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ON CLOSING CODES

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Abstract. We extend Jung's result about the relations among bi-closing, open and constant-to-one codes between general shift spaces to closing codes. We also show that any closing factor code $\varphi : X \to Y$ has a degree d, and it is proved that d is the minimal number of preimages of points in Y. Some other properties of closing codes are provided. Then, we show that any closing factor code is hyperbolic. This enables us to determine some shift spaces which preserved by closing codes.

1. Introduction

Closing codes are important in symbolic dynamics. A classical consideration of them is in [7, 8]. Jung showed that these maps have a close relation with open codes [6]. They also have a very natural description from the viewpoint of hyperbolic dynamics [3]: right-closing codes are injective on unstable sets while left-closing codes are injective on stable sets.

There are some relations among bi-closing, open and constant-to-one codes. In [9], Nasu showed that in the category of irreducible shifts of finite type, these conditions are equivalent. Jung extended this result to the general shift spaces and showed that any two of these properties imply the third [6]. In Section 3, we prove that any closing factor code $\varphi : X \to Y$ has a degree d and also, we show that d is the minimal number of preimages of points in Y (Theorem 3.3). Then, Corollary 3.5 states that any open closing code from a shift space to an irreducible shift space is bi-closing. After that, we extend Jung's result to closing codes (Theorem 3.7).

Closing extensions have been given some attention in coded systems which are subshifts generated by the arbitrary concatenations of a countable set of words. Boyle et al. [2] investigated the irreducible closing extensions of SFT (resp. sofic) shift spaces and showed that they are SFT (resp. sofic). Also in [1], Blanchard proved that the property 'coded' lifts under closing codes. In Section 4, first, we show that any closing factor code is hyperbolic (Theorem 4.5). Then, by Theorem 4.7, we show that synchronized and half-synchronized

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systems which are two well-known subclasses of coded systems, are preserved by closing codes. Finally, Theorem 4.8 gives an equivalence relation by closing codes on some families of shift spaces.

2. Background and notations

This section is devoted to the basic definitions in symbolic dynamics. The notations has been taken from [8]. Given a non-empty finite set $\mathcal{A}$, the full $\mathcal{A}$-shift, denoted by $\mathcal{A}^{\mathbb{Z}}$, is the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. A word (or block) over $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$. The *shift map* on $\mathcal{A}^{\mathbb{Z}}$ is the map $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ where $(\sigma x)_i = x_{i+1}$. The pair $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *full shift.* Let $\mathcal{F}$ be a set of words over $\mathcal{A}$. Define $X_{\mathcal{F}}$ to be the subset of $\mathcal{A}^{\mathbb{Z}}$ which do not contain any word in $\mathcal{F}$. A *shift space* is a closed subset X of $\mathcal{A}^{\mathbb{Z}}$ such that $X = X_{\mathcal{F}}$ for some set $\mathcal{F}$ of forbidden words. A shift space X is called a *shift of finite type* (SFT) if $X = X_{\mathcal{F}}$ for a finite set $\mathcal{F}$ and it is called *irreducible* if for every ordered pair of words $u, v \in \mathcal{B}(X)$, there is a word $w \in \mathcal{B}(X)$ such that $uwv \in \mathcal{B}(X)$.

Let $\mathcal{B}_n(X)$ be the set of all admissible n-words and $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$. Given $u \in \mathcal{B}(X)$, the *cylinder* ${}_l[u]$ is the set $\{x \in X : x_{[l,l+|u|-1]} = u\}$. For $l \geq 0$ and $|u| = 2l+1$, ${}_{-l}[u]$ is called a *central* $2l+1$ *cylinder.*

Let $\mathcal{A}$ and $\mathcal{D}$ be alphabets and X a shift space over $\mathcal{A}$. For $m, n \in \mathbb{Z}$ with $-m \leq n$, the $(m+n+1)$*-block map* $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathcal{D}$ is defined by

$$y_i = \Phi(x_{i-m}x_{i-m+1} \cdots x_{i+n}) = \Phi(x_{[i-m,i+n]}), \tag{2.1}$$

where $y_i \in \mathcal{D}$. Then, the map $\varphi = \Phi_{\infty}^{[-m,n]} : X \to \mathcal{D}^{\mathbb{Z}}$ defined by $\varphi(x) = y$ with y_i given by (2.1) is called the *code induced by* Φ. If $m = n = 0$, then φ is called a 1*-block code* and $\varphi = \Phi_{\infty}$. An onto (resp. invertible) code $\varphi : X \to Y$ is a *factor code* (resp. *conjugacy*). Then, X is called an *extension* of Y.

