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ON SOME GROWTH ANALYSIS OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS FROM THE VIEW POINT OF THEIR RELATIVE (p, q)-TH TYPE AND RELATIVE (p, q)-TH WEAK TYPE

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Abstract. The main aim of this paper is to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their relative (p, q)-th order, relative (p, q)-th lower order, relative (p, q)-th type and relative (p, q)-th weak type where p and q are any two positive integers.

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. If f is non-constant then it has the following property:

Property (A) [1] : A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large values of r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [1].

Also for a non-constant entire function f, $M_f(r)$ is strictly increasing and continuous and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is

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such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$.

The integrated counting function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ of a-points (distinct a-points) of f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(r,a)}{t} dt + \overline{n}_f(0,a) \log r\right) ,$$

where we denote by $n_f(t,a)$ $(\overline{n}_f(t,a))$ the number of a-points (distinct a-points) of f in $|z| \le t$ and an ∞ -point is a pole of f . In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively. The function $N_f(r,a)$ is called the enumerative function. On the other hand, the function $m_f(r) \equiv m_f(r,\infty)$ known as the proximity function is defined as

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

$$\text{where } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0$$

and an ∞ -point is a pole of f .

Analogously, $m_{\frac{1}{f-a}}(r) \equiv m_f(r,a)$ is defined when a is not an ∞-point of f.

Thus the Nevanlinna's characteristic function $T_f(r)$ corresponding to f is defined as

$$T_f(r) = N_f(r) + m_f(r) \ .$$

When f is entire, $T_f(r)$ coincides with $m_f(r)$ as $N_f(r) = 0$.

Moreover, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exist and is such that $\lim_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's Characteristic functions of the meromorphic functions f and g.

Now we state the following notation which will be needed in the sequel:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\log^{[0]} x = x;$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots \text{ and}$$
$$\exp^{[0]} x = x.$$

Taking this into account, Juneja, Kapoor and Bajpai [8] defined the (p, q)-th order and (p, q)-th lower order of an entire function f respectively as follows:

$$\begin{matrix} \rho_f(p,q) \\ \lambda_f(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$. When f is meromorphic one can easily verify that

$$\begin{matrix} \rho_f(p,q) \\ \lambda_f(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$. If $p = l$ and $q = 1$ then we write $\rho_f(l, 1) = \rho_f^{[l]}$ and $\lambda_f(l, 1) = \lambda_f^{[l]}$ where $\rho_f^{[l]}$ and $\lambda_f^{[l]}$ are respectively known as generalized order and generalized lower order of f. Also for $p = 2$ and $q = 1$ we respectively denote $\rho_f(2, 1)$ and $\lambda_f(2, 1)$ by ρ_f and λ_f, where ρ_f and λ_f are the classical growth indicator known as order and lower order of f.

In this connection we just recall the following definition :

Definition 1. [8] An entire function f is said to have index-pair (p, q), $p \geq q \geq 1$ if $b < \rho_f(p, q) < \infty$ and $\rho_f(p-1, q-1)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ if $p > q$. Moreover if $0 < \rho_f(p, q) < \infty$, then

$$\rho_f(p-n, q) = \infty \text{ for } n < p, \ \rho_f(p, q-n) = 0 \text{ for } n < q \text{ and}$$
$$\rho_f(p+n, q+n) = 1 \text{ for } n = 1, 2, ...$$

Similarly for $0 < \lambda_f(p, q) < \infty$, one can easily verify that

$$\lambda_f(p-n, q) = \infty \text{ for } n < p, \ \lambda_f(p, q-n) = 0 \text{ for } n < q \text{ and}$$

$$\lambda_f (p+n, q+n) = 1 \text{ for } n = 1, 2, ...$$

An entire function for which (p,q)-th order and (p,q)-th lower order are the same is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

Analogously one can easily verify that the Definition 1 of index-pair can also be applicable for a meromorphic function f.

