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SCALAR CURVATURE COMPARISONS OF LEVEL HYPERSURFACES OF GEODESIC SPHERES

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Abstract. Using the comparison of differential equations involving Ricci and scalar curvatures obtained by Eschenburg and O'Sullivan, the scalar curvatures of level hypersurfaces of geodesic spheres are compared.

1. Introduction

The classical volume comparisons of geodesic balls with Ricci curvature bounded below by Bishop and Bishop-Gromov give geometric applications ([1], [2], [3]). By using a Jacobi tensor and its differential equation, the above volume comparisons can also be proved by means of the comparison of Jacobi differential equations as in [6] and [7]. The Lorentzian versions are given in [4], [5] and [8]. Some comparison theorems for three dimensional manifolds with Ricci curvature bounded above are proved in [7] by the following differential equation (1) developed by Eschenburg and O'Sullivan. In this paper, the comparisons of the scalar curvature of geodesic spheres are shown (Theorem 1 and corollaries) by combining the Bishop volume comparison Theorem and the comparison of Jacobi differential equations (Lemma 1).

Let M be a Riemannian manifold of dimension $n+1 \geq 3$ and γ_v be a radial geodesic $\gamma_v(t) = \exp_p tv$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ for all $v \in T_pM$. Let $S(r_0)$ be a sphere of radius r_0 in the tangent space T_pM and $H_{r_0} = \exp_p S(r_0)$ be a hypersurface in M. Consider a geodesic variation along a unit-speed radial geodesic $\gamma_v(t)$ orthogonal to H_{r_0} whose shape operator is denoted by $S_{-\gamma(r_0)} = S_N$. For each level hypersurface $H_t = \exp_p S(t)$ of H_{r_0} with $0 < r_0 < t < \mathrm{cut}_v(p)$, we get the following differential equation obtained by Eschenburg and O'Sullivan in [7]

$$(1) \qquad \theta'(t) + \theta^2(t) + s^M - s(H_t) - \mathrm{Ric}(\gamma_v'(t), \gamma_v'(t)) = 0,$$

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where we denote by s^M, $s(H_t)$ the scalar curvature of M at the point $\gamma_v(t)$, the scalar curvature of level hypersurface H_t, respectively and $\theta(t)$ is the mean curvature of H_t along $\gamma_v(t)$ (see (14)).

Definition 1 (cf. [10])**.** Let γ be a unit speed geodesic orthogonal to a hypersurface H at $\gamma(r)$ with $N_{\gamma(r)} = -\gamma'(r)$. A smooth $(1,1)$ tensor field $A : (\gamma')^{\perp} \to (\gamma')^{\perp}$ is called an *H-Jacobi tensor* along γ if it satisfies

$$A'' + R(A, \gamma')\gamma' = 0, \quad \ker A \cap \ker A' = \{0\}, \quad A(r) = \mathrm{Id}, \quad A'(r) = S_N,$$

where S_N is the shape operator of the hypersurface and Id is the identity endomorphism of $(\gamma')^{\perp}$. A point $\gamma(t_0)$ for $t_0 \in (r, R)$ is called a focal point to H if $\det A(t_0) = 0$.

We assume a linear isometry for volume and the scalar curvature comparisons

$$\imath : T_pM \to T_{\bar{p}}\bar{M} \tag{2}$$

such that $\imath(\gamma_v'(0)) = \bar{\gamma}_{\bar{v}}'(0)$ for all $v \in T_pM$. Let $H_{r_0} = \exp_p S(r_0)$, $\bar{H}_{r_0} = \exp_{\bar{p}} S(r_0)$ be hypersurfaces in M, $\bar{M}$ of dimension $n+1$, respectively. Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of $T_{\gamma(r_0)}H_{r_0}$ and $\{E_1, E_2, \dots, E_n\}$ be its parallel basis along $\gamma_v(t)$ with $E_i(r_0) = e_i$ for each i. The induced linear isometry

$$\imath : T_{\gamma_v(r_0)}H_{r_0} \to T_{\bar{\gamma}_{\bar{v}}(r_0)}\bar{H}_{r_0} \tag{3}$$

satisfies $\imath(\gamma_v'(r_0)) = \bar{\gamma}_{\bar{v}}'(r_0)$ for all $v \in T_pM$ and $\imath(E_i(r_0)) = \bar{E}_i(r_0)$. The cut point of $p \in M$ along $\gamma_v(t)$ is denoted by $\mathrm{cut}_v(p)$. The injectivity radius of $p \in M$ is defined by $\mathrm{inj}_{S(1)}(p) = \inf\{\mathrm{cut}_v(p) \mid v \in S(1)\}$. By $\bar{\gamma}_{\bar{v}}(t) = \exp_{\bar{p}} \circ \imath \circ \exp_p^{-1} tv$ for all $v \in T_pM$ and t less than $\min\{\mathrm{inj}_{S(1)}(p), \mathrm{inj}_{S(1)}(\bar{p})\}$, we have one-to-one correspondence between all corresponding points M and $\bar{M}$. Our comparison results are:

Theorem 1. *Let $\bar{M}(k)$ be a Riemannian manifold of constant curvature k with dimension $n+1 \geq 3$ and $\bar{\gamma}_{\bar{v}}(t)$ be a unit speed radial geodesic with $\bar{\gamma}_{\bar{v}}(0) = \bar{p}$ and $\bar{\gamma}_{\bar{v}}'(0) = \bar{v}$ for all $\bar{v} \in T_{\bar{p}}\bar{M}$ orthogonal to a hypersurface $\bar{H}_{r_0} = \exp_{\bar{p}} S(r_0)$. Let M be an arbitrary Riemannian manifold and $\gamma_v(t)$ be a unit speed radial geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ for all $v \in T_pM$ orthogonal to a hypersurface $H_{r_0} = \exp_p S(r_0)$. Assume that $nk = Ric(\bar{\gamma}_{\bar{v}}'(t), \bar{\gamma}_{\bar{v}}'(t)) \leq Ric(\gamma_v'(t), \gamma_v'(t))$ for all $v \in T_pM$, $\theta(r_0) = \bar{\theta}(r_0)$ and $\det A(r_0) = \det \bar{A}(r_0)$ under a linear isometry $\imath : T_pM \to T_{\bar{p}}\bar{M}$ such that $\imath(\gamma_v'(0)) = \bar{\gamma}_{\bar{v}}'(0)$ for all $v \in T_pM$. If $R(\,\cdot\,, \gamma_v')\gamma_v' = R_{\gamma_v'} \neq R_{\bar{\gamma}'} = R(\,\cdot\,, \bar{\gamma}_v')\bar{\gamma}_v'$ or $H_{r_0} = \exp_p S(r_0)$ is not totally umblic, then we get*

$$s^{\bar{M}} - s(\bar{H}_t) < s^M - s(H_t)$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ and $\gamma_v(t)$ by a linear isometry $\imath$ and $r_0 < t < \min\{inj_{\mathrm{S(1)}}(\mathrm{p}), inj_{\mathrm{S(1)}}(\bar{\mathrm{p}})\}$.

If $s^M - s(H_t) - Ric(\gamma_v'(t), \gamma_v'(t)) \leq s^{\bar{M}} - s(\bar{H}_t) - Ric(\bar{\gamma}_{\bar{v}}'(t), \bar{\gamma}_{\bar{v}}'(t))$*, then we get*

$$s^M - s(H_t) < s^{\bar{M}} - s(\bar{H}_t)$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ *and* $\gamma_v(t)$ *and*

$$r_0 < t < \min\{inj_{\mathrm{S}(1)}(\mathrm{p}), inj_{\mathrm{S}(1)}(\bar{\mathrm{p}})\}.$$

Under the conditions of Theorem 1 we have:

Corollary 1. *If* $s^M \leq s^{\bar{M}}$ *under* $nk \leq Ric(\gamma_v'(t), \gamma_v'(t))$ *for all* $v \in T_pM$*, then we get*

$$s(H_t) < s(\bar{H}_t)$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ *and* $\gamma_v(t)$ *and*

$$r_0 < t < \min\{inj_{\mathrm{S}(1)}(\mathrm{p}), inj_{\mathrm{S}(1)}(\bar{\mathrm{p}})\}.$$

The conclusion of Corollary 1 means in another words that

$$s(H_t) \text{ on } H_t \text{ is less than } s(\bar{H}_t) \text{ on } \bar{H}_t$$

for $r_0 < t < \min\{\mathrm{inj}_{\mathrm{S}(1)}(\mathrm{p}), \mathrm{inj}_{\mathrm{S}(1)}(\bar{\mathrm{p}})\}$. So we apply it to comparison.

