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ON POSITIVE DEFINITE SOLUTIONS OF A CLASS OF NONLINEAR MATRIX EQUATION

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Abstract. This paper is concerned with the positive definite solutions of the nonlinear matrix equation $X - A^*\bar{X}^{-1}A = Q$, where A, Q are given complex matrices with Q positive definite. We show that such a matrix equation always has a unique positive definite solution and if A is nonsingular, it also has a unique negative definite solution. Moreover, based on Sherman-Morrison-Woodbury formula, we derive elegant relationships between solutions of $X - A^*\bar{X}^{-1}A = I$ and the well-studied standard nonlinear matrix equation $Y + B^*Y^{-1}B = Q$, where B, Q are uniquely determined by A. Then several effective numerical algorithms for the unique positive definite solution of $X - A^*\bar{X}^{-1}A = Q$ with linear or quadratic convergence rate such as inverse-free fixed-point iteration, structure-preserving doubling algorithm, Newton algorithm are proposed. Numerical examples are presented to illustrate the effectiveness of all the theoretical results and the behavior of the considered algorithms.

1. Introduction

In the past several decades, nonlinear matrix equations (NMEs)

$$X + A^*X^{-1}A = Q \tag{1.1}$$

and

$$X - A^*X^{-1}A = Q \tag{1.2}$$

have been extensively studied because of their wide applications in control theory, dynamic programming, ladder networks, stochastic filtering and statistics (see [8–10, 26]). Several necessary conditions and sufficient conditions on the existence of positive definite solutions of these two kinds of NMEs have been derived and different iterative methods for computing the maximal positive definite solutions with linear and quadratic rate of convergence have been studied in [5, 10, 13, 18, 20, 22, 24]. Moreover, several variations of these two equations such

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as $X \pm A^*X^{-q}A = Q$ with real number $q > 0$ [11,16,21,27], $X^s \pm A^*X^{-t}A = Q$ with positive integers s, t [19, 25], $X + M^*X^{-1}M - N^*X^{-1}N = I$ [1, 7] and $X - \sum_{i=1}^{m} A_i^* F(X)A_i = Q$ [6, 23] have been considered.

Two square complex matrices A and B are said to be con-similar if there exists a nonsingular complex matrix P such that $A = P^{-1}B\bar{P}$. Consimilarity of complex matrices arises as a result of studying an antilinear operator referred to different bases in complex vector spaces and it plays an important role in modern quantum theory [14]. By using the theory of consimilarity, linear matrix equations $AX - XB = C$ and $X - AXB = C$ which are generally derived by the similarity of square matrices have been extended to $AX - \bar{X}B = C$ and $X - A\bar{X}B = C$ [2,3,15], respectively. Similar to the linear case, the NME (1.1) also has been generalized to

$$X + A^*\bar{X}^{-1}A = Q \tag{1.3}$$

by means of cosimilarity [12, 17, 28]. In [28], Bin Zhou and his coauthors established some sufficient conditions and necessary conditions for the existence of positive definite solutions of (1.3). Several iterative methods such as basic fixed-point iterations, an inversion-free algorithm [17] and a structure-preserving-doubling algorithm [12] for the maximal positive definite solution of (1.3) are proposed.

For the NME (1.2), it is proved that there existed a unique positive definite solution which coincided with the unique positive definite solution of a related algebraic Riccati equation arising in Kalman filtering theory [9]. Unfortunately, for all we know, very little research has been done on the solutions of the variation

$$X - A^*\bar{X}^{-1}A = Q$$

of the NME (1.2), where $X \in \mathbb{C}^{n\times n}$ is unknown, and $A, Q \in \mathbb{C}^{n\times n}$ are given complex matrices with Q Hermitian positive definite. Motivated by this fact and the theory of consimilarity, we consider in this paper the solutions of $X - A^*\bar{X}^{-1}A = Q$. Based on the importance of consimilarity in quantum theory and nonlinear matrix equation (1.2) in control theory, Kalman filtering, it is expected that the NME $X - A^*\bar{X}^{-1}A = Q$ will find possible applications in modern quantum theory.

Multiplying the NME $X - A^*\bar{X}^{-1}A = Q$ on both the right and left sides by $Q^{-1/2}$, we obtain $X_Q - A_Q^*\bar{X}_Q^{-1}A_Q = I$, where $A_Q = \bar{Q}^{-1/2}AQ^{-1/2}$ and $X_Q = Q^{-1/2}XQ^{-1/2}$. Therefore, we only need to consider

$$X - A^*\bar{X}^{-1}A = I \tag{1.4}$$

without loss of generality.

The rest of this paper is organized as follows. Several preliminary results to be used are given in Section 2. In Section 3, using real representation of a complex matrix, we show that (1.4) always has a unique positive definite solution. Based on the famous Sherman-Morrison-Woodbury formula, we transform (1.4) into the extensively studied standard NME $Y + B^*Y^{-1}B = Q$ where B and Q

are uniquely determined by A. Then several effective numerical algorithms for obtaining the unique positive definite solution of (1.4) are proposed. Section 4 focus on the relationships between the positive definite solutions of the NMEs (1.1) and (1.2) which have always been studied independently so far. Several numerical examples are offered in Section 5 to illustrate the effectiveness of the theoretical results.

2. Notations and preliminaries

Throughout this paper, $X \geq 0$ ($X > 0$) means that X is Hermitian positive semidefinite (positive definite). For $n \times n$ Hermitian matrices X, Y, we write $X \geq Y (X > Y)$ if $X - Y \geq 0$ (> 0). Moreover, for a matrix A, we use A^T, A^*, $\bar{A}$, $\|A\|$ and $\rho(A)$ to denote the transpose, the conjugate transpose, the conjugate, the spectral norm and the spectral radius of A, respectively. $\mathcal{C}$ is the unit circle of the complex plane. $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ complex matrices and real matrices, respectively.

We first introduce Sherman-Morrison-Woodbury formula and some existing results regarding the positive definite solutions of the NMEs (1.1) and (1.2) in this section.

Lemma 2.1 ([4] Sherman-Morrison-Woodbury formula)**.** *Let A, B, C and D be some matrices of appropriate dimensions. Assume that $A, C, A + BCD$ and $C^{-1} + DA^{-1}B$ are all nonsingular. Then*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Denote the rational matrix equation

$$\psi(\lambda) = \lambda A + Q + \lambda^{-1}A^*$$

defined on the unit circle $\mathcal{C}$ of the complex plane, which is Hermitian for any $\lambda \in \mathcal{C}$. This function is said to be regular if there exists at least a $\lambda \in \mathcal{C}$ such that $\det\psi(\lambda) \neq 0$.