A code $\varphi : X \to Y$ is *finite-to-one* if there is a positive integer M such that $|\varphi^{-1}(y)| \leq M$ for every $y \in Y$ and if the points in the image have the same number of preimages, φ is called *constant-to-one.*

Let G be a directed graph and $\mathcal{V}$ (resp. $\mathcal{E}$) the set of its vertices (resp. edges). An *edge shift*, denoted by X_G, is a subshift which consists of all bi-infinite sequences of edges from $\mathcal{E}$. A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a directed graph and $\mathcal{L} : \mathcal{E} \to \mathcal{A}$ its labeling. Associated to $\mathcal{G}$, a shift space

$$X_{\mathcal{G}} = \text{closure}\{\mathcal{L}_{\infty}(\xi) : \xi \in X_G\} = \overline{\mathcal{L}_{\infty}(X_G)}$$

is defined and $\mathcal{G}$ is called a *cover* (or *presentation*) of $X_{\mathcal{G}}$. When G is a finite graph and hence compact, $X_{\mathcal{G}} = \mathcal{L}_{\infty}(X_G)$ is a *sofic shift.* If the shift space X is a factor of a SFT, then it is sofic.

A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for each vertex I of G the edges starting at I carry different labels. A *Fischer cover* of an irreducible sofic shift X is a right-resolving cover having the fewest vertices among all right-resolving covers of X. It is unique up to isomorphism.

3. An extension of Jung's theorem

In this section, we investigate some properties of closing codes. First we show that any closing factor code $\varphi : X \to Y$ has a degree and it is the minimal number of preimages of points in Y. Then, we extend Jung's result [6] about relations among bi-closing, open and constant-to-one codes to the class of closing codes and conclude that any open closing code is bi-closing.

A 1-block code $\varphi = \Phi_\infty : X \to Y$ is called *right-resolving* if whenever ab, $ac \in \mathcal{B}_2(X)$ with $\Phi(b) = \Phi(c)$, then $b = c$. A pair x, $\overline{x}$ of points in a shift space X is *left-asymptotic* if there exists an integer N for which $x_{(-\infty,N]} = \overline{x}_{(-\infty,N]}$. A code $\varphi : X \to Y$ is called *right-closing* if whenever x and $\overline{x}$ are left-asymptotic and $\varphi(x) = \varphi(\overline{x})$, then $x = \overline{x}$. A right-closing code can be recoded to a right-resolving code. *Right-asymptotic points* and *left-closing* (*resp. resolving*) *codes* are defined similarly. If φ is left or right-closing, it is called *closing* and we call φ *bi-closing* (*resp. bi-resolving*) if it is both left and right-closing (resp. resolving).

A point x in a subshift X is called *doubly transitive* if every word in X appears in x infinitely often to the left and to the right. Let $\varphi : X \to Y$ be a factor code. If there exists a $d \in \mathbb{N}$ such that every doubly transitive point of Y has exactly d preimages, d is called the *degree* of φ. Then, φ is d-to-one almost everywhere.

Even when a factor code $\varphi : X \to Y$ is finite-to-one, the degree of φ need not be defined. Jung [6] proved that bi-closing codes have a degree. Now we extend this result to closing factor codes. For this, we will use quantity d^*.

Definition 3.1. Suppose $\varphi = \Phi_\infty : X \to Y$ is a 1-block code. For $w = w_1 \cdots w_m \in \mathcal{B}_m(Y)$ and $1 \le i \le m$, we define $d^*(w, i)$ to be the number of distinct symbols that you can see at coordinate i in preimages of the word w. Now put

$$d^* = \min\{d^*(w, i) : w \in \mathcal{B}(Y),\ 1 \le i \le |w|\}.$$

A *magic word* is a word w such that $d^*(w, i) = d^*$ for some i. Then, the index i is called a *magic coordinate.*

Theorem 3.2. *Let $\varphi : X \to Y$ be a resolving 1-block factor code and Y an irreducible shift space. Then φ has a degree d and $d = d^*$. Furthermore, $|\varphi^{-1}(y)| \ge d$ for any $y \in Y$.*

Proof. Let w be a magic word with the magic coordinate i. So if $y \in Y$, then for each m, there exist at least d^* distinct symbols that we can see at coordinate 0 in the preimages of $y_{[-m,m]}$. Therefore, by compactness, y has at least d^* preimages.