Corollary 2. *If* $s^M \leq s^{\bar{M}}$ *with* $nk \leq Ric(\gamma_v'(t), \gamma_v'(t))$ *for all* $v \in T_pM$*, then we get for each* $r_0 < t < r < \min\{inj_{\mathrm{S}(1)}(\mathrm{p}), inj_{\mathrm{S}(1)}(\bar{\mathrm{p}})\}$

$$\int_{r_0}^{r} \int_{S(1)} s(H_t) \det A(t)\, dv\, dt < \int_{r_0}^{r} \int_{S(1)} s(\bar{H}_t) \det \bar{A}(t)\, dv\, dt$$

and

$$\int_{S(1)} s(H_t)\, dH_t < \int_{S(1)} s(\bar{H}_t)\, d\bar{H}_t,$$

where $dH_t = \det A(t)\, dv$ *is the volume element of each hypersurface* $H_t = \exp_p S(t)$ *and* dv *is the volume element of* $S(1)$*.*

In Section 4, we show that a Lorentzian analogue of Theorem 1 does not hold mainly due to the differential equation (22) along a unit timelike geodesic γ instead of (14). But a Lorentzian analogue of Lemma 2 can be similarly stated (Lemma 3).

2. Preliminaries

The shape operator $S_{-\gamma'(t)}$ of each level hypersurface H_t of H_{r_0} is given by

$$A'A^{-1}(t) = S_{-\gamma'(t)} = S_t \tag{4}$$

as in [7]. We denote by $\theta(t) = \mathrm{tr} S_t$ the mean curvature of H_t along a radial geodesic $\gamma_v(t)$. Put $B = A'A^{-1}$ for the H-Jacobi tensor $A(t)$ along $\gamma_v(t)$, then we have

$$B' = A''A^{-1} - A'A^{-1}A'A^{-1} = -R_{\gamma_v'} - B \circ B, \tag{5}$$

where we put $R(A,\gamma_v')\gamma_v' = R_{\gamma_v'}A$. The *shear tensor* σ is defined by $\sigma = B - \frac{\theta}{n}\mathrm{Id}$. Taking the trace of (5), we get the Raychaudhuri equation

$$\theta' + \frac{\theta^2}{n} + \mathrm{Ric}(\gamma_v',\gamma_v') + \mathrm{tr}\sigma^2 = 0, \tag{6}$$

where $\mathrm{Ric}(\gamma_v',\gamma_v') = \sum_{i=1}^n g(R(e_i,\gamma_v')\gamma_v', e_i)$ for an orthonormal basis $\{e_i\}_{i=1}^n$ of ${\gamma_v'}^{\perp}$. The expansion θ of A along γ_v is also expressed as

$$\theta = \mathrm{tr}(B) = \frac{(\det(A))'}{\det(A)}. \tag{7}$$

Put $x = (\det A)^{\frac{1}{n}}$. Then we see

$$x' = \frac{1}{n}x\theta, \quad x'' = \frac{1}{n}(\theta' + \frac{\theta^2}{n})x. \tag{8}$$

So we obtain the Jacobi equation by (6) and (8)

$$x'' + \frac{1}{n}(\mathrm{Ric}(\gamma_v',\gamma_v') + \mathrm{tr}\sigma^2)x = 0. \tag{9}$$

We use the following lemma in [7] for the comparison of the Jacobi equations. Note that the inequality $\le$ in Lemma 1 can be replaced by $\ge$.

Lemma 1 ([7])**.** *For a smooth function $f : \mathbb{R} \to \mathbb{R}$, let x, $\bar{x}$ be a smooth function such that x, $\bar{x}$ is a solution of the differential inequality $x'' + fx \le 0$, $\bar{x}'' + f\bar{x} = 0$, respectively with $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \le \bar{x}'(t_0)$ and x and $\bar{x}$ are both positive in some interval $[t_0, t)$. Let s, $\bar{s}$ be the first positive zero of x, $\bar{x}$, respectively. Then $s \le \bar{s}$ and $x \le \bar{x}$ and $\frac{x'}{x} \le \frac{\bar{x}'}{\bar{x}}$ on $[t_0, s]$.*

Proof. Put $h = \frac{x}{\bar{x}}$ and $g = h'\bar{x}^2 = x'\bar{x} - x\bar{x}'$. If $\bar{x}(t_0) = x(t_0)$ and $x'(t_0) \le \bar{x}'(t_0)$, then we have

$$g(t_0) = x'(t_0)\bar{x}(t_0) - x(t_0)\bar{x}'(t_0) = (x'(t_0) - \bar{x}'(t_0))\bar{x}(t_0) \le 0$$

and $g' = x''\bar{x} - x\bar{x}'' = (x'' + fx)\bar{x} \le 0$. So $g \le 0$, hence $h' \le 0$. Since $h(t_0) \le 1$, we see $h \le 1$. Therefore $x \le \bar{x}$. It follows from $g(t_0) \le 0$, $g' \le 0$ and

$$\frac{x'}{x} - \frac{\bar{x}'}{\bar{x}} = \frac{x'\bar{x} - x\bar{x}'}{x\bar{x}}$$

that $\frac{x'}{x} \le \frac{\bar{x}'}{\bar{x}}$. □

For the linear isometry (2), we get $\bar{H}_{r_0} = \exp_{\bar{\gamma}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma(r_0)} H_{r_0}$. A Riemannian volume between level hypersurfaces H_{r_0} and H_r is defined by

$$V_{r_0}(r) = \int_{r_0}^{r}\int_{S(1)} |\det A| \, dv\, dt,$$

where dv is the volume element of $S(1)$ and $r_0 < r$ is less than the injectivity radius $\mathrm{inj}_{S(1)}(p) = \inf\{\mathrm{cut}_v(p) \,|\, v \in S(1)\}$.