Lemma 2.2 ([8])**.** *Equation $X + A^*X^{-1}A = Q$ has a positive definite solution X if and only if $\psi(\lambda)$ is regular and $\psi(\lambda) \geq 0$ for all $\lambda \in \mathcal{C}$. Moreover, if equation $X + A^*X^{-1}A = Q$ has a positive definite solution, then it has a maximal solution X_L and a minimal solution X_l such that $0 < X_l \leq X \leq X_L$ for any positive definite solution X. In addition, the maximal solution X_L satisfies $\rho(X_L^{-1}A) \leq 1$ and it can be found by the following basic fixed point iteration*:

$$\begin{cases} X_0 = Q, \\ X_{n+1} = Q - A^*X_n^{-1}A, \quad n = 0, 1, 2, \ldots. \end{cases}$$

Lemma 2.3 ([20])**.** *Suppose that A is nonsingular. Then X solves $X + A^*X^{-1}A = Q$ if and only if $Y = Q - X$ solves $Y + AY^{-1}A^* = Q$. In particular, if Y_L is the maximal positive definite solution of $Y + AY^{-1}A^* = Q$, then $X_l = Q - Y_L$ is the minimal positive definite solution of $X + A^*X^{-1}A = Q$.*

Concerning equation $X - A^*X^{-1}A = Q$, we have the following fundamental result.

Lemma 2.4 ([9]). *The set of solutions of $X - A^*X^{-1}A = Q$ is nonempty, and admits a maximal element X_+ and a minimal element X_-, where X_+ satisfies $\rho(X_+^{-1}A) < 1$. Moreover, X_+ is the unique positive definite solution and it can be found by the following fixed point iteration*

$$\begin{cases} X_0 = Q, \\ X_{n+1} = Q + A^*X_n^{-1}A, \quad n = 0, 1, 2, \ldots. \end{cases}$$

In addition, if A is nonsingular, X_- is the unique negative definite solution and $X_- = Q - Y_+$, where Y_+ is the maximal solution of the equation

$$Y - AY^{-1}A^* = Q.$$

3. Solutions of the NME $X - A^*\bar{X}^{-1}A = I$

For a complex matrix $A = A_1 + iA_2 \in \mathbb{C}^{n\times n}$ where $A_1, A_2 \in \mathbb{R}^{n\times n}$, denote $A^\triangledown$ and $A^\blacktriangledown$ as

$$A^\triangledown = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \quad A^\blacktriangledown = \begin{pmatrix} A_2 & A_1 \\ A_1 & -A_2 \end{pmatrix}.$$

Obviously, both $A^\triangledown$ and $A^\blacktriangledown$ are real matrices and for $A, B \in \mathbb{C}^{n\times n}$

$$A = B \Longleftrightarrow A^\triangledown = B^\triangledown \Longleftrightarrow A^\blacktriangledown = B^\blacktriangledown.$$

Moreover, we define two unitary matrices E_{2n} and P_{2n} as follows.

$$E_{2n} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \quad P_{2n} = \frac{\sqrt{2}}{2}\begin{pmatrix} iI_n & I_n \\ I_n & iI_n \end{pmatrix}.$$

Now we present some properties of the operators $(\cdot)^\triangledown$ and $(\cdot)^\blacktriangledown$.

Lemma 3.1 ([28]). *Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$ be two given complex matrices.*

(i) *The following equalities are true.*

$$(AB)^\triangledown = A^\triangledown B^\triangledown, \quad (A^{-1})^\triangledown = (A^\triangledown)^{-1}, \quad (A^T)^\triangledown = E_{2n}(A^\triangledown)^T E_{2n}$$

$$(A^*)^\triangledown = (A^\triangledown)^T, \quad \bar{A}^\triangledown = E_{2n}A^\triangledown E_{2n}, \quad A^\blacktriangledown = E_{2n}A^\triangledown.$$

(ii) $A^\triangledown = P_{2n}\left(\begin{smallmatrix} A & 0 \\ 0 & \bar{A} \end{smallmatrix}\right)P_{2n}^*$.

(iii) $A \geq 0$ $(A > 0)$ *if and only if* $A^\triangledown \geq 0$ $(A^\triangledown > 0)$.

(iv) *For any $A \in \mathbb{C}^{n\times n}$, there holds $\|A^\triangledown\| = \|A^\blacktriangledown\| = \|A\|$.*

(v) *For any $A \in \mathbb{C}^{n\times n}$, there hold $\rho(A^\triangledown) = \rho(A)$ and $\rho(A^\blacktriangledown) = \rho^{1/2}(A\bar{A})$.*

Theorem 3.1. *The NME in* (1.4) *always has a unique positive definite solution X_+ and the sequence $\{X_n\}$ generated by*

$$\begin{cases} X_0 = I, \\ X_{n+1} = I + A^*\bar{X}_n^{-1}A, \quad n = 1, 2, \ldots, \end{cases} \tag{3.2}$$

converges to X_+. Moreover, if A is nonsingular, (1.4) *has a unique negative definite solution X_- and $X_- = I - Z_+$ where Z_+ is the unique positive definite solution of the NME $Z - A\bar{Z}^{-1}A^* = I$.*

Proof. Consider the nonlinear matrix equation in the following form

$$W - (A^{\blacktriangledown})^T W^{-1} A^{\blacktriangledown} = I_{2n}. \tag{3.3}$$

Applying Lemma 2.4, (3.3) always has a unique positive definite solution, denoted by W_+, and the sequence obtained by the following iteration

$$\begin{cases} W_0 = I_{2n}, \\ W_{k+1} = I_{2n} + (A^{\blacktriangledown})^T W_k^{-1} A^{\blacktriangledown}, \quad k = 1, 2, \ldots, \end{cases}$$

converges to W_+.

In the following, for the sequences $\{X_k\}$ in (3.2) and $\{W_k\}$ in (3.3), we show by induction that for any nonnegative integer $k > 0$, $W_k = X_k^{\triangledown}$. Obviously, $W_0 = X_0^{\triangledown}$. Assume that $W_k = X_k^{\triangledown}$. Then for $k+1$,

$$\begin{aligned} W_{k+1} &= I_{2n} + (A^{\blacktriangledown})^T W_k^{-1} A^{\blacktriangledown} \\ &= I_{2n} + (A^{\blacktriangledown})^T (X_k^{\triangledown})^{-1} A^{\blacktriangledown} \\ &= I_{2n} + (A^{\triangledown})^T E_{2n} (X_k^{\triangledown})^{-1} E_{2n} A^{\triangledown} \\ &= I_{2n} + (A^{\triangledown})^T (E_{2n} X_k^{\triangledown} E_{2n})^{-1} A^{\triangledown} \\ &= I_{2n} + (A^{\triangledown})^T (\bar{X}_k^{\triangledown})^{-1} A^{\triangledown} \\ &= (I_n + A^* \bar{X}_k^{-1} A)^{\triangledown} \\ &= X_{k+1}^{\triangledown}. \end{aligned}$$

Hence, there exists a $X_+ > 0$ such that

$$W_+ = \lim_{k\to\infty} W_k = \lim_{k\to\infty} X_k^{\triangledown} = X_+^{\triangledown}.$$

Then $X_+^{\triangledown} = I_{2n} + (A^{\blacktriangledown})^T (X_+^{\triangledown})^{-1} A^{\blacktriangledown} = (I_n + A^* \bar{X}_+^{-1} A)^{\triangledown}$ which implies that $X_+ = I_n + A^* \bar{X}_+^{-1} A$, i.e., X_+ is a positive definite solution of (1.4) and X_+ can be obtained from iteration (3.2).