In order to compare the growth of entire functions having the same (p,q)-th order, Juneja, Kapoor and Bajpai [9] also introduced the concepts of (p,q)-th type and (p,q)-th lower type in the following manner :

Definition 2. [9] The (p,q)-th type and the (p,q)-th lower type of entire function f having finite positive (p,q)-th order $\rho_f (p,q)$ $(b < \rho_f (p,q) < \infty)$ $(p, q$ are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q)$ are defined as :

$$\begin{matrix} \sigma_f (p,q) \\ \overline{\sigma}_f (p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_f (r)}{\left(\log^{[q-1]} r\right)^{\rho_f (p,q)}},$$

$$0 \leq \overline{\sigma}_f (p,q) \leq \sigma_f (p,q) \leq \infty .$$

If f is meromorphic function with $0 < \rho_f (p,q) < \infty$, then

$$\begin{matrix} \sigma_f (p,q) \\ \overline{\sigma}_f (p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} T_f (r)}{\left(\log^{[q-1]} r\right)^{\rho_f (p,q)}},$$

$$0 \leq \overline{\sigma}_f (p,q) \leq \sigma_f (p,q) \leq \infty .$$

Likewise, to compare the growth of entire functions having the same (p,q)-th lower order, one can also introduced the concepts of (p,q)-th weak type in the following manner :

Definition 3. The (p,q)-th weak type of entire function f having finite positive (p,q)-th lower order $\lambda_f (p,q)$ $(b < \lambda_f (p,q) < \infty)$ is defined as :

$$\tau_f (p,q) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} M_f (r)}{\left(\log^{[q-1]} r\right)^{\lambda_f (p,q)}}$$

where p, q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$.

Similarly one may define the growth indicator $\overline{\tau}_f(p,q)$ of an entire function f in the following way :

$$\overline{\tau}_f(p,q) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_f(p,q)}} \ , \ b < \lambda_f(p,q) < \infty$$

where p, q are any two positive integers, $b = 1$ if $p = q$ and $b = 0$ for $p > q$.

If f is meromorphic function with $0 < \lambda_f(p,q) < \infty$, then

$$\begin{matrix} \overline{\tau}_f(p,q) \\ \tau_f(p,q) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-2]} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_f(p,q)}}.$$

It is obvious that $0 \le \tau_f(p,q) \le \overline{\tau}_f(p,q) \le \infty$.

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$ its maximum modulus function and Nevanlinna's characteristic function are strictly increasing and continuous. Hence there exists its inverse functions $M_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$ and $T_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ with $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection, Bernal [1, 2] introduced the definition of relative order of an entire function f with respect to another entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0.\} \\ &= \overline{\lim_{r\to\infty}} \frac{\log M_g^{-1} M_f(r)}{\log r} \ . \end{aligned}$$

The definition coincides with the classical one [11] if $g(z) = \exp z$. Similarly one can define the relative lower order of an entire function f with respect to another entire function g denoted by $\lambda_g(f)$ as follows :

$$\lambda_g(f) = \underline{\lim_{r\to\infty}} \frac{\log M_g^{-1} M_f(r)}{\log r} \ .$$

Extending this notion, Ruiz et. al. [10] introduced the definition of (p,q) -th relative order of a entire function with respect to an entire function in the light of index pair which is as follows :

Definition 4. [10] Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$. Then the relative (p, q)-th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

Analogously, the relative (p, q)-th lower order of f with respect to g is defined by:

$$\lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

In order to refine the above growth scale, now we intend to introduce the definitions of an another growth indicators, such as relative (p, q)-th type and relative (p, q)-th lower type of entire function with respect to another entire function in the light of their index-pair which are as follows:

Definition 5. Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are all positive integers such that $m \geq \max\{p, q\}$. The relative (p, q)-th type and relative (p, q)-th lower type of entire function f with respect to the entire function g having finite positive relative (p, q) th order $\rho_g^{(p,q)}(f)$ $\left(0 < \rho_g^{(p,q)}(f) < \infty\right)$ are defined as :

$$\begin{matrix} \sigma_g^{(p,q)}(f) \\ \overline{\sigma}_g^{(p,q)}(f) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}}.$$

Analogously, to determine the relative growth of two entire functions having same non zero finite relative (p, q)-th lower order with respect to another entire function, one can introduced the definition of relative (p, q)-th weak type of an entire function f with respect to another entire function g of finite positive relative (p, q) -th lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 6. Let f and g be any two entire functions having finite positive relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ $\left(0 < \lambda_g^{(p,q)}(f) < \infty\right)$ where p and q are any two positive integers. Then the relative (p, q)-th weak type of entire function f with respect to the entire function g is defined as :

$$\tau_g^{(p,q)}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}} .$$

Similarly one can define another growth indicator $\overline{\tau}_g^{(p,q)}(f)$ in the following way:

$$\overline{\tau}_g^{(p,q)}(f) = \overline{\lim}_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}} .$$

