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EXISTENCE AND LONG-TIME BEHAVIOR OF SOLUTIONS TO NAVIER-STOKES-VOIGT EQUATIONS WITH INFINITE DELAY

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Abstract. In this paper we study the first initial boundary value problem for the 3D Navier-Stokes-Voigt equations with infinite delay. First, we prove the existence and uniqueness of weak solutions to the problem by combining the Galerkin method and the energy method. Then we prove the existence of a compact global attractor for the continuous semigroup associated to the problem. Finally, we study the existence and exponential stability of stationary solutions.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. In this paper we consider the following three-dimensional Navier-Stokes-Voigt (sometimes written Voight) equations with infinite delay:

(1.1)

$$\begin{cases} \partial_t u - \nu\Delta u - \alpha^2\Delta(\partial_t u) + (u\cdot\nabla)u + \nabla p & = f + F(u_t) \text{ in } (0,+\infty)\times\Omega, \\ \nabla\cdot u & = 0 \text{ in } (0,+\infty)\times\Omega, \\ u(t,x) & = 0 \text{ on } (0,+\infty)\times\partial\Omega, \\ u(s,x) & = \phi(s,x),\ s\in(-\infty,0], x\in\Omega, \end{cases}$$

where $u = u(t,x) = (u_1,u_2,u_3)$ is the unknown velocity vector, $p = p(t,x)$ is the unknown pressure, $\nu > 0$ is the kinematic viscosity coefficient, α is a length-scale parameter characterizing the elasticity of the fluid, $f = f(x)$ is a given force field, u_t is the function defined by the relation $u_t(s) = u(t+s), s\in(-\infty,0]$.

The Navier-Stokes-Voigt equations were introduced by Oskolkov in [23] as a model of motion of certain linear viscoelastic fluids. This system was also proposed by Cao, Lunasin and Titi in [4] as a regularization, for small values

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of α, of the 3D Navier-Stokes equations for the sake of direct numerical simulations. In the past years, the existence and long-time behavior of solutions in terms of existence of attractors for the 3D Navier-Stokes-Voigt equations has attracted the attention of many mathematicians in both autonomous and non-autonomous cases (cf. [2, 9, 10, 12, 16, 17, 23, 24, 28]). We also refer the interested reader to [3, 22, 29] for recent results on the decay rates of solutions to the Navier-Stokes-Voigt equations in the whole space $\mathbb{R}^3$.

On the other hand, there are situations in which the model is better described if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system (in a certain sense) by applying a force which takes into account not only the present state but its history, either the finite time history (bounded delay) or the whole past (unbounded or infinite delay). In the past years, equations of Navier-Stokes type with delays have been studied in [6–8], etc, for the case of finite delays, see also a recent survey paper [5]; and in [1, 13, 14, 19–21, 25] for the case of infinite delays. The Navier-Stokes-Voigt equations with finite delays or with memory have been studied recently in [11, 18, 26]. However, as far as we know, the infinite delay case, the more delicate case due to the unboundedness of the delay involved, has not been studied before for the Navier-Stokes-Voigt equations. This is the main motivation of our study in the present paper.

As noticed in [17], the presence of the regularizing term $-\alpha^2\Delta u_t$ in the Navier-Stokes-Voigt equations has some important consequences. First, the natural energy space (for weak solutions) of the Navier-Stokes-Voigt equations is V instead of H in the case of the Navier-Stokes equations, and we can prove the global well-posedness even in the case of three dimensions. However, the Navier-Stokes-Voigt equations do not have a parabolic character, as Navier-Stokes equations do, behaving instead as a damped hyperbolic system. Thus, the associated semigroup is only weakly dissipative, and this leads to the fact that the proof of existence of a global attractor is more involved because we have to check the asymptotic compactness of the associated semigroup directly. On the other hand, it is known that there are numerous technical difficulties in dealing with partial differential equations with infinite delays due to the unboundedness of the delay involved. These introduces a major obstacle for studying the existence and asymptotic behavior of solutions, in particular, in passing to the limit in the nonlinear term and especially in the delay term. In this paper, to overcome these difficulties, we try to exploit and combine the Galerkin method and the energy method, and particularly, the techniques dealing with the infinite delay used in [1, 13, 19–21, 25].

Let us recall functions spaces, operators, inequalities and notations which are frequently used in the paper.

Denote

$$(u, v) := \int_\Omega \sum_{j=1}^{3} u_j v_j \, dx, \;\; u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in (L^2(\Omega))^3,$$

and

$$((u,v)) := \int_\Omega \sum_{j=1}^{3} \nabla u_j \cdot \nabla v_j \, dx, \quad u=(u_1,u_2,u_3), v=(v_1,v_2,v_3) \in (H_0^1(\Omega))^3,$$

and the associated norms $|u|^2 := (u,u), \|u\|^2 := ((u,u))$.

Let

$$\mathcal{V} = \{ u \in (C_0^\infty(\Omega))^3 \ : \ \nabla \cdot u = 0 \}.$$

Denote by H the closure of $\mathcal{V}$ in $(L^2(\Omega)^3)$, and by V the closure of $\mathcal{V}$ in $(H_0^1(\Omega)^3)$. It follows that $V \subset H \equiv H' \subset V'$, where the injections are dense and continuous. We will use $\|\cdot\|_*$ for the norm in V', and $\langle \cdot, \cdot \rangle$ for the duality pairing between V and V'. Denote by P the Helmholtz-Leray orthogonal projection in $(H_0^1(\Omega))^3$ onto the space V.

We now define the trilinear form b by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j \, dx,$$

whenever the integrals make sense. It is easy to check that if $u,v,w \in V$, then

$$b(u,v,w) = -b(u,w,v).$$

Hence

$$b(u,v,v) = 0, \ \forall\, u,v \in V.$$

Set $A : V \to V'$ by $\langle Au, v \rangle = ((u,v))$, $B : V \times V \to V'$ by $\langle B(u,v), w \rangle = b(u,v,w)$. Then $Au = -P\Delta u$ for all $u \in D(A)$.

Using Hölder's inequality and Ladyzhenskaya's inequality in three dimensions:

$$\|u\|_{L^4} \le c|u|^{1/4}\|u\|^{3/4}, \quad \forall\, u \in H_0^1(\Omega),$$

one can prove the following lemma.

Lemma 1.1 ([27]). *We have for all $u,v,w \in V$,*

$$|b(u,v,w)| \le c \begin{cases} |u|^{1/4}\|u\|^{3/4}\|v\||w|^{1/4}\|w\|^{3/4} \\ \lambda_1^{-1/4}\|u\|\|v\|\|w\|, \end{cases}$$

where $\lambda_1 > 0$ is the first eigenvalue of the Stokes operator A defined above. In particular,

$$b(u,v,u)| \le c|u|^{1/2}\|u\|^{3/2}\|v\| \quad \textit{for all } u,v \in V. \tag{1.2}$$

As we know, the first step in studying differential equations with infinite delays is to choose a suitable phase space. There are several phase spaces which allow us to deal with infinite delays (see e.g. [15]). In this paper, inspired by recent works [1, 13, 19–21, 25], we will use the following phase space

$$C_\gamma(V) = \{\varphi \in C((-\infty,0];V) : \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in V\} \text{ for some } \gamma > 0.$$

This is a Banach space with the norm

$$\|\varphi\|_\gamma := \sup_{s\in(-\infty,0]} e^{\gamma s}\|\varphi(s)\|.$$

In order to study problem (1.1), we make the following assumptions:

(H1) $f \in V'$;

(H2) $F : C_\gamma(V) \to (L^2(\Omega))^3$ satisfies

(i) $F(0) = 0$,

(ii) there exists a constant $L_F > 0$ such that for all $\xi, \eta \in C_\gamma(V)$,

$$|F(\xi) - F(\eta)| \le L_F\|\xi - \eta\|_\gamma.$$

It is noticed that the assumption $F(0) = 0$ is not restrictive because it is made for the convenience of the presentation only. Indeed, if $F(0) \neq 0$, then we put $\hat{F}(u_t) = F(u_t) - F(0)$ and $\hat{f} = f + F(0)$, then $\hat{F}$ and $\hat{f}$ will satisfy the assumptions **(H1)-(H2)** above. In particular, from **(H2)** we will have $|F(u_t)| \le L_F\|u_t\|_\gamma$ for all $u_t \in C_\gamma(V)$.

We now give an example of the delay term $F(u_t)$ (cf. [21]). Let $F : C_\gamma(V) \to (L^2(\Omega))^3$ be defined as follows

$$F(\xi) = \int_{-\infty}^0 G(s, \xi(s))ds,\ \forall \xi \in C_\gamma(V),$$

where the function $G : (-\infty, 0) \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies the following assumptions:

(1) $G(s, 0) = 0$ for all $s \in (-\infty, 0)$;

(2) There exists a function $\kappa : (-\infty, 0) \to (0,\infty)$ such that

$$\|G(s, u) - G(s, v)\|_{\mathbb{R}^3} \le \kappa(s)\|u - v\|_{\mathbb{R}^3},\ \ \forall u, v \in \mathbb{R}^3, \forall s \in (-\infty, 0),$$

and the function κ satisfies that $\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot} \in L^2(-\infty, 0)$ for some $\varepsilon > 0$.