The uniqueness of X_+ follows from the uniqueness of the positive definite solution of (3.3). In fact, suppose that X ($X \neq X_+$) is another positive definite solution of $X - A^* \bar{X}^{-1} A = I$. Taking $(\cdot)^{\triangledown}$ on both sides of $X - A^* \bar{X}^{-1} A = I$ and using Lemma 3.1 yields $X^{\triangledown} > 0$ and

$$\begin{aligned} I_n^{\triangledown} &= X^{\triangledown} - (A^* \bar{X}^{-1} A)^{\triangledown} \\ &= X^{\triangledown} - (A^*)^{\triangledown} (\bar{X}^{-1})^{\triangledown} A^{\triangledown} \\ &= X^{\triangledown} - (A^{\triangledown})^T (\bar{X}^{\triangledown})^{-1} A^{\triangledown} \\ &= X^{\triangledown} - (E_{2n} A^{\blacktriangledown})^T (\bar{X}^{\triangledown})^{-1} E_{2n} A^{\blacktriangledown} \\ &= X^{\triangledown} - (A^{\blacktriangledown})^T E_{2n} (\bar{X}^{\triangledown})^{-1} E_{2n} A^{\blacktriangledown} \\ &= X^{\triangledown} - (A^{\blacktriangledown})^T (E_{2n} \bar{X}^{\triangledown} E_{2n})^{-1} A^{\blacktriangledown} \end{aligned}$$

$$= X^{\triangledown} - (A^{\blacktriangledown})^T (X^{\triangledown})^{-1} A^{\blacktriangledown}.$$

Then $X^{\triangledown}$ ($X^{\triangledown} \neq X_+^{\triangledown}$) is also a positive definite solution of (3.3), which is a contradiction with the fact that the NME in (3.3) has unique positive definite solution.

Moreover, if A is nonsingular, then so is $A^{\blacktriangledown}$ from Lemma 3.1. Applying Lemma 2.4, we obtain that equation (3.3) has a unique negative definite solution W_- and $W_- = I_{2n} - Y_+$ where Y_+ is the unique positive definite solution of the equation $Y - A^{\blacktriangledown} Y^{-1} (A^{\blacktriangledown})^T = I_{2n}$. By the discussions above, $Y_+ = Z_+^{\triangledown}$ where Z_+ is the unique positive definite solution of $Z - A\bar{Z}^{-1}A^* = I$. Then we get

$$W_- = I_{2n} - Y_+ = I_{2n} - Z_+^{\triangledown} = (I_n - Z_+)^{\triangledown} < 0.$$

Notice that $W_-^{-1} = [(I - Z_+)^{\triangledown}]^{-1} = [(I - Z_+)^{-1}]^{\triangledown}$. Then

$$\begin{aligned}
I_{2n} &= W_- - (A^{\blacktriangledown})^T W_-^{-1} A^{\blacktriangledown} \\
&= (I_n - Z_+)^{\triangledown} - (A^{\blacktriangledown})^T [(I - Z_+)^{-1}]^{\triangledown} A^{\blacktriangledown} \\
&= (I_n - Z_+)^{\triangledown} - (A^{\triangledown})^T E_{2n} [(I - Z_+)^{-1}]^{\triangledown} E_{2n} A^{\triangledown} \\
&= (I_n - Z_+)^{\triangledown} - (A^*)^{\triangledown} [(I - \bar{Z}_+)^{-1}]^{\triangledown} A^{\triangledown} \\
&= (I_n - Z_+)^{\triangledown} - (A^*(I - \bar{Z}_+)^{-1} A)^{\triangledown},
\end{aligned}$$

which gives

$$(I_n - Z_+) - A^*(I_n - \bar{Z_+})^{-1} A = I_n.$$

Denote $X_- = I - Z_+$. Then $X_- < 0$ and $X_- - A^* \bar{X}_-^{-1} A = I_n$ which implies that X_- is a negative definite solution of (1.4). The uniqueness of X_- follows from the uniqueness of the negative definite solution of (3.3). $\square$

Remark 3.1. Consider NME (1.4) with any given complex A, taking $\mathfrak{F}(X) = -\bar{X}$ and $B = I + A^*A$, we have $I - B = -A^*A \leq -A^*\bar{X}^{-1}A \leq 0$ for any $X \in [I, B]$. From Lemma 2.1 in [22] for matrix equation $X + A^*\mathfrak{F}(X)A = I$, (1.4) has a positive definite solution in $[I, B]$. But here in Theorem 3.1, we show not only the existence, but also the uniqueness of the positive definite solution X_+. Moreover, we offer the basic fixed-point iteration for X_+ which will be used to prove our main result Theorem 3.3.

Iteration (3.2) is very simple and intuitive. For the convergence rate of it, we have

$$\begin{aligned}
&X_{k+1} - X_+ \\
&= A^*(\bar{X}_k^{-1} - \bar{X}_+^{-1})A \\
&= A^*\bar{X}_k^{-1}(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A \\
&= A^*(\bar{X}_k^{-1} - \bar{X}_+^{-1} + \bar{X}_+^{-1})(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A \\
&= A^*(\bar{X}_k^{-1} - \bar{X}_+^{-1})(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A + A^*\bar{X}_+^{-1}(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A \\
&= A^*\bar{X}_+^{-1}(\bar{X}_+ - \bar{X}_k)\bar{X}_k^{-1}(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A + A^*\bar{X}_+^{-1}(\bar{X}_+ - \bar{X}_k)\bar{X}_+^{-1}A.
\end{aligned}$$

Then

$$\begin{aligned}\|X_{k+1}-X_+\| &\le \|\bar{X}_+^{-1}A\|^2\cdot\|\bar{X}_k^{-1}\|\cdot\|\bar{X}_+-\bar{X}_k\|^2+\|\bar{X}_+^{-1}A\|^2\cdot\|\bar{X}_+-\bar{X}_k\|\\ &=(\|\bar{X}_+^{-1}A\|^2\cdot\|\bar{X}_k^{-1}\|\cdot\|\bar{X}_+-\bar{X}_k\|+\|\bar{X}_+^{-1}A\|^2)\cdot\|\bar{X}_+-\bar{X}_k\|.\end{aligned}$$

Since $X_k\to X_+$ as $k\to\infty$, for any $\epsilon>0$, there exists an integer $M>0$ such that for any $k>M$, we have

$$\|X_{k+1}-X_+\|\le(\|\bar{X}_+^{-1}A\|^2+\epsilon)\|X_+-X_k\|.$$

Moreover, we can obtain the following.