Recall that we consider a radial geodesic $\gamma_v(t)$ for all $v \in T_pM$. For simplicity, $\gamma(t)$ is referred to as $\gamma_v(t)$. Let $\bar{M}(k)$ be an $(n+1)$-dimensional Riemannian manifold of constant curvature k as the model space of volume comparison and $\bar{\gamma}$ be a unit speed radial geodesic orthogonal to the hypersurface $\bar{H}_{r_0} = \exp_{\bar{p}} S(r_0)$ for $\bar{p} \in \bar{M}$. The shear tensor $\bar{\sigma}(t)$ along a geodesic $\bar{\gamma}(t)$ is zero if and only if the shape operator of the hypersurface $\bar{H}_{r_0}$ is given by $S_{-\bar{\gamma}'(r_0)} = \bar{A}'\bar{A}^{-1}(r_0) = c \text{ Id}$ (4) for some real constant c together with the isotropic curvature tensor $R_{\bar{\gamma}(t)} = k(t) \text{ Id}$ ([10], p. 574). Then the Jacobi equation along a geodesic $\bar{\gamma}$ is given by

$$\bar{x}'' + \frac{1}{n}\text{Ric}(\bar{\gamma}', \bar{\gamma}')\bar{x} = 0,$$

where $\bar{x} = (\det\bar{A})^{\frac{1}{n}}$. So the Ricci inequality $nk = \text{Ric}(\bar{\gamma}', \bar{\gamma}') \leq \text{Ric}(\gamma', \gamma')$ implies that $x \leq \bar{x}$ under initial conditions $\det A(r_0) = \det\bar{A}(r_0)$ and $\theta(r_0) \leq \bar{\theta}(r_0)$ by comparing the following differential equations (Lemma 1)

$$x'' + \frac{1}{n}(\text{Ric}(\gamma', \gamma') + \text{tr}\sigma^2)x = 0, \quad \bar{x}'' + \frac{1}{n}\text{Ric}(\bar{\gamma}', \bar{\gamma}')\bar{x} = 0.$$

Thus we get $V_{r_0}(r_1) \leq \bar{V}_{r_0}(r_1)$, where r_1 is less than the minimum of the first focal values of the hypersurfaces H_{r_0} and $\bar{H}_{r_0}$. The same proof for the volume equality holds as in [8]. The Bishop-Gromov volume comparison between level hypersurfaces with the initial value conditions can be proved like Theorem 4.4 [4] by using the fact that $\theta = \frac{y'}{y} \leq \frac{\bar{y}'}{\bar{y}} = \bar{\theta}$.

As in [7], let R^H, Ric^H denote the curvature, Ricci tensor of H_t induced by M, respectively. For $v, w \in T_{\gamma(t)}H_t$, the trace of the Gauss equation

$$(10)\quad g(R(v,w)w,v) = g(R^H(v,w)w,v) - g(S_tv,v)g(S_tw,w) + g(S_tv,w)g(S_tv,w)$$

gives

$$(11)\quad \text{Ric}(w,w) - g(R(\gamma',w)w,\gamma') = \text{Ric}^H(w,w) - \text{tr}S_t g(S_tw,w) + g(S_t^2w,w).$$

Again, the trace of (11) gives

$$(12)\qquad s^M - 2\text{Ric}(\gamma', \gamma') = s(H_t) - (\text{tr}S_t)^2 + \text{tr}S_t^2,$$

where $s(H_t)$ and s^M denote, respectively, the scalar curvature of H_t and M at the point $\gamma(t)$. By $B = A'A^{-1} = S_t$ and (6), the equation (12) is equal to

$$(13)\quad s^M - 2\text{Ric}(\gamma', \gamma') = s(H_t) - \theta^2 + (\frac{\theta^2}{m} + \text{tr}\sigma^2) = s(H_t) - \theta^2 - \theta' - \text{Ric}(\gamma', \gamma').$$