Then the function F satisfies **(H2)**. Indeed, **(H2-i)** is obviously satisfied, for **(H2-ii)** we have

$$\begin{aligned}
&|F(\xi) - F(\eta)|^2\\
&= \int_\Omega \Big(\int_{-\infty}^0 \kappa(s)\|\xi(s)(x) - \eta(s)(x)\|_{\mathbb{R}^3} ds\Big)^2 dx\\
&\le \int_\Omega \Big(\int_{-\infty}^0 \kappa^2(s)e^{-2(\gamma+\varepsilon)s}ds\Big)\Big(\int_{-\infty}^0 e^{2(\gamma+\varepsilon)s}\|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} ds\Big)dx\\
&= \|\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot}\|^2_{L^2(-\infty,0)} \int_{-\infty}^0\int_\Omega e^{2(\gamma+\varepsilon)s}\|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} dxds\\
&\le \|\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot}\|^2_{L^2(-\infty,0)}\Big[\sup_{s\in(-\infty,0]} e^{2\gamma s}\int_\Omega \|\xi(s)(x) - \eta(s)(x)\|^2_{\mathbb{R}^3} dx\Big]\int_{-\infty}^0 e^{2\varepsilon s}ds\\
&= \|\kappa(\cdot)e^{-(\gamma+\varepsilon)\cdot}\|^2_{L^2(-\infty,0)}\|\xi - \eta\|^2_\gamma \frac{1}{2\varepsilon}
\end{aligned}$$

$= L_F^2 \|\xi - \eta\|_\gamma^2.$

The plan of the paper is as follows. In Section 2, we prove the existence and uniqueness of weak solutions to problem (1.1) by using the Galerkin method, the energy method and the Gronwall inequality technique. In Section 3, we prove the existence of a global attractor for the continuous semigroup generated by weak solutions to the problem. The existence, uniqueness and global stability of stationary solutions are studied in the last section.

2. Existence and uniqueness of weak solutions

Definition 2.1. A weak solution on the interval $(0,T)$ of problem (1.1) is a function $u \in C((-\infty,T];V)$ such that $\frac{du}{dt} \in L^2(0,T;V)$, $u_0 = \phi$, and for all $v \in V$ we have

$$\begin{aligned}&\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + \alpha^2((\partial_t u(t),v)) + b(u(t),u(t),v)\\ &= \langle f,v\rangle + (F(u_t),v)\end{aligned} \tag{2.1}$$

in the sense of distributions in $\mathcal{D}'(0,T)$.

It is noticed that if u is a weak solution of (1.1) on $(0,T)$, then u satisfies the following energy equality

$$\begin{aligned}&|u(t)|^2 + \alpha^2\|u(t)\|^2 + 2\nu\int_s^t \|u(r)\|^2 dr\\ =\ &|u(s)|^2 + \alpha^2\|u(s)\|^2 + 2\int_s^t \Big[\langle f,u(r)\rangle + (F(u_r),u(r))\Big]dr \ \text{ for all } s,t \in [0,T].\end{aligned}$$

Theorem 2.1. *Let $\phi \in C_\gamma(V)$ and $T > 0$ be given. Then, there exists a unique weak solution u of problem (1.1) on the interval $(0,T)$.*

Proof. **(i) Uniqueness.** Let u,v be two weak solutions of problem (1.1) with the same initial condition. Setting $w = u - v$, we have

$$\frac{1}{2}\frac{d}{dt}(|w|^2 + \alpha^2\|w\|^2) + \nu\|w\|^2 + b(u,u,w) - b(v,v,w) = (F(u_t) - F(v_t),w).$$

Integrating from 0 to t and noticing that $b(u,u,w) - b(v,v,w) = b(w,v,w)$, we obtain

$$\begin{aligned}&|w(t)|^2 + \alpha^2\|w(t)\|^2 + 2\nu\int_0^t \|w(s)\|^2 ds\\ =\ &-2\int_0^t b(w(s),v(s),w(s))ds + 2\int_0^t \big(F(u_s) - F(v_s),w(s)\big)ds.\end{aligned}$$

Since $w(s) = 0, \forall s \le 0$, we deduce that

$$\|w_s\|_\gamma = \sup_{\theta\le 0} e^{\gamma\theta}\|w(s+\theta)\| \le \sup_{\theta\in[-s;0]} e^{\gamma\theta}\|w(s+\theta)\| \quad \text{for } 0 \le s \le T.$$

Because of **(H2-ii)**, we have

$$\Big|2\int_0^t \big(F(u_s)-F(v_s),w(s)\big)ds\Big| \le 2L_F\int_0^t \|w_s\|_\gamma|w(s)|ds.$$

On the other hand,

$$\begin{aligned}\Big|2\int_0^t b(w(s),v(s),w(s))ds\Big| &\le 2c\int_0^t \|v(s)\|\,\|w(s)\|^2\,ds\\ &\le 2\nu\int_0^t \|w(s)\|^2ds+\frac{c^2}{2\nu}\int_0^t \|w(s)\|^2\|v(s)\|^2ds.\end{aligned}$$

Therefore,

$$|w(t)|^2+\alpha^2\|w(t)\|^2 \le \frac{c^2}{2\nu}\int_0^t \|w(s)\|^2\|v(s)\|^2ds+2L_F\int_0^t \|w_s\|_\gamma|w(s)|ds.$$

Since

$$\|w_s\|_\gamma|w(s)| \le C_1\sup_{r\in[0,s]}\|w(r)\|^2,$$

we arrive at

$$\begin{aligned}&\sup_{r\in[0,t]}\big(|w(r)|^2+\alpha^2\|w(r)\|^2\big)\\ &\le \int_0^t \big(\frac{c^2}{2\alpha^2\nu}\|v(s)\|^2+\frac{2C_1L_F}{\alpha^2}\big)\sup_{r\in[0,s]}(|w(r)|^2+\alpha^2\|w(r)\|^2)ds.\end{aligned}$$

Applying the Gronwall inequality completes the proof of uniqueness.

(ii) Existence. We split the proof of the existence into several steps.

Step 1: A Galerkin scheme. Let $\{v_1,v_2,\dots\}$ be the basis consisting of eigenfunctions of the Stokes operator A, which is orthonormal in H and orthogonal in V. Denote $V_m=\mathrm{span}\{v_1,\dots,v_m\}$ and consider the projector $P_mu=\sum_{j=1}^m(u,v_j)v_j$. Define also

$$u^m(t)=\sum_{j=1}^m\gamma_{m,j}(t)v_j,$$

where the coefficients $\gamma_{m,j}$ are required to satisfy the following system

$$\begin{aligned}\frac{d}{dt}(u^m(t),v_j)+\nu((u^m(t),v_j))+\alpha^2((\partial_tu^m(t),v_j))+b(u^m(t),u^m(t),v_j)\\ =\langle f,v_j\rangle+(F(u_t^m),v_j),\ \forall j=1,\dots,m,\end{aligned}\tag{2.2}$$

and the initial condition $u^m(s)=P_m\phi(s)$ for $s\in(-\infty,0]$.

The above system of ordinary differential equations with infinite delay in the unknowns $(\gamma_{m,1}(t),\dots,\gamma_{m,m}(t))$ fulfills the conditions for existence and uniqueness of local solutions in [15, Theorem 1.1, p. 36], so the approximate solutions u^m exist.

Step 2: A priori estimates. Multiplying (2.2) by $\gamma_{m,j}(t)$ and summing in j, we obtain

$$\frac{d}{dt}|u^m(t)|^2 + \alpha^2 \frac{d}{dt}\|u^m(t)\|^2 + 2\nu\|u^m(t)\|^2 = 2\langle f, u^m(t)\rangle + 2(F(u_t^m), u^m(t)).$$

Using the Cauchy inequality, we get

$$\begin{aligned}&\frac{d}{dt}\Big(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2\Big) + 2\nu\|u^m(t)\|^2\\ \le\ & \frac{\nu}{2}\|u^m(t)\|^2 + \frac{2\|f\|_*^2}{\nu} + \frac{2L_F}{\lambda_1^{1/2}}\|u_t^m\|_\gamma\|u^m(t)\|,\end{aligned}$$

and hence

$$\frac{d}{dt}\Big(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2\Big) + \nu\|u^m(t)\|^2 \le \frac{2\|f\|_*^2}{\nu} + \frac{2L_F^2}{\nu\lambda_1}\|u_t^m\|_\gamma^2.$$