Theorem 3.2. *For iteration* (3.2), *we have*

$$\|X_{2k}-X_+\|\le\|\bar{X}_+^{-1}A\|^2\cdot\|X_{2k-1}-X_+\| \tag{3.4}$$

for all $k\ge1$ *and*

$$\limsup_{n\to\infty}\sqrt[n]{\|X_n-X_+\|}\le\rho(X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A)<1. \tag{3.5}$$

Proof. Notice that for all $k\ge1$,

$$\begin{aligned}&X_{k+1}-X_+\\ &\quad=A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_k)\bar{X}_k^{-1}(\bar{X}_+-\bar{X}_k)\bar{X}_+^{-1}A+A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_k)\bar{X}_+^{-1}A.\end{aligned} \tag{3.6}$$

Combining (3.6) with the fact that $\{X_{2k}\}$ is increasing and converges to X_+ and $\{X_{2k+1}\}$ is decreasing and converges to X_+, we have

$$0<X_+-X_{2k}\le A^*\bar{X}_+^{-1}(\bar{X}_{2k-1}-\bar{X}_+)\bar{X}_+^{-1}A$$

for all $k\ge1$, from which (3.4) follows.

Moreover, since

$$\begin{aligned}&X_{2k+1}-X_+\\ &=A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_{2k})\bar{X}_+^{-1}A+A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_{2k})\bar{X}_{2k}^{-1}(\bar{X}_+-\bar{X}_{2k})\bar{X}_+^{-1}A\\ &=A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_{2k})^{1/2}[I+(\bar{X}_+-\bar{X}_{2k})^{1/2}\bar{X}_{2k}^{-1}(\bar{X}_+-\bar{X}_{2k})^{1/2}]\\ &\qquad(\bar{X}_+-\bar{X}_{2k})^{1/2}\bar{X}_+^{-1}A,\end{aligned}$$

and for arbitrary $\epsilon>0$, there exists a k_0, such that

$$(\bar{X}_+-\bar{X}_{2k})^{1/2}\bar{X}_{2k}^{-1}(\bar{X}_+-\bar{X}_{2k})^{1/2}\le\epsilon I$$

for all $k\ge k_0$. Then

$$0\le X_{2k+1}-X_+\le(1+\epsilon)A^*\bar{X}_+^{-1}(\bar{X}_+-\bar{X}_{2k})\bar{X}_+^{-1}A$$

for all $k\ge k_0$. Denote $K=(X_+^{-1}\bar{A})\cdot(\bar{X}_+^{-1}A)$. We obtain that for all $k\ge k_0$,

$$\begin{aligned}0<X_+-X_{2k}&\le A^*\bar{X}_+^{-1}(\bar{X}_{2k-1}-\bar{X}_+)\bar{X}_+^{-1}A\\ &\le(1+\epsilon)(A^*\bar{X}_+^{-1})(A^TX_+^{-1})(X_+-X_{2(k-1)})X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A\\ &=(1+\epsilon)K^*(X_+-X_{2(k-1)})K\\ &\le(1+\epsilon)^2(K^2)^*(X_+-X_{2(k-2)})K^2\end{aligned}$$

$$\leq (1+\epsilon)^3 (K^3)^*(X_+ - X_{2(k-3)})K^3$$
$$\leq (1+\epsilon)^{k-k_0}(K^{k-k_0})^*(X_+ - X_{2k_0})K^{k-k_0}.$$

Then

$$0 < X_+ - X_{2k+2} \leq (1+\epsilon)^{k-k_0+1}(K^{k-k_0+1})^*(X_+ - X_{2k_0})K^{k-k_0+1}$$

and

$$\begin{aligned} 0 &< X_{2k+1} - X_+ \\ &\leq (1+\epsilon)A^*\bar{X}_+^{-1}(\bar{X}_+ - \bar{X}_{2k})\bar{X}_+^{-1}A \\ &\leq (1+\epsilon)^{k-k_0+1}(A^*\bar{X}_+^{-1})(\bar{K}^{k-k_0})^*(\bar{X}_+ - \bar{X}_{2k_0})\bar{K}^{k-k_0}(\bar{X}_+^{-1}A) \end{aligned}$$

which gives

$$\|X_+ - X_{2k+2}\| \leq (1+\epsilon)^{k-k_0+1}\|K^{k-k_0+1}\|^2 \cdot \|X_+ - X_{2k_0}\|,$$

and

$$\|X_{2k+1} - X_+\| \leq (1+\epsilon)^{k-k_0+1} \cdot \|\bar{X}_+^{-1}A\|^2 \cdot \|K^{k-k_0}\|^2 \cdot \|X_+ - X_{2k_0}\|.$$

Observe that $\rho(A) = \lim_{k\to\infty} \sqrt[k]{\|A^k\|}$, then we have

$$\limsup_{n\to\infty} \sqrt[n]{\|X_n - X_+\|} \leq \sqrt{1+\epsilon}\cdot\rho(K) = \sqrt{1+\epsilon}\cdot\rho(X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A).$$

Using Lemma 2.4 and Lemma 3.1, we obtain that

$$\begin{aligned} \rho^{1/2}(X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A) &= \rho^{1/2}(AX_+^{-1}\cdot\bar{A}\bar{X}_+^{-1}) = \rho((AX_+^{-1})^{\blacktriangledown}) \\ &= \rho(E_n(AX_+^{-1})^{\triangledown}) = \rho(E_nA^{\triangledown}(X_+^{-1})^{\triangledown}) \\ &= \rho((X_+^{\triangledown})^{-1}E_nA^{\triangledown}) = \rho((X_+^{\triangledown})^{-1}A^{\blacktriangledown}) \\ &= \rho(W_+^{-1}A^{\blacktriangledown}) < 1, \end{aligned}$$

where W_+ is the unique positive definite solution of NME (3.3). Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{\|X_n - X_+\|} \leq \sqrt{1+\epsilon}\cdot\rho(X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A) < 1.$$

Since $\epsilon > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} \sqrt[n]{\|X_n - X_+\|} \leq \rho(X_+^{-1}\bar{A}\cdot\bar{X}_+^{-1}A) < 1.$$

□

In the following, using the famous Sherman-Morrison-Woodbury formula, we give an elegant relationship between NME (1.4) and the extensively studied nonlinear matrix equation $Y + B^*Y^{-1}B = Q$ where $B = \bar{A}A$ and $Q = I + A^*A + \bar{A}\bar{A}^*$.