Now we show that any doubly transitive point has at most d^* preimages. Let $y \in Y$ be a doubly transitive point. By definition, w appears in y infinitely often to the left and to the right. Since φ is resolving, the number of preimages of y is at most d^*. Thus, $d = d^*$. $\square$

Recall that if $\varphi : X \to Y$ is a right-closing code, then there exist a shift space X' and a conjugacy $\psi : X' \to X$ such that $\varphi \circ \psi$ is right-resolving. So by Theorem 3.2, we have:

Theorem 3.3. *Let $\varphi : X \to Y$ be a closing factor code and Y an irreducible shift space. Then φ has a degree d and $|\varphi^{-1}(y)| \geq d$ for any $y \in Y$.*

Jung showed that any open bi-closing code from a shift space to an irreducible subshift is constant-to-one [6, Corollary 2.8]. We give an analogue of this result for closing codes. For this, first recall that any open code from a shift space to an irreducible shift space is onto [6, Lemma 2.1].

Theorem 3.4. *Let $\varphi : X \to Y$ be an open closing code and Y irreducible. Then φ is constant-to-one.*

Proof. By Theorem 3.3, φ has a degree d and $|\varphi^{-1}(y)| \geq d$ for all $y \in Y$. Since φ is closing, it is finite-to-one [8, Proposition 8.1.11]. So openness implies that $|\varphi^{-1}(y)| \leq d$ for all $y \in Y$ [6, Lemma 2.5]. □

Corollary 3.5. *Any open closing code from a shift space to an irreducible shift space is bi-closing.*

Proof. Theorem 3.4 implies that such codes are constant-to-one and by [6, Proposition 2.9], any open constant-to-one code is bi-closing. □

A continuous map $f : Y \to X$ is called a *cross section* of a code $\varphi : X \to Y$ if $\varphi(f(y)) = y$ for all $y \in Y$. We say that φ has d disjoint cross sections if there are d cross sections $f_i : Y \to X$ such that $f_i(Y) \cap f_j(Y) = \emptyset$ for all $i \neq j$. Using cross sections, Jung showed that any constant-to-one bi-closing code is open [6, Proposition 4.5]. Using Theorem 3.3, the main ingredients of the proof of [6, Proposition 4.5] works for closing codes as follows.

Theorem 3.6. *Let $\varphi : X \to Y$ be a d-to-one closing factor code and Y an irreducible shift space. Then φ is open.*

Proof. We claim that φ has d disjoint cross sections such that the union of their images is X. Without loss of generality, we can assume that $\varphi = \Phi_\infty$ is a right-resolving 1-block code. For each $n \geq 0$ and $w \in \mathcal{B}_{2n+1}(Y)$, define

$$\mathcal{D}(w) = \{x_0 : x \in X, \Phi(x_{[-n,n]}) = w\}.$$

Then by Definition 3.1, $d(w) = |\mathcal{D}(w)| \geq d^*$ and so Theorem 3.2 gives $|\mathcal{D}(w)| \geq d$. Now define $Y_n = \{y \in Y : d(y_{[-n,n]}) > d\}$. Each Y_n is closed and the family of Y_n's is nested. If for all n, $Y_n \neq \emptyset$, then there is a $y \in \cap Y_n$. Therefore, for each n, there are at least $d+1$ distinct symbols that appear at coordinate 0 in the preimages of $y_{[-n,n]}$ and hence by compactness, y has at least $d+1$ preimages which is a contradiction. So there is an n such that $Y_n = \emptyset$. Thus, $|\mathcal{D}(w)| = d$ for all $w \in \mathcal{B}_{2n+1}(Y)$.

For this n, we can define $g_1, \ldots, g_d : \mathcal{B}_{2n+1}(Y) \to \mathcal{B}_1(X)$ such that for any $w \in \mathcal{B}_{2n+1}(Y)$, $\{g_1(w), \ldots, g_d(w)\} = \mathcal{D}(w)$. Then, we define cross sections

$f_1, \ldots, f_d : Y \to X$ such that for each $1 \le i \le d$, $f_i(y)$ is the unique point in $\varphi^{-1}(y)$ where $f_i(y)_0 = g_i(y_{[-n,n]})$. Now it is sufficient to show continuity.