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q)-th order of meromorphic functions. Debnath et. al. [4] also introduced such definition in the light of index pair in the following manner:

Definition 7. [4] Let f be any meromorphic function and g be any entire function with index-pairs (m,q) and (m,p) respectively where p,q,m are all positive integers such that $m \geq p$ and $m \geq q$. Then the relative (p,q)th order of f with respect to g is defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim}_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

Similarly, one can define the relative (p,q)-th lower order of a meromorphic function f with respect to an entire function g denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_g^{-1} T_f(r)}{\log^{[q]} r}.$$

Now we state the following two definitions relating to the meromorphic function which are similar to Definition 5 and Definition 6 respectively.

Definition 8. Let f be a meromorphic function and g be an entire function with index-pairs (m,q) and (m,p) respectively where p,q,m are all positive integers such that $m \geq \max\{p,q\}$. The relative (p,q)-th type and relative (p,q)-th lower type of meromorphic function f with respect to the entire function g having finite positive relative (p,q) th order $\rho_g^{(p,q)}(f)$ $\left(0<\rho_g^{(p,q)}(f)<\infty\right)$ are defined as :

$$\begin{matrix}\sigma_g^{(p,q)}(f)\\ \overline{\sigma}_g^{(p,q)}(f)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\left(\log^{[q-1]} r\right)^{\rho_g^{(p,q)}(f)}} .$$

Definition 9. Let f be a meromorphic function and g be an entire function having finite positive relative (p,q)-th lower order $\lambda_g^{(p,q)}(f)$ $\left(0<\lambda_g^{(p,q)}(f)<\infty\right)$ where p and q are any two positive integers. Then the relative (p,q)-th weak type of meromorphic function f with respect to the entire function g is defined as :

$$\tau_g^{(p,q)}(f) = \varliminf_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}} .$$

Similarly one can define another growth indicator $\overline{\tau}_g^{(p,q)}(f)$ in the following way:

$$\overline{\tau}_g^{(p,q)}(f) = \varlimsup_{r\to\infty} \frac{\log^{[p-1]} T_g^{-1} T_f(r)}{\left(\log^{[q-1]} r\right)^{\lambda_g^{(p,q)}(f)}} .$$

In this paper we wish to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their relative (p,q)-th order, relative (p,q)-th type and relative (p,q)-th weak type where p and q are any two positive integers which in fact extend some results of [6]. We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [7] and [12].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [3] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)) .$$

Lemma 2. [5] *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right) .$$

3. Main Results

In this section we present the main results of the paper.

Theorem 1. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, $\sigma_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\sigma_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)} .$$

Proof. Let us suppose that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 2. Since $T_h^{-1}(r)$ is an increasing function r, it follows from Lemma 1, Lemma 2 and the inequality $T_g(r) \leq \log M_g(r)$ $\{cf.$ [7]$\}$ for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1} T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\} T_f(M_g(r))\right] \\ i.e.,\ T_h^{-1} T_{f\circ g}(r) &\leqslant \alpha\left[T_h^{-1} T_f(M_g(r))\right]^{\delta} \end{aligned}$$

$$i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \log^{[p]} T_h^{-1} T_f(M_g(r)) + O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} M_g(r) + O(1)$$

$$i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[m-1]} M_g(r) + O(1)$$

$$\begin{aligned} &i.e.,\ \log^{[p]} T_h^{-1} T_{f\circ g}(r) \\ &\leqslant \left(\rho_h^{(p,q)}(f) + \varepsilon\right)(\sigma_g(m,n) + \varepsilon)\left(\log^{[n-1]} r\right)^{\rho_g(m,n)} + O(1) . \end{aligned} \tag{1}$$

Now from the definition of $\lambda_h^{(p,q)}(f)$, we obtain for all sufficiently large values of r that

$$\begin{aligned} &\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right) \\ &\geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right)\left(\log^{[n-1]} r\right)^{\rho_g(m,n)} . \end{aligned} \tag{2}$$