Hence we obtain

$$(14)\qquad \theta' + \theta^2 + s^M - s(H_t) - \text{Ric}(\gamma', \gamma') = 0.$$

Let M and $\bar{M}$ be $(n+1)$-dimensional Riemannian manifolds. Let A and $\bar{A}$ be H-Jacobi tensors along geodesics γ and $\bar{\gamma}$. By differentiating $\theta = \frac{(\det(A))'}{\det(A)}$, thus $\theta' = \frac{y''}{y} - \theta^2$ for $y = \det A$, we get

$$y'' - (\theta' + \theta^2)y = 0.$$

Hence the following differential equations

$$y'' - (\theta' + \theta^2)y = 0, \quad \bar{y}'' - (\bar{\theta}' + \bar{\theta}^2)\bar{y} = 0$$

become by (14)

$$(15)\quad y''+(s^M - s(H_t) - \mathrm{Ric}(\gamma',\gamma'))y = 0, \quad \bar{y}''+(s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}',\bar{\gamma}'))\bar{y} = 0.$$

Thus the comparison of the above differential equations using Lemma 1 enables us to compare the volumes $V_{r_0}(r)$ and $\bar{V}_{r_0}(r)$ under inequality

$$s^M - s(H_t) - \mathrm{Ric}(\gamma',\gamma') \le s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}'(t),\bar{\gamma}'(t)).$$

Lemma 2. *Let $H_{r_0} = \exp_p S(r_0)$ and $\bar{H}_{r_0} = \exp_{\bar{p}} S(r_0)$ be hypersurfaces orthogonal to geodesics γ and $\bar{\gamma}$ in $(n+1)$-dimensional Riemannian manifolds M and $\bar{M}$. If $\det A(r_0) = \det \bar{A}(r_0)$, $\theta(r_0) = \bar{\theta}(r_0)$ and*

$$s^M - s(H_t) - Ric(\gamma'(t),\gamma'(t)) \le s^{\bar{M}} - s(\bar{H}_t) - Ric(\bar{\gamma}'(t),\bar{\gamma}'(t)),$$

then we get

$$V_{r_0}(r) \ge \bar{V}_{r_0}(r),$$

where $0 < r_0 < r < \min\{inj_{\mathrm{S}(1)}(\mathrm{p}), inj_{\mathrm{S}(1)}(\bar{\mathrm{p}})\}$.

Proof. Consider the differential equation (15) along a geodesic γ, $\bar{\gamma}$

$$y'' + (s^M - s(H_t) - \mathrm{Ric}(\gamma',\gamma'))y = 0, \quad \bar{y}'' + (s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}',\bar{\gamma}'))\bar{y} = 0,$$

respectively. If $s^M - s(H_t) - \mathrm{Ric}(\gamma'(t),\gamma'(t)) \le s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}'(t),\bar{\gamma}'(t))$, then we have

$$\frac{y''}{y} = -(s^M - s(H_t) - \mathrm{Ric}(\gamma',\gamma')) \ge -(s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}',\bar{\gamma}')).$$

Hence we get

$$\bar{y}'' + (s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}',\bar{\gamma}'))\bar{y} = 0, \quad y'' + (s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}',\bar{\gamma}'))y \ge 0.$$

By Lemma 1, we get $y \ge \bar{y}$ under $y(r_0) = \bar{y}(r_0)$ and $y'(r_0) = \bar{y}'(r_0)$ ((4), (7)). Therefore we obtain $V_{r_0}(r) \ge \bar{V}_{r_0}(r)$. □

3. Proof of Theorem 1 and corollaries

We prove Theorem 1 by using the induced linear isometry (3) $\imath : T_{\gamma_v(r_0)} H_{r_0} \to T_{\bar{\gamma}_{\bar{v}}(r_0)} \bar{H}_{r_0}$ such that $\imath(\gamma_v'(r_0)) = \bar{\gamma}_{\bar{v}}'(r_0)$ for all $v \in T_p M$. For simplicity, $\gamma(t)$ is referred to as $\gamma_v(t)$.

First, if $nk = \mathrm{Ric}(\bar{\gamma}'(t),\bar{\gamma}'(t)) \le \mathrm{Ric}(\gamma'(t),\gamma'(t))$, then $V_{r_0}(r) \le \bar{V}_{r_0}(r)$ by Bishop comparison Theorem. The equality $V_{r_0}(r) = \bar{V}_{r_0}(r)$ holds if and only if $R_{\gamma'} = R_{\bar{\gamma}'}$ and the hypersurface $H_{r_0} = \exp_p S(r_0)$ is totally umblic with $\theta(r_0) = \bar{\theta}(r_0)$ (Theorem 4 in [8]). Hence we see that

$$(16)\qquad nk \le \mathrm{Ric}(\gamma'(t),\gamma'(t)) \quad \text{implies} \quad V_{r_0}(r) < \bar{V}_{r_0}(r),$$

since we assume that $R_{\gamma'} \ne R_{\bar{\gamma}'}$ or $H_{r_0} = \exp_p S(r_0)$ is not totally umblic.