For each $1 \le i \le d$, define an open and closed set

$$V_i = \cup\{{}_{-n}[u] : u \in \mathcal{B}_{2n+1}(X),\ (g_i \circ \Phi)(u) = u_0\}.$$

Then, $x \in V_i$ if and only if $x = (f_i \circ \varphi)(x)$. Therefore, $f_i^{-1} = \varphi_{|V_i}$ is an invertible continuous map between compact metric spaces. So, f_i is continuous and the claim is proved.

Now suppose φ is not open. Then, there exists an open set $U \subseteq X$ such that $\varphi(U)$ is not open. So there are a point $x \in U$ and a sequence $\{y^{(n)}\}$ in Y such that $y^{(n)} \to \varphi(x)$ and $y^{(n)} \notin \varphi(U)$ for any $n \in \mathbb{N}$. Let f be a cross section with $x \in f(Y)$ and let $z^{(n)} = f(y^{(n)})$ for all n. Then, we have $z^{(n)} \to x$ and since U is open, $z^{(n)} \in U$. But then, $y^{(n)} \in \varphi(U)$ for all large enough n, which is a contradiction. □

Theorems 3.4, 3.6 and [6, Proposition 2.9] give the following result.

Theorem 3.7. *Let $\varphi : X \to Y$ be a factor code between shift spaces and Y an irreducible shift space. Then, any two of the following conditions imply the third*:

(1) *φ is open*;
(2) *φ is constant-to-one*;
(3) *φ is closing.*

4. Extension by closing codes

In this section, first we show that any closing factor code is hyperbolic. By using this, we investigate some properties which preserved by closing codes. Then, we define an equivalence relation by having a common right-closing extension on some classes of subshifts.

Theorem 4.1 ([2])**.** *Let X be an irreducible shift, Y a SFT* (*resp. sofic*) *and $\varphi : X \to Y$ a right-closing factor code. Then, X is also SFT* (*resp. sofic*).

Blanchard showed that right-closing codes also preserve the property 'coded' [1]. Now we generalize these results to synchronized and half-synchronized systems which are two subclasses of coded systems and defined as follows.

Definition 4.2. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever $uv, vw \in \mathcal{B}(X)$, then we have $uvw \in \mathcal{B}(X)$. An irreducible shift space X with a synchronizing word is called a *synchronized system.*

For $x \in X$, let $x_- = (x_i)_{i<0}$ and $x_+ = (x_i)_{i \in \mathbb{Z}^+}$. Also, let $X^+ = \{x_+ : x \in X\}$. Then, the follower set of x_- (resp. $m \in \mathcal{B}(X)$) is defined as $\omega_+(x_-) = \{x_+ \in X^+ : x_- x_+ \in X\}$ (resp. $\omega_+(m) = \{x_+ \in X^+ : m x_+ \in X^+\}$).

Definition 4.3. An irreducible shift space X is called *half-synchronized*, if there exist a word $m \in \mathcal{B}(X)$ and a left-transitive point x in X where $x_{[-|m|+1,0]}$

$= m$ and $\omega_+(x_{(-\infty,0]}) = \omega_+(m)$. Such a word m is a *half-synchronizing word* for X.

Any synchronizing word is a half-synchronizing word. So any synchronized shift is half-synchronized. Also, half-synchronized systems are coded [5].

The following notation is motivated by the hyperbolic homeomorphism $\varphi : D(X) \to D(Y)$ defined in [11] which $D(X)$ stands for the set of doubly transitive points of X.

Definition 4.4. Let $\varphi : X \to Y$ be a factor code and X be an irreducible shift space. We call φ *hyperbolic* if there exist a $d \in \mathbb{N}$ and a word $w \in \mathcal{B}_{2n+1}(Y)$ and d words $m^{(1)}, m^{(2)}, \ldots, m^{(d)} \in \mathcal{B}_{2k+1}(X)$, so that

(1) if $y \in Y$ such that $y_{[-n,n]} = w$, then we have

$$\varphi^{-1}(y)_{[-k,k]} = \{m^{(1)}, m^{(2)}, \ldots, m^{(d)}\},$$

(2) if $w' = ww''w \in \mathcal{B}(Y)$, then for each $1 \leq i \leq d$, there exists a unique word $a^{(i)} \in \mathcal{B}(X)$, so that for any $x \in X$ with $x_{[-k,k]} = m^{(i)}$ and $\varphi(x)_{[-n,n+p]} = w'$, we have $x_{[-k,k+p]} = a^{(i)}$.

In order to show that the properties 'synchronized' and 'half-synchronized' lift under closing factor codes, first we prove that there is a close relation between hyperbolic maps and closing codes.