Integrating from 0 to t, we obtain

$$\begin{aligned}&|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2 + \nu\int_0^t \|u^m(s)\|^2 ds\\ \le\ & |u(0)|^2 + \alpha^2\|u(0)\|^2 + \int_0^t \Big(\frac{2\|f\|_*^2}{\nu} + \frac{2L_F^2}{\nu\lambda_1}\|u_s^m\|_\gamma^2\Big)ds,\end{aligned}$$

and therefore

$$\begin{aligned}&\alpha^2\|u^m(t)\|^2 + \nu\int_0^t \|u^m(s)\|^2 ds\\ \le\ & \big(\frac{1}{\lambda_1} + \alpha^2\big)\|u(0)\|^2 + \int_0^t \Big(\frac{2\|f\|_*^2}{\nu} + \frac{2L_F^2}{\nu\lambda_1}\|u_s^m\|_\gamma^2\Big)ds.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|u_t^m\|_\gamma^2 \le \max\Big\{ & \sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(\theta+t)\|^2;\\ & \frac{1}{\alpha^2}\sup_{\theta\in[-t,0]}\big[e^{2\gamma\theta}\big(\frac{1}{\lambda_1}+\alpha^2\big)\|u(0)\|^2 + e^{2\gamma\theta}\int_0^{t+\theta}\big(\frac{2\|f\|_*^2}{\nu} + \frac{2L_F^2}{\nu\lambda_1}\|u_s^m\|_\gamma^2\big)\big]\Big\}\\ \le \max\Big\{ & \sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(\theta+t)\|^2; \big(\frac{1}{\lambda_1}+\alpha^2\big)\|u(0)\|^2\\ & + \int_0^t \big(\frac{2\|f\|_*^2}{\nu} + \frac{2L_F^2}{\nu\lambda_1}\|u_s^m\|_\gamma^2\big)ds\Big\}, \quad \forall t\ge 0.\end{aligned}$$

Since

$$\sup_{\theta\in(-\infty,-t]} e^{\gamma\theta}\|\phi(\theta+t)\| = \sup_{\theta\le 0} e^{\gamma(\theta-t)}\|\phi(\theta)\| = e^{-\gamma t}\|\phi\|_\gamma \le \|\phi\|_\gamma,$$

and $\|u(0)\| = \|\phi(0)\| \le \|\phi\|_\gamma$, we deduce that

$$\|u_t^m\|_\gamma^2 \le \big(\frac{1}{\lambda_1} + \alpha^2 + 1\big)\|\phi\|_\gamma^2 + \frac{2\|f\|_*^2}{\nu}T + \frac{2L_F^2}{\nu\lambda_1}\int_0^t \|u_s^m\|_\gamma^2 ds.$$

By the Gronwall inequality we have

$$\|u_t^m\|_\gamma^2 \le e^{\frac{2L_F^2}{\nu\lambda_1}t}\Big[\big(\frac{1}{\lambda_1}+\alpha^2+1\big)\|\phi\|_\gamma^2+\frac{2\|f\|_*^2}{\nu}T\Big].$$

Then we obtain the following estimates: for any $R>0$ such that $\|\phi\|_\gamma \le R$, there exists a constant C_1 depending on $\lambda_1,\nu,\alpha,L_F,f,R,T$, such that

$$\|u_t^m\|_\gamma^2 \le C_1,\ \forall t\in[0,T],\ \ \forall m\ge 1. \tag{2.3}$$

In particular,

$$\{u^m\} \text{ is bounded in } L^\infty(0,T;V). \tag{2.4}$$

Hence, it is easy to check that $\{Au^m\}$ and $\{Bu^m\}$ are bounded in $L^2(0,T;V')$. Now, we prove the boundedness of $\{du^m/dt\}$. Multiplying (2.2) by $\gamma'_{m,j}(s)$, then adding the resulting equations for j and integrating from 0 to t, we obtain

$$\int_0^t \big|\frac{\partial u^m}{\partial s}\big|^2 ds+\nu\int_0^t\int_\Omega \nabla u^m\frac{\partial \nabla u^m}{\partial s}dxds+\alpha^2\int_0^t\int_\Omega|\nabla(\partial_s u^m)|^2dxds$$
$$+\int_0^t b(u^m,u^m,\partial_s u^m)ds=\int_0^t\langle f,\partial_s u^m\rangle ds+\int_0^t(F(u_s^m),\partial_s u^m)ds.$$

Using the Cauchy and Ladyzhenskaya inequalities, we have

$$\begin{aligned}
&\int_0^t \big|\frac{\partial u^m}{\partial s}\big|^2 ds+\frac{\nu}{2}\int_0^t\frac{\partial}{\partial s}|\nabla u^m|^2ds+\alpha^2\int_0^t|\nabla(\partial_s u^m)|^2ds\\
&=\int_0^t b(u^m,\partial_s u^m,u^m)ds+\int_0^t\langle f,\partial_s u^m\rangle ds+\int_0^t(F(u_s^m),\partial_s u^m(s))ds\\
&\le \frac{1}{\alpha^2}\|f\|_*^2t+\frac{\alpha^2}{4}\|\partial_s u^m\|^2_{L^2(0,t;V)}+c\int_0^t|u^m|^{1/2}|\nabla u^m|^{3/2}|\nabla(\partial_s u^m)|ds\\
&\quad+\Big(\int_0^t\frac{1}{\lambda_1}L_F^2\|u_s^m\|_\gamma^2ds\Big)^{1/2}\cdot\|\partial_s u^m\|_{L^2(0,t;V)}\\
&\le \frac{1}{\alpha^2}\|f\|_*^2t+\frac{\alpha^2}{4}\|\partial_s u^m\|^2_{L^2(0,t;V)}+c\int_0^t|\nabla u^m|^2|\nabla(\partial_s u^m)|ds\\
&\quad+\Big(\int_0^t\frac{1}{\lambda_1}L_F^2\|u_s^m\|_\gamma^2ds\Big)^{1/2}\cdot\|\partial_s u^m\|_{L^2(0,t;V)}\\
&\le \frac{1}{\alpha^2}\|f\|_*^2t+\frac{\alpha^2}{4}\|\partial_s u^m\|^2_{L^2(0,t;V)}+c\|u^m\|^4_{L^4(0,t;V)}\\
&\quad+\frac{\alpha^2}{4}\|\partial_s u^m\|^2_{L^2(0,t;V)}+\frac{1}{\lambda_1\alpha^2}\int_0^tL_F^2\|u_s^m\|_\gamma^2ds+\frac{\alpha^2}{4}\|\partial_s u^m\|^2_{L^2(0,t;V)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&4\int_0^t\big|\frac{\partial u^m}{\partial s}\big|^2ds+\alpha^2\int_0^t|\nabla(\partial_s u^m)|^2ds+2\nu|\nabla u^m(t)|^2\\
&\le\frac{4}{\alpha^2}\|f\|_*^2T+c\|u^m\|^4_{L^4(0,t;V)}+\frac{4}{\lambda_1\alpha^2}\int_0^tL_F^2\|u_s^m\|_\gamma^2ds+2\nu|\nabla u(0)|^2
\end{aligned}$$

for all $0 \le t \le T$. Therefore, from (2.3) we deduce that

$$\left\{\frac{du^m}{dt}\right\} \text{ is bounded in } L^2(0,T;V). \tag{2.5}$$

So, it follows from above estimates that there exist $u \in L^\infty(0,T;V)$ with $\frac{du}{dt} \in L^2(0,T;V)$ and a subsequence of $\{u^m\}$, relabeled the same, such that

- $\{u^m\}$ converges weakly-star to u in $L^\infty(0,T;V)$,
- $\{\frac{du^m}{dt}\}$ converges weakly to $\frac{du}{dt}$ in $L^2(0,T;V)$.

Repeating the arguments as in the proof of Theorem 3.2 in [2], we get

$$\int_0^T \int_\Omega (u^m \cdot \nabla) u^m v_j \psi dx dt \to \int_0^T \int_\Omega (u \cdot \nabla) u v_j \psi dx dt \tag{2.6}$$

for any function v_j in the basis and any continuously differentiable function ψ on $[0,T]$.

However, the estimates obtained above are not enough to pass to the limit in the delay term $F(u_t^m)$ because in general the nonlinear term is not continuous with respect to the weak convergence. To overcome this point, we need some kind of strong convergence.

Step 3: Convergence in $C_\gamma(V)$ and existence of a weak solution.

We will prove that

$$u_t^m \to u_t \text{ in } C_\gamma(V),\ \forall t \in (-\infty, T],$$

by showing that

$$P_m\phi \to \phi \text{ in } C_\gamma(V), \tag{2.7}$$

$$u^m \to u \text{ in } C([0,T];V). \tag{2.8}$$

Step 3.1. Approximation in $C_\gamma(V)$ of the initial datum.
Assume contrary that (2.7) is not true. Then there exist $\epsilon > 0$ and a subsequence, relabeled the same, such that

$$e^{\gamma\theta_m} \|P_m\phi(\theta_m) - \phi(\theta_m)\| > \epsilon. \tag{2.9}$$

One can assume that $\theta_m \to -\infty$, otherwise if $\theta_m \to \theta$, then $P_m\phi(\theta_m) \to \phi(\theta)$, since $\|P_m\phi(\theta_m) - \phi(\theta\| \le \|P_m\phi(\theta_m) - P_m\phi(\theta)\| + \|P_m\phi(\theta) - \phi(\theta)\| \to 0$ as $m \to +\infty$. But with $\theta_m \to -\infty$ as $m \to +\infty$, if we denote $x = \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$, we obtain

$$\begin{aligned} e^{\gamma\theta_m}\|P_m\phi(\theta_m) - \phi(\theta_m)\| &= \|P_m(e^{\gamma\theta_m}\phi(\theta_m)) - e^{\gamma\theta_m}\phi(\theta_m)\| \\ &\le \|P_m(e^{\gamma\theta_m}\phi(\theta_m)) - P_m x\| + \|P_m x - x\| + \|x - e^{\gamma\theta_m}\phi(\theta_m)\| \to 0. \end{aligned}$$

This contradicts (2.9), so (2.7) holds.