Second, under the assumptions of $\theta(r_0) = \bar{\theta}(r_0)$ and $\det A(r_0) = \det \bar{A}(r_0)$, if

$$s^M - s(H_t) - \mathrm{Ric}(\gamma', \gamma') \le s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}'),$$

then $V_{r_0}(r) \ge \bar{V}_{r_0}(r)$ by Lemma 2.

Hence the contraposition gives

$$V_{r_0}(r) < \bar{V}_{r_0}(r) \text{ implies } s^M - s(H_t) - \mathrm{Ric}(\gamma', \gamma') > s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') \tag{17}$$

which is equivalent to

$$s^{\bar{M}} - s(\bar{H}_t) - (s^M - s(H_t)) < \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') - \mathrm{Ric}(\gamma', \gamma'). \tag{18}$$

By combining the above two facts (16) and (17), we obtain from (18)

$$s^{\bar{M}} - s(\bar{H}_t) < s^M - s(H_t) \tag{19}$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ and $\gamma_v(t)$ and

$$r_0 < t < \min\{\mathrm{inj}_{S(1)}(p), \mathrm{inj}_{S(1)}(\bar{p})\}.$$

Now we show the second statement in Theorem 1. First, under the assumptions of $\theta(r_0) = \bar{\theta}(r_0)$ and $\det A(r_0) = \det \bar{A}(r_0)$, if $s^M - s(H_t) - \mathrm{Ric}(\gamma', \gamma') \le s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}')$, then we get $V_{r_0}(r) \ge \bar{V}_{r_0}(r)$ by Lemma 2.

Second, recall that $nk = \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') \le \mathrm{Ric}(\gamma', \gamma')$ implies $V_{r_0}(r) < \bar{V}_{r_0}(r)$, since we assume that $R_{\gamma'} \ne R_{\bar{\gamma}'}$ or $H_{r_0} = \exp_p S(r_0)$ is not totally umblic. Hence by contraposition if $V_{r_0}(r) \ge \bar{V}_{r_0}(r)$, then

$$nk = \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') > \mathrm{Ric}(\gamma', \gamma').$$

By combining the above two facts, we obtain that if

$$s^M - s(H_t) - \mathrm{Ric}(\gamma', \gamma') \le s^{\bar{M}} - s(\bar{H}_t) - \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}'), \tag{20}$$

then we get $\mathrm{Ric}(\gamma', \gamma') < nk$. Hence we obtain from (20)

$$0 < nk - \mathrm{Ric}(\gamma', \gamma') = \mathrm{Ric}(\bar{\gamma}', \bar{\gamma}') - \mathrm{Ric}(\gamma', \gamma') \le s^{\bar{M}} - s(\bar{H}_t) - (s^M - s(H_t)).$$

Therefore we get $s^M - s(H_t) < s^{\bar{M}} - s(\bar{H}_t)$. Consider for all radial geodesics. Then we get

$$s^M - s(H_t) < s^{\bar{M}} - s(\bar{H}_t)$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ and $\gamma_v(t)$ and

$$r_0 < t < \min\{\mathrm{inj}_{S(1)}(p), \mathrm{inj}_{S(1)}(\bar{p})\}.$$

Proof of Corollary 1. If $s^M \le s^{\bar{M}}$, then we get

$$s(H_t) < s(\bar{H}_t)$$

for all corresponding points $\bar{\gamma}_{\bar{v}}(t)$ and $\gamma_v(t)$ and

$$r_0 < t < \min\{\mathrm{inj}_{S(1)}(p), \mathrm{inj}_{S(1)}(\bar{p})\},$$

since $0 \le s^{\bar{M}} - s^M < s(\bar{H}_t) - s(H_t)$ by (19). □

Proof of Corollary 2. First, the Ricci inequality $nk \le \mathrm{Ric}(\gamma_v'(t), \gamma_v'(t))$ implies that

$$(\det A)^{\frac{1}{n}} = x \le \bar{x} = (\det \bar{A})^{\frac{1}{n}}$$

under initial conditions $\det A(r_0) = \det \bar{A}(r_0)$ and $\theta(r_0) \le \bar{\theta}(r_0)$ by Lemma 1.