Theorem 4.5. *Let $\varphi : X \to Y$ be a closing factor code. Then φ is hyperbolic.*

Proof. Without loss of generality, suppose that φ is a right-resolving 1-block code. Let $k \in \mathbb{N}$ and $y \in Y$. Then, compactness of X implies that there is a $m \in \mathbb{N}$ such that for any point $x \in X$ with $\varphi(x)_{[-m,m]} = y_{[-m,m]}$, we have that $x_{[-k,k]} \in \varphi^{-1}(y)_{[-k,k]}$. Let

$$d = \min\{|\varphi^{-1}(y')_{[-k,k]}| : y' \in Y \text{ with } y'_{[-m,m]} = y_{[-m,m]}\}$$

and let $\overline{y} \in Y$ with $\overline{y}_{[-m,m]} = y_{[-m,m]}$ and $|\varphi^{-1}(\overline{y})_{[-k,k]}| = d$. By compactness of X, there exists $n > m$ such that for any point $x \in X$ with $\varphi(x)_{[-n,n]} = \overline{y}_{[-n,n]}$, we have that $x_{[-k,k]} \in \varphi^{-1}(\overline{y})_{[-k,k]}$. Now let $w = \overline{y}_{[-n,n]}$ and $\{m^{(1)}, m^{(2)}, \ldots, m^{(d)}\} = \varphi^{-1}(\overline{y})_{[-k,k]}$. Then by the minimality of d, Definition 4.4(1) is satisfied and right-resolving gives (2). $\square$

Remark 4.6. Not every hyperbolic map is closing. Because closing codes are finite-to-one [8, Proposition 8.1.11], while there are some hyperbolic maps which are not finite-to-one [4].

Fiebig showed that 'being coded (or synchronized)' is an invariant for hyperbolic codes [4]. Later we extended this result to half-synchronized systems [10, Theorem 3.3]. So by Theorem 4.5, we have:

Theorem 4.7. *Let $\varphi : X \to Y$ a closing factor code and X an irreducible shift space. Then,*

(1) *X is synchronized if and only if Y is synchronized.*
(2) *X is half-synchronized if and only if Y is half-synchronized.*

Now we can introduce an equivalence relation using closing factor codes among shift spaces. First, we define the notation of fiber product.

Let $\varphi_X : X \to Z$ and $\varphi_Y : Y \to Z$ be codes between shift spaces. The *fiber product* of (φ_X, φ_Y) is (Σ, ψ_X, ψ_Y), where

$$\Sigma = \{(x, y) \in X \times Y : \varphi_X(x) = \varphi_Y(y)\},$$

and $\psi_X : \Sigma \to X$ and $\psi_Y : \Sigma \to Y$ are the projection maps. Some properties of the maps φ_X and φ_Y are inherited by ψ_X and ψ_Y and vice versa. More specifically, if φ_X is right-closing, then ψ_Y is also right-closing.

Theorem 4.8. *Having a common SFT (resp. sofic, synchronized, half- synchronized or coded) extension with right-closing factor codes defines an equivalence relation on the set of irreducible SFT (resp. sofic, synchronized, half-synchronized or coded) systems.*

Proof. First, we consider the set of irreducible shifts of finite type. Let X, Y and Z be irreducible shifts of finite type such that X and Y and Y and Z have the common SFT right-closing extensions $(V, \varphi_X, \varphi_Y)$ and $(W, \varphi'_Y, \varphi'_Z)$, respectively and also, $(\Sigma, \psi_V, \psi'_W)$ be the fiber product of (φ_Y, φ'_Y). Suppose that (v, w) is a point in Σ such that the orbit of $\varphi_Y(v) = \varphi'_Y(w)$ is dense in Y and denote by Γ the orbit closure of (v, w) in Σ. Then, Γ is an irreducible component of Σ such that $\psi_V : \Gamma \to V$ and $\psi'_W : \Gamma \to W$ are onto. Since φ_Y is right-closing, $\psi'_W : \Gamma \to W$ will be right-closing. So by Theorem 4.1, Γ is SFT and since the composition of right-closing codes is a right-closing code, the triple $(\Gamma, \varphi_X \circ \psi_V, \varphi'_Z \circ \psi'_W)$ is a common SFT right-closing extension between X and Z.

The result for sofic (resp. synchronized, half-synchronized or coded) systems holds by Theorem 4.1 (resp. Theorem 4.7 or [1, Proposition 13]) in a similar way. ☐

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