Theorem 3.3. (i) *Let $X_+ > 0$ be the unique positive definite solution of* (1.4). *Then $X_+ = Y_L - \bar{A}\bar{A}^*$ where Y_L is the maximal positive definite solution of*

$$Y + (\bar{A}A)^*Y^{-1}(\bar{A}A) = I + A^*A + \bar{A}\bar{A}^*. \tag{3.7}$$

(ii) *Moreover, if A is nonsingular, let $X_- < 0$ be the unique negative definite solution of* (1.4)*, then $X_- = Z_l - AA^*$, where Z_l is the minimal positive definite solution of*

$$Z + (A\bar{A})^* Z^{-1}(A\bar{A}) = I + AA^* + \bar{A}^*\bar{A}. \tag{3.8}$$

Proof. (i) Consider the nonlinear matrix equation (3.7). Denote $Q = I + A^*A + \bar{A}\bar{A}^*$. Observe that for any $\lambda \in \mathcal{C}$, it holds that

$$\begin{aligned}\psi(\lambda) &= Q + \lambda\bar{A}A + \lambda^{-1}(\bar{A}A)^* \\ &= I + \bar{A}\bar{A}^* + A^*A + \lambda\bar{A}A + \bar{\lambda}(\bar{A}A)^* \\ &= I + \bar{A}\bar{A}^* + |\lambda|^2 A^*A + \lambda\bar{A}A + \bar{\lambda}(\bar{A}A)^* \\ &= I + (\bar{A} + \bar{\lambda}A^*)(\bar{A}^* + \lambda A) \\ &> 0.\end{aligned}$$

Applying Lemma 2.2, we know that the NME (3.7) always has a positive definite solution and thus has the maximal positive definite solution Y_L.

From Theorem 3.1, equation (1.4) always has a unique positive definite solution, denoted by X_+, and the sequence $\{X_n\}$ generated by (3.2) converges to X_+.

Now consider the subsequence $\{X_{2k+1}\}_{k=0}^{\infty}$ consisting of the odd elements of $\{X_n\}$ in (3.2). We have $X_1 = I + A^*A$ and for all $k \geq 1$,

$$\begin{aligned}X_{2k+1} &= I + A^*\bar{X}_{2k}^{-1}A \\ &= I + A^*(I + \bar{A}^* X_{2k-1}^{-1}\bar{A})^{-1}A \\ &= I + A^*[I - \bar{A}^*(X_{2k-1} + \bar{A}\bar{A}^*)^{-1}\bar{A}]A \\ &= I + A^*A - (A^*\bar{A}^*)(X_{2k-1} + \bar{A}\bar{A}^*)^{-1}\bar{A}A,\end{aligned} \tag{3.9}$$

where Sherman-Morrison-Woodbury formula is used in the third equality. Then

$$X_{2k+1} + \bar{A}\bar{A}^* = I + A^*A + \bar{A}\bar{A}^* - (A^*\bar{A}^*)(X_{2k-1} + \bar{A}\bar{A}^*)^{-1}\bar{A}A.$$

Denote

$$Y_k = X_{2k+1} + \bar{A}\bar{A}^*, \quad k = 0, 1, 2, \ldots. \tag{3.10}$$

Obviously, $X_{2k+1} \geq I$. Then $Y_k \geq I + \bar{A}\bar{A}^*$, $k = 1, 2, \ldots$ and $\{Y_k\}$ satisfies

$$\begin{cases} Y_0 = X_1 + \bar{A}\bar{A}^* = Q, \\ Y_k = Q - (\bar{A}A)^* Y_{k-1}^{-1}(\bar{A}A), \quad k = 1, 2, \ldots. \end{cases}$$

Applying Lemma 2.2 to NME (3.7), we obtain that $\{Y_k\}$ converges to Y_L, the maximal positive definite solution of equation (3.7).

Since $\{X_n\}$ converges to the unique positive definite solution X_+ of equation (1.4), so does the odd subsequence $\{X_{2k+1}\}_{k=0}^{\infty}$. Taking limits on both sides of (3.10) gives

$$Y_L = \lim_{k\to\infty} Y_k = \lim_{k\to\infty} X_{2k+1} + \bar{A}\bar{A}^* = X_+ + \bar{A}\bar{A}^*,$$

that is, $X_+ = Y_L - \bar{A}\bar{A}^*$, where Y_L is the maximal positive definite solution of (3.7).

Moreover, according to Theorem 3.1, if A is nonsingular, then (1.4) has a unique negative definite solution X_-, and $X_- = I - Y_+$, where Y_+ is the unique positive definite solution of $Y - A\bar{Y}^{-1}A^* = I$. From (i), $Y_+ + \bar{A}^*\bar{A}$ is the maximal positive definite solution of equation

$$Z + (A\bar{A})Z^{-1}(A\bar{A})^* = P, \quad P = I + AA^* + \bar{A}^*\bar{A}. \tag{3.11}$$

Applying Lemma 2.3 to (3.11) gives that

$$P - (Y_+ + \bar{A}^*\bar{A}) = I + AA^* - Y_+ = X_- + AA^*$$

is the minimal positive definite solution Z_l of (3.8). That is, $X_- = Z_l - AA^*$ which affirms (ii). □

Applying Theorem 3.3, several effective algorithms with elegant numerical performances such as fixed-point iteration, inversion-free iteration, structure-preserving doubling algorithm, B. Meini's Cyclic reduction algorithm and Newton iteration for the unique positive definite solution X_+ of $X - A^*\bar{X}^{-1}A = I$ can be obtained immediately from the corresponding algorithms [10, 18, 20] for the maximal positive definite solution Y_L of the well-studied NME $Y + B^*Y^{-1}B = Q$ with $B = \bar{A}A, Q = I + A^*A + \bar{A}\bar{A}^*$.

Accelerated fixed-point iteration

$$\begin{cases} Y_0 = Q, \\ Y_{n+1} = Q - B^*Y_n^{-1}B, \quad n = 1, 2, \ldots. \end{cases} \tag{3.12}$$

Then we have $X_+ = Y_L - \bar{A}\bar{A}^* = \lim_{n\to\infty} Y_n - \bar{A}\bar{A}^*$.