Step 3.2. Convergence of u^m to u in $C([0,T];V)$.
From the strong convergence of $\{u^m\}$ to u in $L^2(0,T;V)$, we deduce that

$$u^m(t) \to u(t) \text{ in } V \text{ for a.e. } t \in (0,T).$$

Since

$$u^m(t) - u^m(s) = \int_s^t {(u^m)}'(r)dr \text{ in } V',\ \forall s,t \in [0,T],$$

from (2.5) we have that $\{u^m\}$ is equicontinuous on $[0,T]$ with values in V'. By the compactness of the embedding $V \hookrightarrow V'$, from (2.4) and the equicontinuity in V', using the Arzela-Ascoli theorem we have

$$u^m \to u \text{ in } C([0,T];V'). \tag{2.10}$$

Again from (2.4) we obtain that for any sequence $\{t_m\} \subset [0,T]$ with $t_m \to t$,

$$u^m(t_m) \rightharpoonup u(t) \text{ in } V, \tag{2.11}$$

where we have used (2.10) in order to identify which is the weak limit.

Now, we are ready to prove (2.8) by a contrary argument. If it would not be so, then taking into account that $u \in C([0,T];V)$, there would exist $\epsilon > 0$, a value $t_0 \in [0,T]$ and subsequences (relabeled the same) $\{u^m\}$ and $\{t_m\} \subset [0,T]$ with $\lim\limits_{m\to+\infty} t_m = t_0$ such that

$$\|u^m(t_m) - u(t_0)\| \geq \epsilon,\ \forall m. \tag{2.12}$$

To prove that this is absurd, we will use an energy method. Observe that the following energy inequality holds for all u^m:

(2.13)

$$\begin{aligned}&\frac{1}{2}|u^m(t)|^2 + \frac{\alpha^2}{2}\|u^m(t)\|^2 + \nu\int_s^t \|u^m(r)\|^2 dr \\ &\leq \int_s^t \langle f, u^m(r)\rangle dr + \frac{1}{2}|u^m(s)|^2 + \frac{\alpha^2}{2}\|u^m(s)\|^2 + C_3(t-s),\ \forall\ s,t \in [0,T],\end{aligned}$$

where $C_3 = \frac{D}{2\nu\lambda_1}$ and D corresponds to the upper bound

$$\int_s^t |F(u_r^m)|^2 dr \leq D(t-s)\ \forall 0 \leq s < t \leq T.$$

On the other hand, from (2.4) and **(H2)**, there exists $\xi_F \in L^2(0,T;L^2(\Omega)^3)$ such that $\{F(u^m)\}$ converges weakly to ξ_F in $L^2(0,T;L^2(\Omega)^3)$. Thus, we can pass to the limit to deduce that u satisfies

$$\frac{d}{dt}(u(t),v) + \nu((u(t),v)) + \alpha^2((\partial_t u(t),v)) + b(u(t),u(t),v) = \langle f,v\rangle + (\xi_F(t),v)$$

for all $v \in V$. Therefore, u satisfies the energy equality

$$\begin{aligned}&|u(t)|^2 + \alpha^2\|u(t)\|^2 + 2\nu\int_s^t \|u(r)\|^2 dr \\ &= |u(s)|^2 + \alpha^2\|u(s)\|^2 + 2\int_s^t \big(\langle f,u(r)\rangle + (\xi_F(r),u(r))\big)dr,\ \forall s,t \in [0,T],\end{aligned}$$

and for the weak limit ξ_F we have the estimate

$$\int_s^t |\xi_F|^2 dr \leq \liminf_{m\to+\infty}\int_s^t |F(u_r^m)|^2 dr \leq D(t-s),\ \forall 0 \leq s \leq t \leq T.$$

So, we have that u also satisfies inequality (2.13) with the same constant C_3. Now, consider two functions $J_m, J : [0,T] \to \mathbb{R}$ defined by

$$J_m(t) = \frac{1}{2}(|u^m(t)|^2 + \alpha^2\|u^m(t)\|^2) - \int_0^t \langle f, u^m(r)\rangle dr - C_3 t,$$

$$J(t) = \frac{1}{2}(|u(t)|^2 + \alpha^2\|u(t)\|^2) - \int_0^t \langle f, u(r)\rangle dr - C_3 t.$$

It is clear that J_m and J are non-increasing and continuous functions. Moreover, by the convergence of u^m to u a.e. in time with value in V, and weakly in $L^2(0,T;V)$, it holds that

$$J_m(t) \to J(t) \text{ a.e. } t \in [0,T]. \tag{2.14}$$

Now we will prove that

$$u^m(t_m) \to u(t_0) \text{ in } V, \tag{2.15}$$

which contradicts (2.12). First, recall from (2.11) that

$$u^m(t_m) \rightharpoonup u(t_0) \text{ weakly in } V, \tag{2.16}$$

so we have

$$\|u(t_0)\| \leq \liminf_{m\to+\infty} \|u^m(t_m)\|.$$

Therefore, if we show that

$$\limsup_{m\to+\infty} \|u^m(t_m)\| \leq \|u(t_0)\|, \tag{2.17}$$

we will obtain $\lim_{m\to+\infty} \|u^m(t_m)\| = \|u(t_0)\|$, which jointly with (2.16) imply (2.15).

Now, observe that the case $t_0 = 0$ follows directly from (2.13) with $s = 0$ and the definition of $u^m(0) = P_m\phi(0)$. So, we may assume that $t_0 > 0$. This is important, since we will approach this value t_0 from the left by a sequence $\{t'_k\}$, i.e., $\lim_{k\to+\infty} t'_k \nearrow t_0$. Since $u(\cdot)$ is continuous at t_0, there is k_ϵ such that

$$|J(t'_k) - J(t_0)| < \frac{\epsilon}{2}, \ \forall\, k \geq k_\epsilon.$$

On the other hand, taking $m \geq m(k_\epsilon)$ such that $t_m > t'_{k_\epsilon}$, as J_m is non-increasing and for all t'_k the convergence (2.15) holds, one has

$$J_m(t_m) - J(t_0) \leq |J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| + |J(t'_{k_\epsilon}) - J(t_0)|,$$

and obviously, taking $m \leq m'(k_\epsilon)$, it is possible to obtain $|J_m(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| < \frac{\epsilon}{2}$. It can also be deduced from Step 2 that

$$\int_0^{t_m} \langle f, u^m(r)\rangle dr \to \int_0^{t_0} \langle f, u(r)\rangle dr,$$

so we conclude that (2.17) holds. Thus, (2.15) and finally (2.8) are also true, as we wanted to check. Hence, we have

$$F(u_t^m) \to F(u_t) \text{ in } L^2(0,T;(L^2(\Omega))^3). \tag{2.18}$$

Finally, we will show that the convergence results above enable us to conclude that u is a weak solution of problem (1.1). Let $\psi \in \mathcal{D}(0,T)$ be any smooth real-valued function with compact support. Multiplying (2.2) by $\psi(t)$, we have

$$\begin{aligned} &\int_0^T (\frac{du^m(t)}{dt}, \psi(t)v_j)dt + \nu \int_0^T ((u^m(t), v_j\psi(t)))dt \\ &\quad + \alpha^2 \int_0^T ((\partial_t u^m(t), v_j\psi(t)))dt + \int_0^T b(u^m(t), u^m(t), v_j\psi(t))dt \\ = &\int_0^T \langle f, v_j\psi(t)\rangle dt + \int_0^T (F(u_t^m), v_j\psi(t))dt. \end{aligned}$$

Taking a diagonal subsequence, denoted again by u^m, that satisfies (2.6) and (2.18), and passing to the limit, we have

$$\begin{aligned} &\int_0^T (\frac{du(t)}{dt}, v_j\psi(t))dt + \nu \int_0^T ((u(t), v_j\psi(t)))dt \\ &\quad + \alpha^2 \int_0^T ((\partial_t u(t), v_j\psi(t)))dt + \int_0^T b(u(t), u(t), v_j\psi(t))dt \\ = &\int_0^T \langle f, v_j\psi(t)\rangle dt + \int_0^T (F(u_t), v_j\psi(t))dt \end{aligned}$$

holds for all v_j in the basis (and therefore for every $v \in V$ by density) and any function $\psi \in \mathcal{D}(0,T)$, i.e., u satisfies (2.1) in the distribution sense. □

3. Existence of a global attractor

Thanks to Theorem 2.1, we can define a semigroup $S(t) : C_\gamma(V) \to C_\gamma(V)$ by the formula

$$S(t)\phi := u_t,$$

where $u(t)$ is the unique weak solution of (1.1) with the initial datum $\phi \in C_\gamma(V)$.

The aim of this section is to prove the existence of a compact global attractor in the space $C_\gamma(V)$ for the semigroup $S(t)$. First, we prove the continuity of the semigroup $S(t)$.

Proposition 3.1. *Under the conditions* **(H1)**-**(H2)**, *the semigroup $S(t)$ is continuous on $C_\gamma(V)$.*

Proof. Denoting u^i, for $i = 1, 2$, the corresponding solutions to initial data $\phi^i \in C_\gamma(V)$. Consider the equations satisfied by u^i for $i = 1$ and 2, acting on the element $u^1 - u^2$, and take the difference. This gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(|u^1(t) - u^2(t)|^2 + \alpha^2\|u^1(t) - u^2(t)\|^2\right) + \nu\|u^1(t) - u^2(t)\|^2 \\ &+ b(u^1(t), u^1(t), u^1(t) - u^2(t)) - b(u^2(t), u^2(t), u^1(t) - u^2(t)) \end{aligned}$$

$$= (F(u_t^1) - F(u_t^2), u^1 - u^2).$$

Arguing as in the proof of Theorem 2.1 and using the Ladyzhenskaya inequality, we have

$$\begin{aligned} &|b(u^1(t), u^1(t), u^1(t) - u^2(t)) - b(u^2(t), u^2(t), u^1(t) - u^2(t))| \\ &= |b(u^1(t) - u^2(t), u^1(t), u^1(t) - u^2(t))| \\ &\le c\|u^1(t)\|\|u^1(t) - u^2(t)\|^2. \end{aligned}$$

Thus, by the Lipschitz assumption on F, and the fact that, for $s \in [0, t]$, one has

(3.1)

$$\begin{aligned} &\|u_s^1 - u_s^2\|_\gamma \\ &= \sup_{\theta \le 0} e^{\gamma\theta} \|u^1(s+\theta) - u^2(s+\theta)\| \\ &= \max\Big\{ \sup_{\theta \in (-\infty, -s]} e^{\gamma\theta}\|\phi^1(s+\theta) - \phi^2(s+\theta)\|; \sup_{\theta \in [-s,0]} e^{\gamma\theta}\|u^1(s+\theta) - u^2(s+\theta)\| \Big\} \\ &\le \max\Big\{ e^{-\gamma s}\|\phi^1 - \phi^2\|_\gamma; \max_{\theta\in[0,s]} \|u^1(\theta) - u^2(\theta)\| \Big\}, \end{aligned}$$

we conclude that, for all $t \in [0, T]$,

$$\begin{aligned} &|u^1(t) - u^2(t)|^2 + \alpha^2\|u^1(t) - u^2(t)\|^2 + 2\nu \int_0^t \|u^1(s) - u^2(s)\|^2 ds \\ &= 2\int_0^t b(u^1(s) - u^2(s), u^1(s), u^1(s) - u^2(s)) ds \\ &\quad + |u^1(0) - u^2(0)|^2 + \alpha^2\|u^1(0) - u^2(0)\|^2 \\ &\quad + 2\int_0^t (F(u_s^1) - F(u_s^2), u^1(s) - u^2(s)) ds. \end{aligned}$$

Hence

$$\begin{aligned} &|u^1(t) - u^2(t)|^2 + \alpha^2\|u^1(t) - u^2(t)\|^2 \\ &\le |\phi^1(0) - \phi^2(0)|^2 + \alpha^2\|\phi^1(0) - \phi^2(0)\|^2 + \frac{C}{\nu}\int_0^t \|u^1(s)\|^2\|u^1(s) - u^2(s)\|^2 ds \\ &\quad + 2L_F\|\phi^1 - \phi^2\|_\gamma \int_0^t e^{-\gamma s}|u^1(s) - u^2(s)| ds \\ &\quad + 2L_F \int_0^t |u^1(s) - u^2(s)| \max_{\theta\in[0,s]} \|u^1(\theta) - u^2(\theta)\| ds \\ &\le \big(\frac{1}{\lambda_1} + \alpha^2\big)\|\phi^1(0) - \phi^2(0)\|^2 + L_F\|\phi^1 - \phi^2\|_\gamma^2 \\ &\quad \frac{C}{\nu}\int_0^t \|u^1(s)\|^2 \max_{r\in[0,s]} \|u^1(r) - u^2(r)\|^2 ds + \frac{L_F}{\lambda_1}\int_0^t \max_{r\in[0,s]} \|u^1(r) - u^2(r)\|^2 ds \end{aligned}$$

$$+\frac{2L_F}{\lambda_1^{1/2}}\int_0^t \max_{r\in[0,s]}\|u^1(r)-u^2(r)\|^2ds.$$

If we now substitute t by $r\in[0,t]$ and consider the maximum when varying this r, from above we can conclude that

$$\begin{aligned}
&\max_{r\in[0,t]}\|u^1(r)-u^2(r)\|^2\\
\le&\big(1+\frac{1}{\lambda_1\alpha^2}\big)\|\phi^1(0)-\phi^2(0)\|^2+\frac{L_F}{\alpha^2}\|\phi^1-\phi^2\|_\gamma^2\\
&+\int_0^t\big(\frac{L_F}{\lambda_1\alpha^2}+\frac{2L_F}{\lambda_1^{1/2}\alpha^2}+\frac{C}{\nu\alpha^2}\|u^1(s)\|^2\big)\max_{r\in[0,s]}\|u^1(r)-u^2(r)\|^2ds.
\end{aligned}$$

By the Gronwall inequality, we obtain

$$\begin{aligned}
&\max_{r\in[0,t]}\|u^1(r)-u^2(r)\|^2\\
\le&\Big(\big(1+\frac{1}{\lambda_1\alpha^2}\big)\|\phi^1(0)-\phi^2(0)\|^2+\frac{L_F}{\alpha^2}\|\phi^1-\phi^2\|_\gamma^2\Big)e^{\int_0^t\big(\frac{L_F}{\lambda_1\alpha^2}+\frac{2L_F}{\lambda_1^{1/2}\alpha^2}+\frac{C}{\nu\alpha^2}\|u^1(s)\|^2\big)ds}.
\end{aligned}$$

Combining with (3.1), we get

$$\|u_t^1-u_t^2\|_\gamma^2\le\Big(1+\frac{1}{\lambda_1\alpha^2}+\frac{L_F}{\alpha^2}\Big)\|\phi^1-\phi^2\|_\gamma^2e^{\int_0^t\big(\frac{L_F}{\lambda_1\alpha^2}+\frac{2L_F}{\lambda_1^{1/2}\alpha^2}+\frac{C}{\nu\alpha^2}\|u^1(s)\|^2\big)ds}.$$

This completes the proof. □

Next, we prove the existence of a bounded absorbing set for the semigroup $S(t)$.

Lemma 3.1. *Let conditions* **(H1)**-**(H2)** *hold, and let*

$$\frac{2L_F^2}{\nu\lambda_1\alpha^2}<\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}<2\gamma.$$

Then the semigroup $S(t)$ has a bounded absorbing set $\mathcal{B}$ in $C_\gamma(V)$.

Proof. We have

$$\frac{d}{dt}\left(|u(t)|^2+\alpha^2\|u(t)\|^2\right)+2\nu\|u(t)\|^2=2\langle f,u(t)\rangle+2(F(u_t),u(t)).$$

Using the Cauchy inequality and the Poincaré inequality, we get

$$\frac{d}{dt}\Big(|u(t)|^2+\alpha^2\|u(t)\|^2\Big)+2\nu\|u(t)\|^2\le\frac{\nu}{2}\|u(t)\|^2+\frac{2\|f\|_*^2}{\nu}+\frac{2L_F}{\lambda_1^{1/2}}\|u_t\|_\gamma\|u(t)\|,$$

and hence by the Cauchy inequality once again,

$$\frac{d}{dt}\Big(|u(t)|^2+\alpha^2\|u(t)\|^2\Big)+\nu\|u(t)\|^2\le\frac{2\|f\|_*^2}{\nu}+\frac{2L_F^2}{\nu\lambda_1}\|u_t\|_\gamma^2.$$

Noting that

$$\frac{1}{\alpha^2}\|u\|^2\le|u|^2+\alpha^2\|u\|^2\le\frac{1+\lambda_1\alpha^2}{\lambda_1}\|u\|^2,$$

we have

$$\frac{d}{dt}\Big(|u(t)|^2+\alpha^2\|u(t)\|^2\Big)+\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}\Big(|u(t)|^2+\alpha^2\|u(t)\|^2\Big)$$
$$\le\frac{2\|f\|_*^2}{\nu}+\frac{2L_F^2}{\nu\lambda_1}\|u_t\|_\gamma^2.$$

Using the Gronwall inequality we obtain

$$|u(t)|^2+\alpha^2\|u(t)\|^2\le e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}t}(|u(0)|^2+\alpha^2\|u(0)\|^2)+\int_0^t e^{-\frac{\nu\lambda_1}{1+\lambda_1}(t-s)}\Big(\frac{2\|f\|_*^2}{\nu}+\frac{2L_F^2}{\nu\lambda_1}\|u_s\|_\gamma^2\Big)ds, \tag{3.2}$$

so

$$\|u(t)\|^2\le\frac{1+\lambda_1\alpha^2}{\lambda_1\alpha^2}e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}t}\|u(0)\|^2+\int_0^t e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t-s)}\Big(\frac{2\|f\|_*^2}{\nu\alpha^2}+\frac{2L_F^2}{\nu\lambda_1\alpha^2}\|u_s\|_\gamma^2\Big)ds.$$