Therefore from (1) and (2), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p]} T_h^{-1} T_f \left(\exp^{[q]} \left(\log^{[n-1]} r \right)^{\rho_g(m,n)} \right)}$$

$$\leq \frac{\left(\rho_h^{(p,q)}(f) + \varepsilon \right) \left(\sigma_g(m,n) + \varepsilon \right) \left(\log^{[n-1]} r \right)^{\rho_g(m,n)} + O(1)}{\left(\lambda_h^{(p,q)}(f) - \varepsilon \right) \left(\log^{[n-1]} r \right)^{\rho_g(m,n)}},$$

$$i.e., \varlimsup_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p]} T_h^{-1} T_f \left(\exp^{[q]} \left(\log^{[n-1]} r \right)^{\rho_g(m,n)} \right)} \leq \frac{\sigma_g(m,n) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)} .$$

Thus the theorem is established. □

In the line of Theorem 1 the following theorem can be proved :

THEOREM 2. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\varlimsup_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p]} T_h^{-1} T_f \left(\exp^{[q]} \left(\log^{[n-1]} r \right)^{\rho_g(m,n)} \right)} \leq \frac{\overline{\tau}_g(m,n) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)} .$$

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 1 and Theorem 2 respectively.

THEOREM 3. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\sigma_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\varlimsup_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p]} T_h^{-1} T_g \left(\exp^{[n]} \left(\log^{[n-1]} r \right)^{\rho_g(m,n)} \right)} \leq \frac{\sigma_g(m,n) \cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} .$$

THEOREM 4. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\overline{\tau}_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

Using the notion of (p,q)-th lower type we may state the following two theorems without proof because it can be carried out in the line of Theorem 1 and Theorem 3 respectively.

THEOREM 5. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\overline{\sigma}_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

THEOREM 6. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\overline{\sigma}_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\overline{\sigma}_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}.$$

Further using the notion of (p,q)-th weak type we may also state the following two theorems without proof because it can be carried out in the line of Theorem 2 and Theorem 4 respectively.

THEOREM 7. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\tau_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\tau_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\lambda_h^{(p,q)}(f)}.$$

Theorem 8. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\tau_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\tau_g(m,n)\cdot \rho_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} \ .$$

Now we state the following six theorems without their proofs as those can easily be carried out in the line of Theorem 1.

Theorem 9. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\sigma_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \sigma_g(m,n) \ .$$

Theorem 10. *Let f be meromorphic, g and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) < \infty$ or $0 < \rho_h^{(p,q)}(f) < \infty$ and $\overline{\tau}_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_f\left(\exp^{[q]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \overline{\tau}_g(m,n) \ .$$

Theorem 11. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g) > 0$, $\lambda_h^{(p,q)}(f) < \infty$ and $\sigma_g(m,n) < \infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m - 1 = q$. Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p]} T_h^{-1} T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\sigma_g(m,n)\cdot \lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)} \ .$$

Theorem 12. *Let f be meromorphic, g and h be any two entire functions such that $\rho_h^{(p,n)}(g) > 0$, $\rho_h^{(p,q)}(f) < \infty$ and $\sigma_g(m,n) < \infty$ and*

h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m-1=q$. Then

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}T_{f\circ g}(r)}{\log^{[p]} T_h^{-1}T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\sigma_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}\ .$$

Theorem 13. *Let f be meromorphic, g and h be any two entire functions such that $\lambda_h^{(p,n)}(g)>0$, $\lambda_h^{(p,q)}(f)<\infty$ and $\overline{\tau}_g(m,n)<\infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m-1=q$. Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}T_{f\circ g}(r)}{\log^{[p]} T_h^{-1}T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\overline{\tau}_g(m,n)\cdot\lambda_h^{(p,q)}(f)}{\lambda_h^{(p,n)}(g)}\ .$$

Theorem 14. *Let f be meromorphic, g and h be any two entire functions such that $\rho_h^{(p,n)}(g)>0$, $\rho_h^{(p,q)}(f)<\infty$ and $\overline{\tau}_g(m,n)<\infty$ and h satisfies the Property (A) where p, q, m, n are all positive integers with $m \geq n$ and $m-1=q$. Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}T_{f\circ g}(r)}{\log^{[p]} T_h^{-1}T_g\left(\exp^{[n]}\left(\log^{[n-1]} r\right)^{\rho_g(m,n)}\right)} \leq \frac{\overline{\tau}_g(m,n)\cdot\rho_h^{(p,q)}(f)}{\rho_h^{(p,n)}(g)}\ .$$