Second, recall that if $s^M \le s^{\bar{M}}$ with $nk \le \mathrm{Ric}(\gamma_v'(t), \gamma_v'(t))$, then

$$s(H_t) \text{ on } H_t \text{ is less than } \ s(\bar{H}_t) \text{ on } \bar{H}_t$$

by Corollary 1.

By combining the above two facts for $0 < r_0 < r < \min\{\mathrm{inj}_{\mathrm{S(1)}}(\mathrm{p}), \mathrm{inj}_{\mathrm{S(1)}}(\bar{\mathrm{p}})\}$, we obtain

$$\int_{r_0}^{r} \int_{S(1)} s(H_t) \ \det A(t) \, dv \, dt < \int_{r_0}^{r} \int_{S(1)} s(\bar{H}_t) \ \det \bar{A}(t) \, dv \, dt,$$

where dv is the volume element of $S(1)$. Let us denote by dH_t, $d\bar{H}_t$ the volume element of a hypersurface $H_t = \exp_p S(t)$, $\bar{H}_t = \exp_{\bar{p}} S(t)$, respectively. Note that $dH_t = \det A(t) \, dv$ and $d\bar{H}_t = \det \bar{A}(t) \, dv$. So we get

$$\int_{S(1)} s(H_t) \, dH_t < \int_{S(1)} s(\bar{H}_t) \, d\bar{H}_t.$$

□

4. The Lorentzian case

We introduce a Lorentzian version of the volume comparison as in [4], [5]. Let $\mathrm{Fut}(T_pM)$ be the set of all future directed timelike vectors $v \in T_pM$ such that $\exp_p(v)$ is defined for a fixed point $p \in M$. Put

$$H(1) = \{v \in \mathrm{Fut}(T_pM) \,|\, \langle v, v\rangle = -1\}$$

and denote by $H^*(1)$ a compact subset of $H(1)$. Consider a geodesic variation along a unit timelike geodesic γ starting from p which gives level hypersurfaces of a geodesic sphere $\exp_p H^*(1)$. Put

$$(B_p^H)_{r_0}^{r_1} = \{\exp_p(tv) \,|\, v \in H^*(1), \ \ r_0 \le t \le r_1\}$$

for $0 < r_0 < r_1 < \mathrm{cut}_v(p)$. Let M be a globally hyperbolic Lorentzian manifold and γ_v be a unit timelike radial geodesic $\gamma_v(t) = \exp_p tv$ with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$ for all $v \in \mathrm{Fut}(T_pM)$. The Lorentzian volume $V_{r_0}(r_1) = \mathrm{Vol}((B_p^H)_{r_0}^{r_1})$ between $H_{r_0}^* = \exp_p H^*(r_0)$ and $H_{r_1}^* = \exp_p H^*(r_1)$ for $H^*(r_0) = \{v \in \mathrm{Fut}(T_pM) \,|\, \langle v, v\rangle = -r_0^2, \, v \in H^*(1)\}$ is naturally given by

$$V_{r_0}(r_1) = \int_{r_0}^{r_1} \int_{H^*(1)} |\det A| \ dv dt,$$

where dv is the volume element of $H^*(1)$.

By taking the sign into the account for the trace of the Gauss equation (10) along a unit timelike geodesic orthogonal to a spacelike hypersurface H in a Lorentzian manifold M, we get

$$\mathrm{Ric}(w, w) + g(R(\gamma', w)w, \gamma') = \mathrm{Ric}^H(w, w) - \mathrm{tr} S_t g(S_t w, w) + g(S_t^2 w, w). \tag{21}$$

Again, the trace of (21) gives

$$s^M + 2\mathrm{Ric}(\gamma', \gamma') = s(H_t) - (\mathrm{tr}S_t)^2 + \mathrm{tr}S_t^2,$$

which is equal to

$$s^M + 2\mathrm{Ric}(\gamma', \gamma') = s(H_t) - \theta^2 + (\frac{\theta^2}{n-1} + \mathrm{tr}\sigma^2) = s(H_t) - \theta^2 - \theta' - \mathrm{Ric}(\gamma', \gamma').$$

Hence we obtain as in [9]

$$\theta' + \theta^2 + s^M - s(H_t) + 3\mathrm{Ric}(\gamma', \gamma') = 0. \tag{22}$$