We have

$$\|u_t\|_\gamma^2\le\max\Big\{\sup_{\theta\in(-\infty,-t]}e^{2\gamma\theta}\|\phi(\theta+t)\|^2;\ \sup_{\theta\in[-t,0]}\Big[\frac{1+\lambda_1\alpha^2}{\lambda_1\alpha^2}e^{2\gamma\theta-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t+\theta)}\|u(0)\|^2+e^{2\gamma\theta}\int_0^{t+\theta}e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t+\theta-s)}\left(\frac{2\|f\|_*^2}{\nu\alpha^2}+\frac{2L_F^2}{\nu\lambda_1\alpha^2}\|u_s\|_\gamma^2\right)ds\Big]\Big\}.$$

By the assumption $2\gamma>\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}$, we get

$$\sup_{\theta\in[-t,0]}e^{2\gamma\theta}\int_0^{t+\theta}e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t+\theta-s)}\big(\frac{2\|f\|_*^2}{\nu\alpha^2}+\frac{2L_F^2}{\nu\lambda_1\alpha^2}\|u_s\|_\gamma^2\big)ds$$
$$\le\sup_{\theta\in[-t,0]}\int_0^{t+\theta}e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t-s)}\big(\frac{2\|f\|_*^2}{\nu\alpha^2}+\frac{2L_F^2}{\nu\lambda_1\alpha^2}\|u_s\|_\gamma^2\big)ds.$$

Since

$$\sup_{\theta\in(-\infty,-t]}e^{\gamma\theta}\|\phi(\theta+t)\|=\sup_{\theta\le0}e^{\gamma(\theta-t)}\|\phi(\theta)\|=e^{-\gamma t}\|\phi\|_\gamma,$$

and $\|u(0)\|=\|\phi(0)\|\le\|\phi\|_\gamma$, we deduce that

$$\|u_t\|_\gamma^2\le\frac{1+\lambda_1\alpha^2}{\lambda_1\alpha^2}e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}t}\|\phi\|_\gamma^2+\int_0^t e^{-\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}(t-s)}\big(\frac{2\|f\|_*^2}{\nu\alpha^2}+\frac{2L_F^2}{\nu\lambda_1\alpha^2}\|u_s\|_\gamma^2\big)ds.$$

By the Gronwall inequality, we have

(3.3)
$$\begin{aligned}\|u_t\|_\gamma^2 \leq & \frac{1+\lambda_1\alpha^2}{\lambda_1\alpha^2} e^{-(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2})t}\|\phi\|_\gamma^2 + \frac{2\|f\|_*^2}{\nu\alpha^2}\int_0^t e^{-(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2})(t-s)}ds \\ \leq & \frac{1+\lambda_1\alpha^2}{\lambda_1\alpha^2} e^{-(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2})t}\|\phi\|_\gamma^2 + \frac{2\|f\|_*^2}{\nu\alpha^2(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2})}.\end{aligned}$$

By the condition $\frac{2L_F^2}{\nu\lambda_1\alpha^2} < \frac{\nu\lambda_1}{1+\lambda_1\alpha^2}$, it implies that the ball

$$\mathcal{B} = \left\{ v \in C_\gamma(V) : \|v\|_\gamma \leq \sqrt{\frac{4\|f\|_*^2}{\nu\alpha^2\big(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2}\big)}} \right\}$$

is a bounded absorbing set in $C_\gamma(V)$ for the semigroup $S(t)$.

By combining (3.2) and (3.3) we can see that

$$|u(t)|^2 + \alpha^2\|u(t)\|^2 \leq \ C = C(\nu, \lambda, \alpha_1, \|u_0\|, \|\phi\|_\gamma, \|f\|_*), \ \ \forall t \geq 0. \tag{3.4}$$ □

To show the existence of a global attractor for the semigroup $S(t)$, it remains to prove the asymptotic compactness of $S(t)$.

Lemma 3.2. *Under the assumptions of Lemma 3.1, the semigroup $S(t)$ is asymptotically compact.*

Proof. Let B be a bounded set in $C_\gamma(V)$ and $u^n(\cdot)$ be a sequence of solutions in $[0, +\infty)$ with initial data $\phi^n \in B$. Consider the sequence $\xi^n = u^n_{t_n}$, where $t_n \to +\infty$ as $n \to +\infty$. We have to prove that this sequence is relatively compact in $C_\gamma(V)$. To do this, we will use energy method as in Step 3 in the proof of Theorem 2.1.

Step 1: Consider two arbitrary values $0 < \overline{T} < T$. We will prove that $\xi^n|_{[-\overline{T},0]}$ is relatively compact in $C([-\overline{T}, 0]; V)$. It follows from (3.3) that there exists n_0 such that $t_n \geq T$ for all $n \geq n_0$ and

$$\|\xi^n\|_\gamma \leq R, \tag{3.5}$$

where

$$R = \sqrt{\frac{4\|f\|_*^2}{\nu\alpha^2\big(\frac{\nu\lambda_1}{1+\lambda_1\alpha^2}-\frac{2L_F^2}{\nu\lambda_1\alpha^2}\big)}},$$

so

$$\begin{aligned}\|u^n(t)\| \leq R, \ \ \forall t \in [0, T], \ \forall n \geq n_0, \\ \|u^n_{t_n-T}\|_\gamma \leq R, \ \ \forall n \geq n_0.\end{aligned} \tag{3.6}$$

Let $y^n(\cdot) = u^n_{t_n-T}(\cdot) = u^n(\cdot + t_n - T)$. Then for each $n \geq 1$ such that $t_n \geq T$, the function y^n is a solution on $[0, T]$ of a similar problem to (1.1), namely

$$\frac{d}{dt}y^n(t) + \nu A y^n(t) + \alpha^2 A(\partial_t y^n(t)) + B(y^n(t), y^n(t)) = F(y^n_t) + f \tag{3.7}$$

with $y_0^n = u_{t_n-T}^n$, $y_T^n = \xi^n$. Then y_0^n satisfies the estimate (3.5) for all $n \geq n_0$.

Applying the estimate (3.4), one can see that $\{y^n\}$ is bounded in $L^\infty(0,T;V)$, and therefore $\{Ay^n\}, \{By^n\}$ are bounded in $L^2(0,T;V')$. So, $\{dy^n/dt\}$ is bounded in $L^2(0,T;V)$. Thus, as in the proof of Theorem 2.1, up to a subsequence (relabeled the same), for some function $y(\cdot)$ we have

- y^n converges weakly-star to y in $L^\infty(0,T;V)$,
- $\dfrac{dy^n}{dt}$ converges weakly to $\dfrac{dy}{dt}$ in $L^2(0,T;V)$,
- y^n converges to y in $L^2(0,T;H)$,
- $y^n(t)$ converges to $y(t)$ in V for a.e. $t \in (0,T)$.

Moreover, reasoning as in the proof of Theorem 2.1, we obtain that $y^n(t_n)$ converges weakly to $y(t_0)$ in V if $t_n \to t_0 \in [0,T]$. Also, by **(H2)** and (3.5), we obtain

$$\int_0^t |F(y_s^n)|^2 ds \leq Ct, \ \ \forall 0 \leq t \leq T,$$

where $C > 0$ does not depend either on n or t. Since

$$F(y_t^n) \rightharpoonup \xi$$

in $L^2(0,T;(L^2(\Omega))^3)$, we get

$$\int_s^t |\xi(r)|^2 dr \leq \liminf_{n\to+\infty} \int_s^t |F(y_r^n)|^2 dr \leq C(t-s), \ \forall 0 \leq s \leq t \leq T.$$

Then we can prove that y is a solution of

$$\begin{cases} \partial_t y - \nu\Delta y - \alpha^2\Delta(\partial_t y) + (y\cdot\nabla)y = -\nabla p + f + \xi \text{ in } (0,T)\times\Omega, \\ \nabla\cdot y = 0 \ \text{ in } (0,T)\times\Omega, \\ y(t,x) = 0 \ \text{ on } (0,T)\times\partial\Omega, \\ y(0,x) = u(0,x) \ \text{ in } \Omega. \end{cases}$$

Thus, we obtain the energy inequality

$$\begin{aligned} &|z(t)|^2 + \alpha^2\|z(t)\|^2 + \nu\int_s^t \|z(r)\|^2 dr \\ &\leq |z(s)|^2 + \alpha^2\|z(s)\|^2 + 2\int_s^t \langle f, z(r)\rangle dr + 2C_3(t-s), \ \forall 0 \leq s \leq t \leq T, \end{aligned}$$

where $z = y^n$ or $z = y$.

Now, consider two functions $J_m, J : [0,T] \to \mathbb{R}$ defined by

$$J_m(t) = \frac{1}{2}\left(|y^m(t)|^2 + \alpha^2\|y^m(t)\|^2\right) - \int_0^t \langle f, y^m(r)\rangle dr - C_3 t,$$

$$J(t) = \frac{1}{2}\left(|y(t)|^2 + \alpha^2\|y(t)\|^2\right) - \int_0^t \langle f, y(r)\rangle dr - C_3 t.$$

It is clear that J_m and J are non-increasing and continuous functions.