Theorem 15. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\rho_g(m,n)$, (iii) $\sigma_g(m,n)<\infty$ and (iv) $0<\sigma_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]} T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1}T_f(r)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\sigma_h^{(p,q)}(f)}\ .$$

Proof. In view of condition (ii),we obtain from (1) for all sufficiently large values of r that

$$\begin{aligned}&\log^{[p]} T_h^{-1}T_{f\circ g}(r)\\ &\quad\leqslant\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\sigma_g(m,n)+\varepsilon\right)\left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)}+O(1)\ .\end{aligned}\tag{3}$$

Again in view of Definition 8 we get for a sequence of values of r tending to infinity that

$$\log^{[p-1]} T_h^{-1} T_f(r) \geq \left(\sigma_h^{(p,q)}(f) - \varepsilon\right) \left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)} . \tag{4}$$

Now from (3) and (4), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \left(\sigma_g(m,n) + \varepsilon\right) \left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)} + O(1)}{\left(\sigma_h^{(p,q)}(f) - \varepsilon\right) \left[\log^{[n-1]} r\right]^{\rho_h^{(p,q)}(f)}} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\underline{\lim}_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f) \cdot \sigma_g(m,n)}{\sigma_h^{(p,q)}(f)} .$$

□

Using the notion of (p,q)-th lower type and relative (p,q)-th lower type , we may state the following theorem without its proof as it can be carried out in the line of Theorem 15.

THEOREM 16. *Let f be meromorphic, g and h be any two entire functions such that (i) $0 < \rho_h^{(p,q)}(f) < \infty$, (ii) $\rho_h^{(p,q)}(f) = \rho_g(m,n)$, (iii) $\overline{\sigma}_g(m,n) < \infty$ and (iv) $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $m - 1 = n = q$. Also let h satisfies the Property (A). Then*

$$\underline{\lim}_{r \to \infty} \frac{\log^{[p]} T_h^{-1} T_{f \circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f) \cdot \overline{\sigma}_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)} .$$

Similarly using the notion of (p,q)-th type and relative (p,q)-th lower type one may state the following two theorems without their proofs because those can also be carried out in the line of Theorem 15.

THEOREM 17. *Let f be meromorphic, g and h be any two entire functions such that (i) $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$, (ii) $\rho_h^{(p,q)}(f) = \rho_g(m,n)$, (iii) $\sigma_g(m,n) < \infty$ and (iv) $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ where*

p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\lambda_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Theorem 18. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\rho_g(m,n)$, (iii) $\sigma_g(m,n)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varlimsup_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)}.$$

Now using the concept of relative (p,q)-th weak type, we may state the subsequent four theorems without their proofs since those can be carried out in the line of Theorem 15, Theorem 16, Theorem 17 and Theorem 18 respectively.

Theorem 19. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda_g(m,n)$, (iii) $\overline{\tau}_g(m,n)<\infty$ and (iv) $0<\overline{\tau}_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}_g(m,n)}{\overline{\tau}_h^{(p,q)}(f)}.$$

Theorem 20. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda_g(m,n)$, (iii) $\tau_g(m,n)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f)\cdot\tau_g(m,n)}{\tau_h^{(p,q)}(f)}.$$

Theorem 21. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\lambda_g(m,n)$, (iii) $\overline{\tau}_g(m,n)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where p, q, m, n are all* *positive integers with* $m-1=n=q$. *Also let* h *satisfies the Property (A). Then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\tau}_g(m,n)}{\tau_h^{(p,q)}(f)}\ .$$

Theorem 22. *Let* f *be meromorphic,* g *and* h *be any two entire functions such that (i)* $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, *(ii)* $\lambda_h^{(p,q)}(f)=\lambda_g(m,n)$, *(iii)* $\overline{\tau}_g(m,n)<\infty$ *and (iv)* $0<\tau_h^{(p,q)}(f)<\infty$ *where* p,q,m,n *are all positive integers with* $m-1=n=q$. *Also let* h *satisfies the Property (A). Then*

$$\overline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}_g(m,n)}{\tau_h^{(p,q)}(f)}\ .$$