Here $\gamma(t)$ is referred to as $\gamma_v(t)$ for all $v \in \mathrm{Fut}(T_pM)$. Let $M, \bar{M}$ be globally hyperbolic Lorentzian manifolds of dimension $n+1$ and γ, $\bar{\gamma}$ be unit timelike geodesics orthogonal to hypersurfaces $H^*_{r_0} = \exp_p H^*(r_0)$, $\bar{H}^*_{r_0} = \exp_{\bar{p}} H^*(r_0)$, for a fixed point $p \in M$, $\bar{p} \in \bar{M}$ respectively. Let A, $\bar{A}$ be an $H^*_{r_0}$, $\bar{H}^*_{r_0}$-Jacobi tensor along γ, $\bar{\gamma}$, respectively. Assume a linear isometry

$$\imath : T_{\gamma(r_0)} H^*_{r_0} \to T_{\bar{\gamma}(r_0)} H^*(r_0)$$

such that $\bar{H}^*(r_0) = \exp_{\bar{\gamma}(r_0)} \circ \imath \circ \exp^{-1}_{\gamma(r_0)} H^*_{r_0}$ and $\imath(\gamma'(r_0)) = \bar{\gamma}'(r_0)$, $\imath(E_i(r_0)) = \bar{E}_i(r_0)$ for an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_{\gamma(r_0)} H^*_{r_0}$ and its parallel basis $\{E_1, E_2, \ldots, E_n\}$ along γ with $E_i(r_0) = e_i$ for each i. Using a linear isometry $\imath : T_{\gamma(r_0)} H^*_{r_0} \to T_{\bar{\gamma}(r_0)} \bar{H}^*_{r_0}$ and the following differential equations along γ and $\bar{\gamma}$,

$$y'' + (s^M - s(H_t) + 3\mathrm{Ric}(\gamma', \gamma'))y = 0, \quad \bar{y}'' + (s^{\bar{M}} - s(\bar{H}_t) + 3\mathrm{Ric}(\bar{\gamma}', \bar{\gamma}'))\bar{y} = 0.$$

We get a Lorentzian analogue of Lemma 2.

Lemma 3. *Let $H^*_{r_0} = \exp_p H^*(r_0)$ and $\bar{H}^*_{r_0} = \exp_{\bar{p}} H^*(r_0)$ be hypersurfaces orthogonal to timelike geodesics γ and $\bar{\gamma}$ in $(n+1)$-dimensional globally hyperbolic Lorentzian manifolds M and $\bar{M}$. If $\theta(r_0) = \bar{\theta}(r_0)$, $\det A(r_0) = \det \bar{A}(r_0)$ and*

$$s^M - s(H_t) + 3Ric(\gamma'(t), \gamma'(t)) \leq s^{\bar{M}} - s(\bar{H}_t) + 3Ric(\bar{\gamma}'(t), \bar{\gamma}'(t)),$$

then we get

$$V_{r_0}(r) \geq \bar{V}_{r_0}(r),$$

where $0 < r_0 < r < \min\{cut_{\mathrm{v}}(\mathrm{p}), cut_{\bar{\mathrm{v}}}(\bar{\mathrm{p}})\}$.

The comparison results of Theorem 1 do not hold in Lorentzian geometry. First, we have

$$nk \leq \mathrm{Ric}(\gamma'(t), \gamma'(t)) \quad \text{implies} \quad V_{r_0}(r) < \bar{V}_{r_0}(r) \tag{23}$$

under the assumption that $R_{\gamma'} \neq R_{\bar{\gamma}'}$ or $H_{r_0} = \exp_p S(r_0)$ is not totally umblic by the same arguments in the proof of Theorem 1.

Second, if $s^M - s(H_t) + 3\mathrm{Ric}(\gamma'(t), \gamma'(t)) \leq s^{\bar{M}} - s(\bar{H}_t) + 3\mathrm{Ric}(\bar{\gamma}'(t), \bar{\gamma}'(t))$ under $\theta(r_0) = \bar{\theta}(r_0)$ and $\det A(r_0) = \det \bar{A}(r_0)$, then we have $V_{r_0}(r) \geq \bar{V}_{r_0}(r)$ by Lemma 3. Hence by contraposition if $V_{r_0}(r) < \bar{V}_{r_0}(r)$, then we get

$$s^M - s(H_t) + 3\mathrm{Ric}(\gamma'(t), \gamma'(t)) > s^{\bar{M}} - s(\bar{H}_t) + 3\mathrm{Ric}(\bar{\gamma}'(t), \bar{\gamma}'(t))$$

which is equivalent to

$$s^M - s(H_t) - (s^{\bar{M}} - s(\bar{H}_t)) > 3(\mathrm{Ric}(\bar{\gamma}'(t), \bar{\gamma}'(t)) - \mathrm{Ric}(\gamma'(t), \gamma'(t))). \tag{24}$$

Thus we note that $nk < \mathrm{Ric}(\gamma', \gamma')$ does not imply

$$s^M - s(H_t) - (s^{\bar{M}} - s(\bar{H}_t)) > 0.$$

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