Since $y^n(t)$ converges to y a.e. $t \in (0,T)$, we obtain that

$$J_m(t) \to J(t) \text{ a.e. } t \in [0,T].$$

Analogously as we did in Step 3 of the proof in Theorem 2.1, for a fixed $t_0 > 0$, using a sequence $\{\tilde{t}_k\}$ with $\tilde{t}_k \nearrow t_0$, we are able to establish the convergence of the norms

$$\lim_{n\to\infty} \|y^n(t_n)\| = \|y(t_0)\|.$$

And therefore, jointly with the weak convergence already proved, deduce that $y^n \to y$ in $C([\delta,T];V)$ for any $\delta > 0$.

Now, because $T > \overline{T}$ and $y^n \to y$ in $C([\delta,T];V)$, we obtain that $\xi^n \to \psi$ in $C([-\overline{T},0];V)$, where $\psi(s) = y(s+T)$ for $s \in [-\overline{T},0]$. Repeating the same procedure for $2\overline{T}, 3\overline{T}$, etc, for a diagonal subsequence (relabeled the same) we can obtain a continuous function $\psi : (-\infty,0] \to V$ and a subsequence such that $\xi^n \to \psi$ in $C([-\overline{T},0];V)$ on every interval $[-\overline{T},0]$.

Moreover, for a fixed $T > 0$, we also have

$$|\psi(s)| \le R, \ \forall s \in [-T,0], \ \forall T > 0.$$

Step 2: We claim that ξ_n converges to ψ in $C_\gamma(V)$. Indeed, we have to see that for every $\epsilon > 0$ there exists n_ϵ such that

$$\sup_{s\in(-\infty,0]} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \le \epsilon \quad \forall n \ge n_\epsilon. \tag{3.8}$$

Fix $T_\epsilon > 0$ such that $e^{-2\gamma T_\epsilon} R^2 \le \frac{\epsilon}{4}$.

In Step 1, we proved that $\xi^n \to \psi$ in $C([-T_\epsilon,0];V)$, so there exists $n_\epsilon = n_\epsilon(T_\epsilon)$ such that for all $n \ge n_\epsilon$, we have

$$\sup_{s\in[-T_\epsilon,0]} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \le \epsilon, \ \forall t_n \ge T_\epsilon.$$

(This is possible since the convergence of ξ^n to ψ holds in compact intervals of time.) So, in order to prove (3.8) we only have to check that

$$\sup_{s\in(-\infty,T_\epsilon)} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \le \epsilon \quad \forall n \ge n_\epsilon.$$

By (3.5) and the choice of T_ϵ, it is not difficult to check that, for all $k \in \mathbb{N}\cup\{0\}$, and for all $s \in [-(T_\epsilon+k+1), -(T_\epsilon+k)]$, it holds that

$$\begin{aligned}\sup_{s\in[-(T_\epsilon+k+1),-(T_\epsilon+k)]} e^{2\gamma s}\|\psi(s)\|^2 &\le \sup_{s\in[-1,0]} e^{2\gamma(s-T_\epsilon-k)}\|\psi(s-T_\epsilon-k)\|^2 \\ &\le e^{-2\gamma(T_\epsilon+k)}R^2 \\ &\le \frac{\epsilon}{4}.\end{aligned}$$

So, it suffices to prove the following

$$\sup_{s\in(-\infty,-T_\epsilon]} e^{2\gamma s}\|\xi^n(s)\|^2 \le \epsilon/4, \quad \forall n \ge n_\epsilon.$$

We remember that ξ^n has two parts:

$$\xi^n(s) = \begin{cases} \phi^n(s+t_n) & \text{if } s \in (-\infty, -t_n], \\ u^n(s+t_n) & \text{if } s \in [-t_n, 0]. \end{cases}$$

Thus, the proof is finished if we prove that

$$\max\{\sup_{s\in(-\infty,-t_n]} e^{2\gamma s}\|\phi^n(s+t_n)\|^2, \sup_{s\in[-t_n,-T_\epsilon]} e^{2\gamma s}\|u^n(s+t_n)\|^n\} \leq \epsilon/4.$$

The first term above can be estimated as follows

$$\begin{aligned}\sup_{s\leq -t_n} e^{2\gamma s}\|\phi^n(s+t_n)\|^2 &= \sup_{s\leq -t_n} e^{2\gamma(s+t_n)}e^{-2\gamma t_n}\|\phi^n(s+t_n)\|^2 \\ &= e^{-2\gamma T_\epsilon}\|\phi^n\|_\gamma^2 \\ &\leq \epsilon/4,\end{aligned}$$

thanks to the choice of n_ϵ. And finally, for the second term, we have

$$\begin{aligned}\sup_{s\in[-t_n,-T_\epsilon]} e^{2\gamma s}\|u^n(s+t_n)\|^2 &= \sup_{\theta\in[-t_n+T_\epsilon,0]} e^{2\gamma(s-T_\epsilon)}\|u^n(t_n-T_\epsilon+s)\|^2 \\ &\leq e^{-2\gamma T_\epsilon}||u^n_{t_n-T_\epsilon}||_\gamma^2 \\ &\leq e^{-2\gamma T_\epsilon}R^2 \\ &\leq \epsilon/4,\end{aligned}$$

where we have used (3.6) with $T = T_\epsilon$. □

From Lemma 3.1 and Lemma 3.2, we get the main result of this section.

Theorem 3.2. *Under the assumptions of Lemma 3.1, the semigroup $S(t)$ has a compact global attractor $\mathcal{A}$ in the space $C_\gamma(V)$.*

4. Existence and stability of stationary solutions

A stationary solution to problem (1.1) is an element $u^* \in V$ such that

$$\nu((u^*,v)) + b(u^*,u^*,v) = \langle f, v\rangle + (F(u^*),v), \ \forall v \in V.$$

Theorem 4.1. *Let the assumptions* **(H1)-(H2)** *hold. If*

$$\nu > \frac{L_F}{\lambda_1^{1/2}},$$

then

(a) *Problem* (1.1) *admits at least one stationary solution u^*. Moreover, any such stationary solution satisfies the following estimate*

$$\|u^*\| \leq \frac{1}{(\nu - \frac{L_F}{\lambda_1^{1/2}})}\|f\|_*. \tag{4.1}$$

(b) *If the following condition holds*

$$\Big[\nu - \frac{L_F}{\lambda_1^{1/2}}\Big]^2 > \frac{c_0}{\lambda_1^{1/4}}\|f\|_*, \tag{4.2}$$

where c_0 *is the best constant in the inequality* (1.2)*, then the stationary solution of* (1.1) *is unique.*

Proof. **(i) Existence.** The estimate (4.1) can be obtained taking into account that in particular any stationary solution u^*, if it exists, should verify

$$\nu\langle Au^*, u^*\rangle = \langle f, u^*\rangle + (F(u^*), u^*),$$

and therefore

$$\nu\|u^*\|^2 \le \|f\|_*\|u^*\| + \frac{L_F}{\lambda_1^{1/2}}\|u^*\|^2.$$

Hence we get the desired estimate.

For the existence, let $\{v_j\}_{j=1}^{\infty}$ be the basis of V consisting of eigenfunctions of the Stokes operator A. For each $m \ge 1$, let us denote $V_m = \text{span}\{v_1, \dots, v_m\}$ and we would like to define an approximate stationary solution u^m of (1.1) by

$$\begin{aligned} & u^m = \sum_{i=1}^{m} \gamma_{mi} v_i, \\ & \nu((u^m, v_i)) + +b(u^m, u^m, v_i) = \langle f, v_i\rangle + (F(u^m), v_i), \ i = 1, \dots, m. \end{aligned} \tag{4.3}$$

To prove the existence of u^m, we define operators $R_m : V_m \to V_m$ by

$$[R_m u, v] := \nu\langle Au, v\rangle + b(u, u, v) - \langle f, v\rangle - (F(u), v), \ \forall u, v \in V_m.$$

For all $u \in V_m$,

$$\begin{aligned} [R_m u, u] &= \nu\langle Au, u\rangle - \langle f, u\rangle - (F(u), u) \\ &\ge \nu\|u\|^2 - \|f\|_*\|u\| - \frac{L_F}{\lambda_1^{1/2}}\|u\|^2 \\ &= \big(\nu - \frac{L_F}{\lambda_1^{1/2}}\big)\|u\|^2 - \|f\|_*\|u\|. \end{aligned}$$

Thus, if we take

$$\beta > \frac{\|f\|_*}{\nu - \frac{L_F}{\lambda_1^{1/2}}},$$

we obtain $[R_m u, u] \ge 0$ for all $u \in V_m$ such that $\|u\| = \beta$. Consequently, by a corollary of the Brouwer fixed point theorem (see [27, Chapter 2, Lemma 1.4]), for each $m \ge 1$ there exists $u_m \in V_m$ such that $R_m(u_m) = 0$, with $\|u_m\| \le \beta$.

Replacing v_i by u^m in (4.3) and taking into account that $b(u^m, u^m, u^m) = 0$, we get

$$\nu\|u^m\|^2 = \langle f, u^m\rangle + (F(u^m), u^m)$$

$$\leq \|f\|_*\|u^m\| + \frac{L_F}{\lambda_1^{1/2}}\|u^m\|^2.$$

Hence

$$(\nu - \frac{L_F}{\lambda_1^{1/2}})\|u^m\| \leq \|f\|_*.$$

We extract from $\{u^m\}$ a subsequence $\{u^{m'}\}$, which converges weakly in V to some limit u. Since the domain Ω is bounded, the injection of V into H is compact. Thus,

$$u^{m'} \to u \text{ weakly in } V, \text{ and strongly in } H,$$

up to a subsequence. Passing to the limit in (4.3) with the sequence m', we find that u is a stationary solution of (1.1).