Inversion-free iteration

$$\begin{cases} Y_0 = Q, \quad 0 < X_0 \le Q^{-1}, \\ X_{n+1} = X_n(2I - Y_nX_n), \\ Y_{n+1} = Q - B^*X_{n+1}B, \quad n = 1, 2, \ldots. \end{cases} \tag{3.13}$$

Then we have $X_+ = Y_L - \bar{A}\bar{A}^* = \lim_{n\to\infty} Y_n - \bar{A}\bar{A}^*$.

Structure-preserving Doubling Algorithm (SDA)

$$\begin{cases} A_0 = \bar{A}A, \quad Q_0 = Q, \quad P_0 = 0, \\ A_{n+1} = A_n(Q_n - P_n)^{-1}A_n, \\ Q_{n+1} = Q_n - A_n^*(Q_n - P_n)^{-1}A_n, \\ P_{n+1} = P_n + A_n(Q_n - P_n)^{-1}A_n^*, \quad n = 1, 2, \ldots. \end{cases} \tag{3.14}$$

Then we have $X_+ = Y_L - \bar{A}\bar{A}^* = \lim_{n\to\infty} Q_n - \bar{A}\bar{A}^*$.

Cyclic Reduction Algorithm (CRA)

$$\begin{cases} A_0 = \bar{A}A, \quad Q_0 = Q, \quad Y_0 = Q, \\ A_{n+1} = A_nQ_n^{-1}A_n, \\ Q_{n+1} = Q_n - A_n^*Q_n^{-1}A_n - A_nQ_n^{-1}A_n^*, \\ Y_{n+1} = Y_n - A_n^*Q_n^{-1}A_n, \quad n = 1, 2, \ldots. \end{cases} \tag{3.15}$$

Then we have $X_+ = Y_L - \bar{A}\bar{A}^* = \lim_{n\to\infty} Y_n - \bar{A}\bar{A}^*$.

Newton's iteration

Take $Y_0 = Q$, $\ B = \bar{A}A$. For $n = 1, 2, \dots$, compute $L_n = Y_{n-1}^{-1}B$, and solve

$$Y_n - L_n^* Y_n L_n = Q - 2L_n B \tag{3.16}$$

until $\|Y_n - Y_{n+1}\| < \epsilon$. Then $X_+ = Y_L - \bar{A}\bar{A}^* = \lim_{n\to\infty} Y_n - \bar{A}\bar{A}^*$.

Remark 3.2. As to the convergence rate of these algorithms, we have

(i) Iteration (3.2) is actually the basic fixed-point theorem, and it is linear convergent from (3.4) in Theorem 3.2;

(ii) Iteration (3.12) can be regarded as an accelerated version of iteration (3.2). It is also linear convergent with a convergence rate about twice of that of (3.2) provided $\|\bar{X}_+^{-1}A\| < 1$. In fact, from (3.10) we know that the sequence $\{Y_n\}$ generated by (3.12) is actually the odd elements in the sequence generated by (3.2) plus $\bar{A}\bar{A}^*$. Moreover, we have

$$\begin{aligned}\|Y_{n+1} - Y_L\| &= \|X_{2n+3} + \bar{A}\bar{A}^* - (X_+ + \bar{A}\bar{A}^*)\| = \|X_{2n+3} - X_+\| \\ &\le \|\bar{X}_+^{-1}A\|^2 \|X_{2n+2} - X_+\| \le \|\bar{X}_+^{-1}A\|^4 \|X_{2n+1} - X_+\| \\ &= \|\bar{X}_+^{-1}A\|^4 \|Y_n - Y_L\|\end{aligned}$$

according to Theorem 3.2;

(iii) Iteration (3.13) can be regarded as the inversion-free variation of algorithm (3.12). It is also linear convergent with roughly the same convergence rate with that of (3.12), while (3.13) does not require the computation of matrix inversion which may turn it to be more numerically reliable.

(iv) Algorithms SDA and CRA are actually the same and proved to have quadratically convergence rate, low computational cost per step and good numerical stability in [18, 20], so do iterations (3.14) and (3.15). Newton's iteration is proved to have quadratical convergence rate and good numerical stability [10].

Remark 3.3. As to the minimal solution (unique negative definite solution) X_- of NME (1.4) when A is nonsingular, according to Theorem 3.3, $X_- = Z_l - AA^*$, where Z_l is the minimal positive definite solution of

$$Z + (A\bar{A})^* Z^{-1}(A\bar{A}) = G, \quad G = I + AA^* + \bar{A}^*\bar{A},$$

and $G - Z_l$, denoted by Y_+, is the maximal positive definite solution of

$$Y + (A\bar{A})Y^{-1}(A\bar{A})^* = G. \tag{3.17}$$

Thus we have

$$X_- = Z_l - AA^* = G - Y_+ - AA^* = (I + \bar{A}^*\bar{A}) - Y_+,$$

which allows us to obtain similar efficient algorithms for X_- since we can obtain the maximal positive definite solution Y_+ of (3.17) by the algorithms (3.12)-(3.16).

4. Results on nonlinear matrix equation $X - A^*X^{-1}A = I$

Inspired by Theorem 3.3, we offer in this section an elegant relationship between the maximal positive definite solutions of the well studied nonlinear matrix equations

$$X - A^*X^{-1}A = I \tag{4.1}$$

and

$$Y + B^*Y^{-1}B = Q, \tag{4.2}$$

which are always studied independently so far. Hopefully, it may offer a new way and a new perspective to consider the solutions of these two matrix equations and their corresponding applied problems.

Consider the following matrix equation

$$Y + (A^2)^*Y^{-1}A^2 = I + AA^* + A^*A. \tag{4.3}$$

Denote $K = I + AA^* + A^*A$. Observe that for any $\lambda \in \mathcal{C}$,

$$\begin{aligned}\psi(\lambda) &= K + \lambda A^2 + \lambda^{-1}(A^2)^* \\ &= I + AA^* + A^*A + \lambda A^2 + \bar{\lambda}A^*A^* \\ &= I + (A + \bar{\lambda}A^*)(A^* + \lambda A) > 0.\end{aligned}$$

Applying Lemma 2.2, we obtain that (4.3) always has a positive definite solution and thus has the maximal positive definite solution Y_L and the minimal positive definite solution Y_l.

Theorem 4.1. *Let $X_+ > 0$ be the maximal solution (i.e., the unique positive definite solution) of (4.1). Then $X_+ = Y_L - AA^*$; Moreover, if A is nonsingular, let $X_- < 0$ be the minimal solution (i.e., the unique negative definite solution) of (4.1), then $X_- = Y_l - AA^*$, where Y_L and Y_l are the maximal and minimal positive definite solutions of (4.3), respectively.*

Proof. The proof is similar to that of Theorem 3.3 which is omitted here. □

Theorem 4.1 offers a new and simple way to consider matrix equation $X - A^*X^{-1}A = I$. For example, applying Theorem 4.1, several new effective algorithms for the unique positive definite solution X_+ of $X - A^*X^{-1}A = I$ can be obtained immediately from the existed algorithms for the maximal positive definite solution of $Y + B^*Y^{-1}B = K$ with $B = A^2$ and $K = I + AA^* + A^*A$.