We may now state the following theorems without their proofs based on relative (p,q)-th type and relative (p,q)-th weak type:

Theorem 23. *Let* f *be meromorphic,* g *and* h *be any two entire functions such that (i)* $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, *(ii)* $\lambda_h^{(p,q)}(f)=\rho_g(m,n)$, *(iii)* $\sigma_g(m,n)<\infty$ *and (iv)* $0<\overline{\tau}_h^{(p,q)}(f)<\infty$ *where* p,q,m,n *are all positive integers with* $m-1=n=q$. *Also let* h *satisfies the Property (A). Then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\overline{\tau}_h^{(p,q)}(f)}\ .$$

Theorem 24. *Let* f *be meromorphic,* g *and* h *be any two entire functions such that (i)* $0<\rho_h^{(p,q)}(f)<\infty$, *(ii)* $\rho_h^{(p,q)}(f)=\lambda_g(m,n)$, *(iii)* $\overline{\tau}_g(m,n)<\infty$ *and (iv)* $0<\sigma_h^{(p,q)}(f)<\infty$ *where* p,q,m,n *are all positive integers with* $m-1=n=q$. *Also let* h *satisfies the Property (A). Then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\tau}_g(m,n)}{\sigma_h^{(p,q)}(f)}\ .$$

Theorem 25. *Let* f *be meromorphic,* g *and* h *be any two entire functions such that (i)* $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, *(ii)* $\lambda_h^{(p,q)}(f)=\rho_g(m,n)$, *(iii)* $\overline{\sigma}_g(m,n)<\infty$ *and (iv)* $0<\tau_h^{(p,q)}(f)<\infty$ *where*

p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\overline{\sigma}_g(m,n)}{\tau_h^{(p,q)}(f)}\,.$$

Theorem 26. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\lambda_g(m,n)$, (iii) $\tau_g(m,n)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\tau_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)}\,.$$

Theorem 27. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\rho_g(m,n)$, (iii) $\sigma_g(m,n)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\tau_h^{(p,q)}(f)}\,.$$

Theorem 28. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\rho_h^{(p,q)}(f)=\lambda_g(m,n)$, (iii) $\overline{\tau}_g(m,n)<\infty$ and (iv) $0<\overline{\sigma}_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varliminf_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\lambda_h^{(p,q)}(f)\cdot\overline{\tau}_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)}\,.$$

Theorem 29. *Let f be meromorphic, g and h be any two entire functions such that (i) $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$, (ii) $\lambda_h^{(p,q)}(f)=\rho_g(m,n)$, (iii) $\sigma_g(m,n)<\infty$ and (iv) $0<\tau_h^{(p,q)}(f)<\infty$ where p, q, m, n are all positive integers with $m-1=n=q$. Also let h satisfies the Property (A). Then*

$$\varlimsup_{r\to\infty}\frac{\log^{[p]}T_h^{-1}T_{f\circ g}(r)}{\log^{[p-1]}T_h^{-1}T_f(r)}\leq\frac{\rho_h^{(p,q)}(f)\cdot\sigma_g(m,n)}{\tau_h^{(p,q)}(f)}\,.$$

THEOREM 30. *Let f be meromorphic, g and h be any two entire functions such that (i) $0 < \rho_h^{(p,q)}(f) < \infty$, (ii) $\rho_h^{(p,q)}(f) = \lambda_g(m,n)$, (iii) $\overline{\tau}_g(m,n) < \infty$ and (iv) $0 < \overline{\sigma}_h^{(p,q)}(f) < \infty$ where p, q, m, n are all positive integers with $m - 1 = n = q$. Also let h satisfies the Property (A). Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r)}{\log^{[p-1]} T_h^{-1} T_f(r)} \leq \frac{\rho_h^{(p,q)}(f) \cdot \overline{\tau}_g(m,n)}{\overline{\sigma}_h^{(p,q)}(f)} .$$

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References

[1] L. Bernal *Orden relative de crecimiento de funciones enteras*, Collect. Math. **39** (1988), 209–229.

[2] L. Bernal *Crecimiento relativo de funciones enteras. