Korean J. Math. **26** (2018), No. 1, pp. 53–60
https://doi.org/10.11568/kjm.2018.26.1.53

ROTA-BAXTER OPERATORS OF 3-DIMENSIONAL HEISENBERG LIE ALGEBRA

Guangzhi Ji† and Xiuying Hua*

Abstract. In this paper, we consider the question of the Rota-Baxter operators of 3-dimensional Heisenberg Lie algebra on $\mathbb{F}$, where $\mathbb{F}$ is an algebraic closed field. By using the Lie product of the basis elements of Heisenberg Lie algebras, all Rota-Baxter operators of 3-dimensional Heisenberg Lie algebras are calculated and left symmetric algebras of 3-dimensional Heisenberg Lie algebra are determined by using the Yang-Baxter operators.

1. Introduction

Baxter proposed the concept of Rota-Baxter operator in 1960 (see [3]), while Rota further promoted the study of Baxter operator (see [8]). Rota-Baxter operator in various fields of mathematics has been widely used (see [2,4]). This year, many people have described the Rota-Baxter operator on low-dimensional algebra, for example, in [1,6] give the Rota-Baxter operators on low-dimensional pre-Lie algebras, in [7,9] give all Rota-Baxter operators on finite-dimensional Hamilton algebras and 3-, 4- and 5-dimensional Heisenberg Superalgebras. In [4] gives the Rota-Baxter operators on exterior algebras of two variables. By using the

Received December 19, 2017. Revised March 13, 2018. Accepted March 15, 2018.
2010 Mathematics Subject Classification: 17B05, 17B65.
Key words and phrases: Rota-Baxter operators, Heisenberg Lie algebra, Yang-Baxter operators.
† This work was supported by College of Science, Harbin University of Science and Technology.
* Corresponding author.

Lie product of the basis elements of Heisenberg Lie algebras, all Rota-Baxter operators of 3-dimensional Heisenberg Lie algebras are calculated and left symmetric algebras of 3-dimensional Heisenberg Lie algebra are determined by using the Yang-Baxter operators.

2. Definition and basic properties

DEFINITION 2.1. Let G be Lie algebra on $\mathbb{F}$ where $\mathbb{F}$ is a field, we say that R is a Rota-Baxter operator on G, if the following condition holds for any x, y in G:

$$[R(x), R(y)] + \lambda R([x, y]) = R([R(x), y]) + R([(x), R(y)], \tag{1}$$

$\forall x, y \in G, \lambda \in \mathbb{F}$.
In particular, we say that R is a Yang-Baxter operator of G it is the Rota-Baxter operator of the weight $\lambda = 0$. In this case the equation (1) becomes

$$[R(x), R(y)] = R([R(x), y]) + R([(x), R(y)], \qquad \forall x, y \in G \tag{2}$$

which is called the classical Yang-Baxter equation of G and the Rota-Baxter of weight $\lambda = 0$ will be a solution of the classical Yang-Baxter equation of G.

Obviously, $\lambda^{-1}R$ is the Rota-Baxter operator of the weight 1 when $\lambda \neq 0$, hence, We can get all Rota-Baxter operators of non-zero weight by applying the Rota-Baxter operator of weight 1. Hence, we only need to calculate Rota-Baxter operators of the weights 0 and 1.

One of the applications of the Yang-Baxter operators is to construct left symmetric algebras by using these operators and defining a new operation on G as Lemma 2.2.

LEMMA 2.2. *Let G be a Lie algebra and R a solution of the classical Yang-Baxter equation of G. We define a new operation on G as follows:*

$$\begin{aligned} * : G \times G &\longrightarrow \mathbb{F} \\ (x, y) &\longrightarrow x * y := [R(x), y] \qquad \forall x, y \in G \end{aligned}$$

then $(G, *)$ will be a left symmetric algebra.

Now let us to consider the 3-dimensional Heisenberg Lie algebra G with base elements $\{c, e, f\}$ satisfying in the relation

$$\begin{cases} [e,f] = -[f,e] = c \\ [x,y] = 0 \qquad if\ x,y \notin \{c,e,f\} \end{cases}$$

Now let R be a linear operator on G such that

$$\begin{cases} R(c) = a_{11}c + a_{21}e + a_{31}f \\ R(e) = a_{12}c + a_{22}e + a_{32}f \\ R(f) = a_{13}c + a_{23}e + a_{33}f \end{cases}$$

where $a_{ij} \in \mathbb{F}$ for $i,j \in \{1,2,3\}$.