Accelerated fixed-point iteration

$$\begin{cases} Y_0 = K, \\ Y_{n+1} = K - B^*Y_n^{-1}B, \quad n = 1, 2, \ldots. \end{cases} \tag{4.8}$$

Then we have $X_+ = \lim_{n\to\infty} Y_n - AA^*$.

Inversion-free iteration

$$\left\{\begin{array}{l} Y_0 = K, \quad 0 < X_0 \leq K^{-1}, \\ X_{n+1} = X_n(2I - Y_n X_n), \\ Y_{n+1} = K - B^* X_{n+1} B, \quad n = 1, 2, \ldots. \end{array}\right. \tag{4.9}$$

Then we have $X_+ = \lim_{n\to\infty} Y_n - AA^*$.

Structure-preserving Doubling Algorithm (SDA)

$$\left\{\begin{array}{l} A_0 = A^2, \quad Q_0 = K, \quad P_0 = 0, \\ A_{n+1} = A_n(Q_n - P_n)^{-1} A_n, \\ Q_{n+1} = Q_n - A_n^*(Q_n - P_n)^{-1} A_n, \\ P_{n+1} = P_n + A_n(Q_n - P_n)^{-1} A_n^*, \quad n = 1, 2, \ldots. \end{array}\right. \tag{4.10}$$

Then we have $X_+ = \lim_{n\to\infty} Q_n - AA^*$.

Cyclic Reduction Algorithm (CRA)

$$\left\{\begin{array}{l} A_0 = A^2, \quad Q_0 = K, \quad Y_0 = K, \\ A_{n+1} = A_n Q_n^{-1} A_n, \\ Q_{n+1} = Q_n - A_n^* Q_n^{-1} A_n - A_n Q_n^{-1} A_n^*, \\ Y_{n+1} = Y_n - A_n^* Q_n^{-1} A_n, \quad n = 1, 2, \ldots. \end{array}\right. \tag{4.11}$$

Then we have $X_+ = \lim_{n\to\infty} Y_n - AA^*$.

It is easy to find that (4.11) can be slightly adjusted to

Cyclic Reduction Algorithm (CRA)

$$\left\{\begin{array}{l} A_0 = A^2, \quad Q_0 = K, \quad X_0 = K - AA^* = I + A^*A, \\ A_{n+1} = A_n Q_n^{-1} A_n, \\ Q_{n+1} = Q_n - A_n^* Q_n^{-1} A_n - A_n Q_n^{-1} A_n^*, \\ X_{n+1} = X_n - A_n^* Q_n^{-1} A_n, \quad n = 1, 2, \ldots, \end{array}\right. \tag{4.11$'$}$$

which gives $X_+ = \lim_{n\to\infty} X_n$.

Newton's iteration

Take $Y_0 = K$. For $n = 1, 2, \ldots$, compute $L_n = Y_{n-1}^{-1} B$, and solve

$$Y_n - L_n^* Y_n L_n = K - 2L_n B, \tag{4.12}$$

until $\|Y_n - Y_{n+1}\| < \epsilon$. Then $X_+ = \lim_{n\to\infty} Y_n - AA^*$.

Remark 4.1. We point out that here (4.8), (4.9), (4.11) and (4.12) are new, and (4.10) and (4.11)$'$ have been obtained in [18] and [20] respectively with much more complicated analysis and proofs. As to the convergence rates of these algorithms, we have same results with Remark 3.2.

Remark 4.2. As to the minimal solution (unique negative definite solution) X_- of NME (4.1), applying theorem 4.1, $X_- = Y_l - AA^*$, where Y_l is the minimal positive definite solution of

$$Y + (A^2)^* Y^{-1} A^2 = H, \quad H = I + AA^* + A^*A,$$

and $H - Y_l$, denoted by Z_+, is the maximal positive definite solution of

$$Z + A^2 Z^{-1} (A^2)^* = H. \tag{4.13}$$

Thus we have

$$X_- = Y_l - AA^* = H - Z_+ - AA^* = (I + A^*A) - Z_+,$$

which allows us to obtain similar efficient algorithms for X_-.

5. Numerical experiments

In this section, we carry out several numerical experiments to examine the effectiveness of all the theoretical results. All the programming is implemented on a PC with 1.7 GHz Pentium IV using MATLAB 7.1.

Example 5.1. Consider a NME in (1.4) with the coefficient matrix A given by

$$A = \begin{pmatrix} 0.6294 - 0.1565i & 0.2647 + 0.3115i & 0.9150 + 0.3575i & 0.9143 + 0.3110i \\ 0.8116 + 0.8315i & -0.8049 - 0.9286i & 0.9298 + 0.5155i & -0.0292 - 0.6576i \\ -0.7460 + 0.5844i & -0.4430 + 0.6983i & -0.6848 + 0.4863i & 0.6006 + 0.4121i \\ 0.8268 + 0.9190i & 0.0938 + 0.8680i & 0.9412 - 0.2155i & -0.7162 - 0.9363i \end{pmatrix},$$

which is generated randomly by the function $(2{*}rand(4){-}1){+}i{*}(2{*}rand(4){-}1)$ in MATLAB. We compute the unique positive definite solution X_+ of equation (1.4) by algorithms (3.2) and (3.12)-(3.16). We use the spectral norm of the residuals

$$Res(X_k) = \|X_k - A^* \bar{X}_k^{-1} A - I\|, \quad k = 0, 1, 2, \ldots,$$

to denote the iteration error. A sufficiently accurate unique positive definite solution of this NME is obtained as

$$X_+ = \begin{pmatrix} 2.7315 & 0.1200 + 0.5372i & 1.5523 - 0.3407i & -0.5077 - 0.4204i \\ 0.1200 - 0.5372i & 2.5649 & -0.2717 - 0.1620i & 0.2258 + 0.6021i \\ 1.5523 + 0.3407i & -0.2717 + 0.1620i & 3.2606 & 0.0473 - 1.2157i \\ -0.5077 + 0.4204i & 0.2258 - 0.6021i & 0.0473 + 1.2157i & 2.3578 \end{pmatrix}.$$

To illustrate the convergence rates of these different algorithms, the iteration errors are recorded in Fig. 1. It's clear that Newton's iteration find a good approximation in the smaller number of iterations and the basic fixed-point iteration gives the more accurate approximation. Moreover, algorithms SDA(3.14)/CRA(3.15) and Newton's iteration (3.16) converge much faster than algorithms (3.2), (3.12) and (3.13) which coincided with the theoretical analysis that (3.14)-(3.16) are convergent quadratically while (3.2), (3.12) and (3.13) are convergent linearly. Besides, the convergence rate of the inversion-free iteration (3.13) is almost the same as that of the accelerated iteration (3.12) since they are both about twice of the convergence rate of the basic fixed-point iteration (3.2). All coincide with the theoretical analysis in Remark 3.2.

