;q) = \phi_2(zq^4;q) + zq^3\phi_2(zq^2;q) + zq^4\phi_2(zq^4;q) + q^{-1}\phi_2(zq^2;q) - q^{-1}\phi_2(zq^6;q).$$

Now proceeding in the same manner as the previous theorem we get our result. □

3.3. Proof of Theorem 3.4

Proof. Let $S(\mu)$ denote the number of n–color overpartitions of μ enumerated by $R_1(\mu)$ with the added restriction that the smallest part is of the form a_a and let $S(m,\mu)$ denote the number of n–color overpartitions of μ enumerated by $S(\mu)$ into m parts. Further let

$$\psi_3(q) = \sum_{\mu=0}^{\infty} S(\mu)q^\mu,$$

$$\psi_3(z;q) = \sum_{m,\mu=0}^{\infty} S(m,\mu)z^m q^\mu.$$

Using the arguments of the proof of Theorem 3.1, we see that

$$S(m,\mu) = R_1(m-1,\mu-2m+1) + \frac{1}{2}\Big[R_1(m-1,\mu-4m+2) + R_1(m,\mu-2m+1) - R_1(m,\mu-6m+1)\Big]. \tag{11}$$

Translating (11) into a q–functional equation, we get

$$\psi_3(z;q) = zq\phi_1(zq^2;q) + \frac{1}{2}zq^2\phi_1(zq^4;q) + \frac{1}{2}q^{-1}\phi_1(zq^2;q) - \frac{1}{2}q^{-1}\phi_1(zq^6;q) \tag{12}$$

setting

$$\psi_3(z;q) = \sum_{\lambda=0}^{\infty}\beta_\lambda(q)z^\lambda$$

and then comparing the coefficients of z^λ in (12), we get

$$2\beta_\lambda(q) = 2q^{2\lambda-1}\alpha_{\lambda-1}(q) + q^{4\lambda-2}\alpha_{\lambda-1}(q) + q^{2\lambda-1}\alpha_\lambda(q) - q^{6\lambda-1}\alpha_\lambda(q).$$

Substituting the value of $\alpha_\lambda(q)$ from (10) and then simplifying, we get

$$\beta_\lambda(q) = \frac{q^{\lambda^2}(-q;q^2)_{\lambda-1}}{(q^4;q^4)_{\lambda-1}(q;q^2)_\lambda}.$$

Thus

$$\psi_3(z;q) = \sum_{\lambda=0}^{\infty}\frac{q^{(\lambda+1)^2}(-q;q^2)_\lambda z^{\lambda+1}}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}} = zq\sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda z^\lambda}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}} = zq\chi(z;q), \tag{13}$$

where,

$$\chi(z;q) = \sum_{\lambda=0}^{\infty}\frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda z^\lambda}{(q^4;q^4)_\lambda(q;q^2)_{\lambda+1}}.$$

Now, define $T(m,\mu)$ by

$$\chi(z;q) = \sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty}T(m,\mu)z^m q^\mu.$$

We see by coefficient comparison in (13) that

$$S(m+1,\mu+1) = T(m,\mu).$$

Now if we replace the part a_a in an overpartition enumerated by $S(m+1,\mu+1)$ by $(a-1)_{a+1}$, we see that the resulting overpartition is enumerated by $R_3(m+1,\mu)$. Thus we have

$$T(m,\mu) = R_3(m+1,\mu)$$

and so

$$\sum_{\mu=0}^{\infty}\sum_{m=0}^{\infty}R_3(m+1,\mu)z^m q^\mu = \chi(z;q).$$

Now

$$\sum_{\mu=0}^{\infty} R_3(\mu) q^{\mu} = \sum_{\mu=0}^{\infty} \left(\sum_{m=1}^{\infty} R_3(m,\mu) \right) q^{\mu} = \sum_{\mu=0}^{\infty} \left(\sum_{m=0}^{\infty} R_3(m+1,\mu) \right) q^{\mu}$$
$$= \sum_{\mu=0}^{\infty} \left(\sum_{m=0}^{\infty} T(m,\mu) \right) q^{\mu}$$
$$= \chi(1;q) = \chi(q) = \sum_{\lambda=0}^{\infty} \frac{q^{\lambda^2+2\lambda}(-q;q^2)_\lambda}{(q^4;q^4)_\lambda (q;q^2)_{\lambda+1}}.$$

This completes the proof of Theorem 3.4. □

3.4. Outline of the proofs of Theorems 3.5, 3.7 and 3.8

Since the proofs of Theorems 3.5, 3.7 and 3.8 are similar to that of Theorems 3.1 and 3.4, we omit the details and give only the identities analogous to (8) and the q–functional equations used in each case. The interested readers can easily supply the details or obtain them from the author.

$$R_4(m,\mu) = R_4(m,\mu-2m) + R_4(m-1,\mu-2m+1) + R_4(m-1,\mu-4m+2)$$
$$+ R_4(m,\mu-2m+1) - R_4(m,\mu-4m+1).$$

$$R_5(m,\mu) = R_4(m-1,\mu-2m+1) + \frac{1}{2}R_4(m-1,\mu-4m+2)$$
$$+ \frac{1}{2}R_4(m,\mu-2m+1) - \frac{1}{2}R_4(m,\mu-4m+1).$$

$$R_6(m,\mu) = R_4(m-1,\mu-2m+1) + \frac{1}{2}R_4(m-1,\mu-4m+2)$$
$$+ \frac{1}{2}R_4(m,\mu-2m+1) - \frac{1}{2}R_4(m,\mu-4m+1).$$

$$\phi_4(z;q) = \phi_4(zq^2;q) + zq\phi_4(zq^2;q) + zq^2\phi_4(zq^4;q)$$
$$+ q^{-1}\phi_4(zq^2;q) - q^{-1}\phi_4(zq^4;q).$$

$$zq\phi_5(zq;q) = zq\phi_4(zq^2;q) + \frac{1}{2}zq^2\phi_4(zq^4;q) + \frac{1}{2}q^{-1}\phi_4(zq^2;q)$$
$$- \frac{1}{2}q^{-1}\phi_4(zq^4;q).$$

$$zq\phi_6(z;q) = zq\phi_4(zq^2;q) + \frac{1}{2}zq^2\phi_4(zq^4;q) + \frac{1}{2}q^{-1}\phi_4(zq^2;q)$$
$$- \frac{1}{2}q^{-1}\phi_4(zq^4;q).$$

4. Equinumerous classes of $(n+t)$–color overpartitions and generalized lattice paths

In this section, we will establish a $1-1$ correspondence between the lattice paths enumerated by $P_i(\mu)$ and the n–color overpartitions enumerated by $R_i(\mu)$ for all $1 \leq i \leq 6$. It is also noticeable that the peak which is immediately proceeded by a S step and then followed by a H step at height $h > 0$ corresponds to the overlined part in the corresponding $(n+t)$–color overpartition. As the proofs are similar for all values of i, so we will discuss the case $i = 1$ in detail and one can easily supply proofs of the remaining cases $2 \leq i \leq 6$.

Theorem 4.1. *For all $\mu \geq 0$, we have*

$$\sum_{\mu=0}^{\infty} P_1(\mu)q^{\mu} = \sum_{\mu=0}^{\infty} R_1(\mu)q^{\mu}.$$

Proof. We will do this by encoding each path as the sequence of the weights of the peaks with each weight subscripted by the height of the respective peak.