In other words we can write

$$(R(c), R(e), R(f)) = (c,e,f) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3. Main Results

THEOREM 3.1. *There is three types of the Rota-Baxter operators of weight 0 for the 3-dimensional Heisenberg Lie algebra G, which are as follows:*

$$R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & \frac{a_{11}a_{22}+a_{23}a_{32}}{a_{22}-a_{11}} \end{bmatrix} \text{where } a_{22} - a_{11} \neq 0$$

$$R_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & \frac{-a_{11}^2}{a_{23}} & a_{33} \end{bmatrix} \text{where } a_{22} = a_{11}, a_{23} \neq 0$$

$$R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{where } a_{ij} \in \mathbb{F}$$

Proof. Since R is linear operator, so we only need to consider the base elements which are satisfying in the equation (2) which come from the equation (1) by substituting 0 in stead of λ and also we have the equations:

$$(3) \qquad \begin{cases} a_{21}=0 \\ a_{31}=0 \\ (a_{22}-a_{11})a_{33}=a_{11}a_{22}+a_{23}a_{32} \end{cases}$$

where

$$[R(c),R(e)]=R([R(c),e])+R([c,R(e)]) \implies a_{31}=0$$
$$[R(c),R(f)]=R([R(c),f])+R([c,R(f)]) \implies a_{21}=0$$
$$[R(e),R(f)]=R([R(e),f])+R([e,R(f)])$$
$$\implies (a_{22}-a_{11})a_{33}=a_{11}a_{22}+a_{23}a_{32}$$

Discuss the situation:
Situation 1: If $a_{22}\neq a_{11}$, then (3) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{33}=\frac{a_{11}a_{22}+a_{23}a_{32}}{a_{22}-a_{11}} \end{cases}$$

and this will yield us to the Rota-Baxter operator

$$R_1=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & \frac{a_{11}a_{22}+a_{23}a_{32}}{a_{22}-a_{11}} \end{bmatrix} \text{where } a_{22}-a_{11}\neq 0.$$

Situation 2: If $a_{22}=a_{11}$, $a_{23}\neq 0$, then (3) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{32}=\frac{-a_{11}^2}{a_{23}} \end{cases}$$

and this will yield us to the Rota-Baxter operator

$$R_2=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & \frac{-a_{11}^2}{a_{23}} & a_{33} \end{bmatrix} \text{where } a_{22}=a_{11}, a_{23}\neq 0.$$

Situation 3: If $a_{11}=a_{22}$, $a_{23}=0$, then (3) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{11}=a_{22}=a_{23} \end{cases}$$

and this will yield us to the Rota-Baxter operator

$$R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}.$$

□

THEOREM 3.2. *The Rota-Baxter operators of weight 1 of 3-dimensional Heisenberg Lie algebra G are the following:*

$$R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & \frac{a_{11}a_{22}-a_{11}+a_{23}a_{32}}{a_{22}-a_{11}} \end{bmatrix} \textit{where } a_{22}-a_{11} \neq 0$$

$$R_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & \frac{a_{11}-a_{11}^2}{a_{23}} & a_{33} \end{bmatrix} \textit{where } a_{22} = a_{11}, a_{23} \neq 0$$

$$R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \textit{where } a_{ij} \in \mathbb{F}$$

$$R_4 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \textit{where } a_{ij} \in \mathbb{F}$$

Proof. Since R is linear operator, hence we only need to consider the base elements which are satisfying in the equation

$$[R(x), R(y)] + R([x, y]) = R([R(x), y]) + R([(x), R(y)]$$

which come from the equation (1) by substituting 1 in stead of λ and also we have the equations:

$$(4) \qquad \begin{cases} a_{21} = 0 \\ a_{31} = 0 \\ (a_{22} - a_{11})a_{33} = a_{11}a_{22} - a_{11} + a_{23}a_{32} \end{cases}$$

where

$$[R(c), R(e)] = R([R(c), e]) + R([c, R(e)]) \implies a_{31} = 0$$
$$[R(c), R(f)] = R([R(c), f]) + R([c, R(f)]) \implies a_{21} = 0$$
$$[R(e), R(f)] = R([R(e), f]) + R([e, R(f)])$$

$$\Longrightarrow\ (a_{22}-a_{11})a_{33}=a_{11}a_{22}-a_{11}+a_{23}a_{32}$$

Discuss the situation:
Situation 1: If $a_{22}\neq a_{11}$, then (4) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{33}=\frac{a_{11}a_{22}-a_{11}+a_{23}a_{32}}{a_{22}-a_{11}} \end{cases}$$

and this will yield us to the Rota-Baxter operator

$$R_1=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & \frac{a_{11}a_{22}-a_{11}+a_{23}a_{32}}{a_{22}-a_{11}} \end{bmatrix} \text{where } a_{22}-a_{11}\neq 0.$$

Situation 2: If $a_{22}=a_{11}, a_{23}\neq 0$, then (4) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{32}=\frac{a_{11}-a_{11}^2}{a_{23}} \end{cases}$$

and this will yield us to the Rota-Baxter operator

$$R_2=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & \frac{a_{11}-a_{11}^2}{a_{23}} & a_{33} \end{bmatrix} \text{where } a_{22}=a_{11}, a_{23}\neq 0.$$

Situation 3: If $a_{11}=a_{22}$, $a_{23}=0$, then $a_{11}^2-a_{11}=0$
(1). If $a_{11}=0$, then (4) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{11}=a_{22}=a_{23}=0 \end{cases}$$

which will yield us to the Rota-Baxter operator

$$R_3=\begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{where } a_{ij}\in\mathbb{F}.$$

(2). If $a_{11}=1$, then (4) becomes

$$\begin{cases} a_{21}=0 \\ a_{31}=0 \\ a_{23}=0 \\ a_{11}=a_{22}=1 \end{cases}$$

which will yield us to the Rota-Baxter operator

$$R_4 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}.$$

□

THEOREM 3.3. *The structure of left symmetric algebra of 3-dimensional Heisenberg Lie algebra*
1) $e * e = -a_{32}c$, $f * f = a_{23}c$, $e * f = a_{22}c$, $f * e = \frac{a_{23}a_{32} - a_{11}a_{22}}{a_{22} - a_{11}}c$.
2) $e * e = \frac{a_{11}^2}{a_{23}}c$, $f * f = a_{23}c$, $e * f = a_{22}c$, $f * e = -a_{33}c$.
3) $e * e = -a_{32}c$, $f * e = -a_{33}c$.

Proof. Considering the application of Yang-Baxter operators, we can calculate directly the structure of left symmetric algebra of Heisenberg Lie algebra by lemma 2.2 and theorem 3.1. □

COROLLARY 3.4. *The homomorphic operator of 3-dimensional Heisenberg Lie algebra of weight 0 is*

$$R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

where $a_{ij} \in \mathbb{F}$

The homomorphic operator of 3-dimensional Heisenberg Lie algebra of weight 1 is

$$R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

where $a_{ij} \in \mathbb{F}$.

COROLLARY 3.5. *Neither of the 3-dimensional Heisenberg Lie algebra of weight 0 and weight 1 have isomorphic operators.*

References

[1] H.H.An and C.M.Bai *From Rota-Baxter algebras to pre-Lie algebras*, J. Phys. A: Math. Theor. **41** (2008), 015201–015219.
[2] F.V.Atkinson, *Some aspects of Baxter's functional equation*, J. Math. Anal. Appl. **07** (1963), 1–30.

[3] G.Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pacific J. Math. **10** (1960), 731–742.
[4] G.C.Rota, *Ten mathematics problems I will never solve*, Mitt. Dtsch. Math,-Ver. **02** (1998), 45–52.
[5] X.Y.Hua and W.D.Liu, *Rota-Baxter Operators on Exterior Algebra*, Journal of Harbin University of Science and Technology. **18** (4) (2013), 125–128.
[6] G.Li and Keigher W, *Baxter algebras and shuffle products*, Adv. Math. **150** (2000), 117–149.
[7] X.X.Li, D.P.Hou, and C.M.Bai, *Rota-Baxter operators on pre-lie algebras*, Nonlinear Math. Phys. **14** (2) (2007), 269–289.
[8] S.Q.Lv and W.D.Liu, *Rota-Baxter Operators on Heisenberg Lie Superalgebras of Dimension Five*, Journal of Mathematics in Practice and Theory. **46** (20) (2016), 248–258.
[9] H.Y.Wen, *Rota-Baxter Operators on Hamilton algebra and Heisenberg Lie Superalgebras*, Harbin Normal University. (2014).

Guangzhi Ji
No.52 xuefu road, nangang district, Harbin city, heilongjiang province
College of Science
Harbin University of Science and Technology
Harbin 150080, China
E-mail: 245368609@qq.com

Xiuying Hua
No.52 xuefu road, nangang district, Harbin city, heilongjiang province
College of Science
Harbin University of Science and Technology
Harbin 150080, China
E-mail: huaxiuyingnihao@163.com