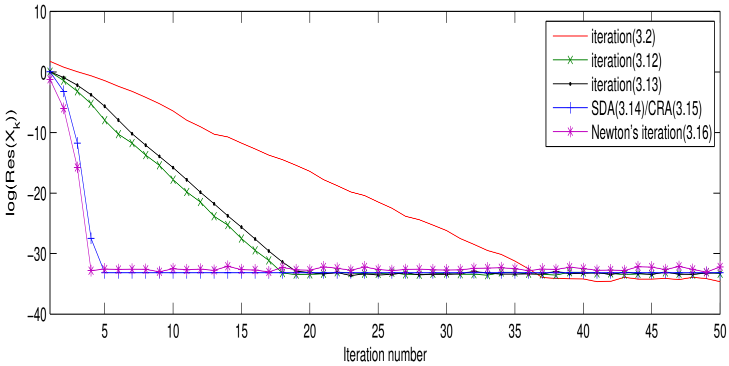

FIGURE 1. The comparison of convergence rates of six different iterations for NME (1.4)

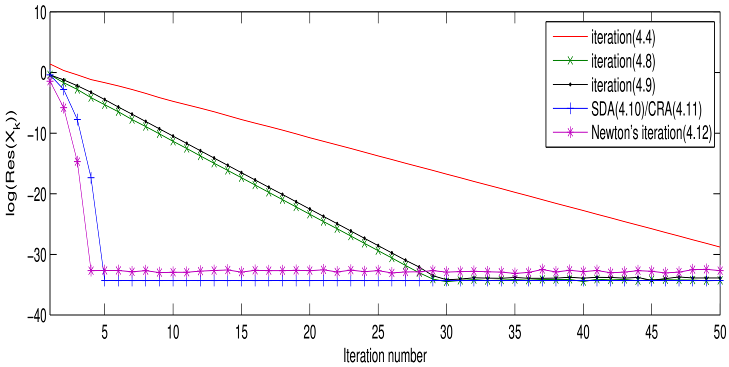

FIGURE 2. The comparison of convergence rates of six different iterations for NME (4.1)

Example 5.2. Consider a NME in (4.1) with the coefficient matrix A given by

$$A = \begin{pmatrix} 0.7818 - 0.2967i & -0.7014 + 0.8344i & 0.6286 - 0.2391i & -0.6068 + 0.0616i \\ 0.9186 + 0.6617i & -0.4850 - 0.4283i & -0.5130 + 0.1356i & -0.4978 + 0.5583i \\ 0.0944 + 0.1705i & 0.6814 + 0.5144i & 0.8585 - 0.8483i & 0.2321 + 0.8680i \\ -0.7228 + 0.0994i & -0.4914 + 0.5075i & -0.3000 - 0.8921i & -0.0534 - 0.7402i \end{pmatrix},$$

which is generated randomly by the function $(2*rand(4)-1)+i*(2*rand(4)-1)$ in MATLAB. We compute the unique positive definite solution X_+ of equation (4.1) by algorithms (4.4) and (4.8)-(4.12). We use the spectral norm of the residuals $Res(X_k) = \|X_k - A^* X_k^{-1} A - I\|$, $k = 0, 1, 2, \dots$, to denote the iteration error.

TABLE 1. The comparison of running time for 6 different iterations for X_+ for NME (1.4)

n	25	30	35	40	45	50	55
Iteration (3.2)	0.007372	0.015134	0.015461	0.018847	0.024541	0.030151	0.033979
Iteration (3.12)	0.003870	0.008686	0.007000	0.009382	0.012124	0.015733	0.017241
Iteration (3.13)	0.005230	0.007568	0.009751	0.013121	0.017220	0.019860	0.025749
Iteration (3.15)	0.003606	0.004953	0.006944	0.087765	0.011463	0.015439	0.018471
Iteration (3.16)	0.611129	1.517971	3.463479	7.202781	13.810753	24.988062	44.566569

TABLE 2. The comparison of running time for 6 different iterations for X_+ for NME (4.1)

n	25	30	35	40	45	50	55
Iteration (4.4)	0.009595	0.016807	0.019832	0.025675	0.032112	0.041134	0.053184
Iteration (4.8)	0.006339	0.007689	0.010795	0.012792	0.017033	0.021357	0.025466
Iteration (4.9)	0.005408	0.010015	0.013097	0.014836	0.015886	0.019715	0.023974
Iteration (4.11)	0.003775	0.005281	0.006776	0.008599	0.012154	0.014409	0.018157
Iteration (4.12)	0.672660	1.534046	3.471230	7.127257	13.828581	25.361517	57.602918

A sufficiently accurate unique positive definite solution of this NME is obtained as

$$X_+ = \begin{pmatrix} 2.7202 & -0.1254+0.4030i & 0.0044+0.3785i & -0.3870+1.2663i \\ -0.1254-0.4030i & 2.3438 & -0.1387-0.3634i & 0.5443+0.7080i \\ 0.0044-0.3785i & -0.1387+0.3634i & 2.1001 & 0.2547+0.3469i \\ -0.3870-1.2663i & 0.5443-0.7080i & 0.2547-0.3469i & 2.8258 \end{pmatrix}.$$

The convergence rates of these algorithms are illustrated in Fig. 2.

Example 5.3. In this example, we compare the time required to compute a sufficient accurate solution by these six different algorithms. We first construct a circular matrix $C \in \mathbb{R}^{n\times n}$ with the first row $c(n)$ given by

$$c(n) = [1-i,\ \ \frac{1}{\sqrt{2}} + (-1)^2\sqrt{2}i,\ \dots\ ,\ \frac{1}{\sqrt{n}} + (-1)^n\sqrt{n}i],$$

where $n \geq 1$ is an integer. Then we consider two NMEs in the form of (1.4) and (4.1) with $A = \frac{C}{2\|C\|}$. Here we use the practical stopping criterion $\log(Res(X_k)) \leq -32$, where $Res(X_k)$ are defined as in Examples 5.1 and 5.2 for NMEs (1.4) and (4.1), respectively. Now we compute the unique positive definite solution X_+ of NMEs (1.4) and (4.1) for different value of n. The running time (unit: second) needed are respectively recorded in Table 1 and Table 2, which show that in both cases, the structure-preserving doubling algorithm (SDA) and the cyclic reduction algorithm (CRA) are the most efficient.

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