(ii) Uniqueness. Suppose that u^* and v^* are two stationary solutions of (1.1). Then

$$\nu\langle Au^* - Av^*, v\rangle + b(u^*, u^*, v) - b(v^*, v^*, v) = (F(u^*) - F(v^*), v)$$

for all $v \in V$. Choosing $v = u^* - v^*$, we have

$$\nu\langle Au^* - Av^*, u^* - v^*\rangle = b(u^* - v^*, v^*, u^* - v^*) + (F(u^*) - F(v^*), u^* - v^*).$$

Hence

$$\nu\|u^* - v^*\|^2 \leq c_0\lambda_1^{-1/4}\|u^* - v^*\|^2\|v^*\| + \frac{L_F}{\lambda_1^{1/2}}\|u^* - v^*\|^2,$$

where we have used inequality (1.2). Therefore,

$$\Big(\nu - \frac{L_F}{\lambda_1^{1/2}}\Big)\|u^* - v^*\|^2 \leq c_0\lambda_1^{-1/4}\|u^* - v^*\|^2\|v^*\|.$$

Using estimate (4.1) we deduce that

$$\Big(\nu - \frac{L_F}{\lambda_1^{1/2}}\Big)^2\|u^* - v^*\|^2 \leq c_0\lambda_1^{-1/4}\|f\|_*\|u^* - v^*\|^2,$$

and hence the uniqueness follows from the condition (4.2). □

We now study the stability of the stationary solution.

Theorem 4.2. *Let* **(H1)-(H2)** *and* (4.2) *hold. Then there exists a value* $\lambda \in (0, 2\gamma)$ *such that for the solution* $u(t)$ *of* (1.1) *with initial datum* $\phi \in C_\gamma(V)$*, the following estimates hold for all* $t \geq 0$:

$$\begin{aligned} |u(t) - u^*|^2 + \alpha^2\|u(t) - u^*\|^2 \leq e^{-\lambda t}\big(&|\phi(0) - u^*|^2 + \alpha^2\|\phi(0) - u^*\|^2 \\ &+ \frac{L_F}{(2\gamma - \lambda)\lambda_1^{1/2}}\|\phi - u^*\|_\gamma^2\big), \end{aligned} \tag{4.4}$$

$$\|u_t - u^*\|_\gamma^2 \leq \max\Big\{e^{-2\gamma t}\|\phi - u^*\|_\gamma^2; e^{-\lambda t}\Big(\frac{1}{\alpha^2}|\phi(0) - u^*|^2 + \|\phi(0) - u^*\|^2$$

$$+\frac{L_F}{\alpha^2(2\gamma-\lambda)\lambda_1^{1/2}}\|\phi-u^*\|_\gamma^2\Big)\Big\}, \tag{4.5}$$

where u^* *is the unique stationary solution to problem* (1.1).

Proof. Denote $w(t) = u(t) - u^*$, one has

$$\frac{d}{dt}(w(t),v)+\nu((w(t),v))+\alpha^2((\partial_t w(t),v))+b(u(t),u(t),v)-b(u^*,u^*,v)$$
$$=(F(u_t)-F(u^*),v),\ \forall t>0,\forall v\in V.$$

From the energy equality, **(H2-ii)** and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed later on, we obtain

$$\begin{aligned}&\frac{d}{dt}\big(e^{\lambda t}(|w(t)|^2+\alpha^2\|w(t)\|^2)\big)\\&=e^{\lambda t}\Big[\lambda(|w(t)|^2+\alpha^2\|w(t)\|^2)-2\nu\|w(t)\|^2\\&\quad+2b(w(t),u^*,w(t))+2(F(u_t)-F(u^*),w(t))\Big]\\&\le e^{\lambda t}\Big[\lambda(|w(t)|^2+\alpha^2\|w(t)\|^2)-2\nu\|w(t)\|^2\\&\quad+\frac{2c_0}{\lambda_1^{1/4}}\|w(t)\|^2\|u^*\|+2L_F\|w_t\|_\gamma|w(t)|\Big].\end{aligned}$$

Hence, using the Cauchy inequality with $\delta > 0$ to be fixed later on and (4.1), we have

$$\begin{aligned}&\frac{d}{dt}\big(e^{\lambda t}(|w(t)|^2+\alpha^2\|w(t)\|^2)\big)\\&\le e^{\lambda t}\frac{L_F}{\delta}\|w_t\|_\gamma^2+e^{\lambda t}\Big[\lambda\lambda_1^{-1}+\lambda\alpha^2-2\nu+\frac{\delta L_F}{\lambda_1}+\frac{2c_0\|f\|_*}{\lambda_1^{1/4}\big(\nu-\frac{L_F}{\lambda_1^{1/2}}\big)}\Big]\|w(t)\|^2.\end{aligned}$$

Therefore, integrating from 0 to t, we have

$$\begin{aligned}&e^{\lambda t}(|w(t)|^2+\alpha^2\|w(t)\|^2)\\&\le|w(0)|^2+\alpha^2\|w(0)\|^2+\frac{L_F}{\delta}\int_0^t e^{\lambda s}\|w_s\|_\gamma^2ds\\&\quad+\Big[\lambda(\lambda_1^{-1}+\alpha^2)-2\nu+\frac{\delta L_F}{\lambda_1}+\frac{2c_0\|f\|_*}{\lambda_1^{1/4}\big(\nu-\frac{L_F}{\lambda_1^{1/2}}\big)}\Big]\int_0^t e^{\lambda s}\|w(s)\|^2ds.\end{aligned} \tag{4.6}$$

In order to control the term $\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds$, we proceed as follows

$$\begin{aligned}&\int_0^t e^{\lambda s}\sup_{\theta\le 0}e^{2\gamma\theta}\|w(s+\theta)\|^2ds\\&=\int_0^t e^{\lambda s}\max\{\sup_{\theta\le -s}e^{2\gamma\theta}\|w(s+\theta)\|^2;\ \sup_{\theta\in[-s,0]}e^{2\gamma\theta}\|w(s+\theta)\|^2\}ds\end{aligned}$$

$$= \int_0^t \max\{e^{-(2\gamma-\lambda)s}\|\phi - u^*\|_\gamma^2; \sup_{\theta\in[-s,0]} e^{(2\gamma-\lambda)\theta}e^{\lambda(s+\theta)}\|w(s+\theta)\|^2\}ds.$$

So, if $\lambda < 2\gamma$, using the above equality in (4.6), we obtain

$$\begin{aligned} &e^{\lambda t}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \\ &\leq |w(0)|^2 + \alpha^2\|w(0)\|^2 + \frac{L_F}{\delta}\|\phi - u^*\|_\gamma^2 \int_0^t e^{(\lambda-2\gamma)s}ds \\ &\quad + \Big[\lambda(\lambda_1^{-1} + \alpha^2) - 2\nu + \frac{\delta L_F}{\lambda_1} + \frac{2c_0\|f\|_*}{\lambda_1^{1/4}\big(\nu - \frac{L_F}{\lambda_1^{1/2}}\big)} + \frac{L_F}{\delta}\Big] \int_0^t \max_{r\in[0,s]} e^{\lambda r}\|w(r)\|^2 ds. \end{aligned}$$

Observe that the choice of $\delta = {\lambda_1}^{1/2}$ makes that $\delta\lambda_1^{-1}L_F + L_F\delta^{-1}$ is minimal and the consider coefficient of the last integral becomes

$$\lambda(\lambda_1^{-1} + \alpha^2) - 2\nu + \frac{2L_F}{{\lambda_1}^{1/2}} + \frac{2c_0\|f\|_*}{\lambda_1^{1/4}\big(\nu - \frac{L_F}{\lambda_1^{1/2}}\big)}. \tag{4.7}$$

By condition (4.2), we can choose $\lambda \in (0, 2\gamma)$ such that (4.7) is negative. Thus, we can deduce that

$$e^{\lambda t}(|w(t)|^2 + \alpha^2\|w(t)\|^2) \leq |w(0)|^2 + \alpha^2\|w(0)\|^2 + \frac{L_F}{(2\gamma-\lambda)\lambda_1^{1/2}}\|\phi - u^*\|_\gamma^2,$$

whence (4.4) follows.

To prove (4.5), we proceed as follows

$$\begin{aligned} \|w_t\|_\gamma^2 &= \sup_{\theta\leq 0} e^{2\gamma\theta}\|w(t+\theta)\|^2 \\ &= \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|w(t+\theta)\|^2; \sup_{\theta\in[-t,0])} e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\} \\ &= \max\Big\{e^{-2\gamma t}\|\phi - u^*\|_\gamma^2; \sup_{\theta\in[-t,0]} e^{2\gamma\theta}\|w(t+\theta)\|^2\Big\}, \end{aligned}$$

and the second term can be estimated by using (4.4) and the fact that $e^{(2\gamma-\lambda)\theta} \leq 1$ when $\theta \leq 0$. □

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