Thus, if we say that X_u is the i–th peak and Y_v is the $(i+1)$–st one, respectively, in Fig. 6 or Fig. 7 or Fig. 8, then

$$\begin{aligned} X &= (2i-1) + \alpha_{m-i+1} + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+2}) + \beta_{m-i+1} + 2(\gamma_m \\ &\quad + \gamma_{m-1} + \cdots + \gamma_{m-i+2}) + \gamma_{m-i+1}, \\ u &= 1 + \beta_{m-i+1} + \gamma_{m-i+1}, \\ Y &= (2i+1) + \alpha_{m-i} + 2(\beta_m + \beta_{m-1} + \cdots + \beta_{m-i+1}) + \beta_{m-i} + 2(\gamma_m + \gamma_{m-1} \\ &\quad + \cdots + \gamma_{m-i+1}) + \gamma_{m-i}, \\ v &= 1 + \beta_{m-i} + \gamma_{m-i}. \end{aligned}$$

The weighted difference of these two parts is $((Y_v - X_u)) = Y - X - u - v = \alpha_{m-i} - \alpha_{m-i+1}$ which is nonnegative and is $\equiv 0 \pmod 4$.

Obviously, if (X, u) is the first peak in the generalized lattice path, then it will correspond to the smallest part in the corresponding n–color overpartition or to the singleton part if the n–color overpartition has only one part and in both cases $X - u = \alpha_m \equiv 0 \pmod 4$. Also, when height of the peak is 1, only *NESE* step will occur once, so there cannot be any S step followed by H step. This confirms that the n–color part of the form a_1 cannot be overlined. To see the reverse implication, we consider two n–color parts of an overpartition enumerated by $R_1(\mu)$, say, C_r and D_s.

Let $M_1 \equiv (C, r)$ and $M_2 \equiv (D, s)$ be the corresponding peaks in the generalized lattice path. The length of the plain between the two peaks is $D - C - r - s$ which is the weighted difference between the two parts C_r and D_s and is therefore nonnegative and is $\equiv 0 \pmod 4$. Also, there cannot be a valley above height 0. This can be proved by contradiction.

Suppose, there is a valley V of height h $(h > 0)$ between the peaks M_1 and M_2. In this case there is a descent of $r - h$ from M_1 to V and an ascent of $s - h$

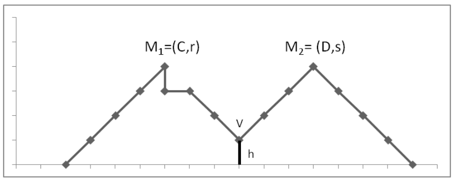


FIGURE 12

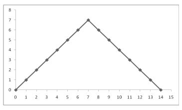
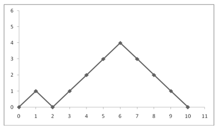
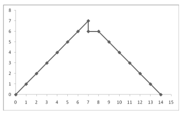
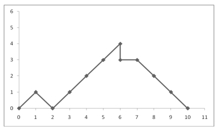
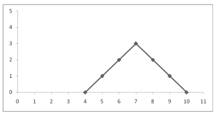
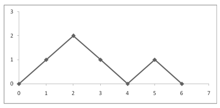
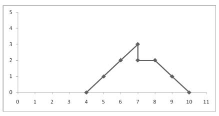
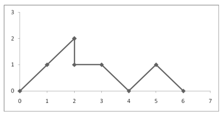
from V to M_2. This implies

$$D = C + (r - h) + (s - h) \Rightarrow D - C - r - s = -2h.$$

But since the weighted difference is nonnegative, therefore h=0.
This completes the proof of Theorem 4.1. □

To illustrate the constructed bijections we consider generalized lattice paths enumerated by $P_1(7)$ and the corresponding n–color overpartitions enumerated by $R_1(7)$ in the following Table 1.

TABLE 1

n–color overpartitions	gen. lattice paths	n–color overpartitions	gen. lattice paths
7_7		$6_4 + 1_1$	
$\overline{7}_7$		$\overline{6}_4 + 1_1$	
7_3		$5_1 + 2_2$	
$\overline{7}_3$		$5_1 + \overline{2}_2$	

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Megha Goyal
Department of Mathematical Sciences
I. K. Gujral Punjab Technical University Jalandhar
Main Campus, Kapurthala-144603, India
Email address: meghagoyal2021@gmail.com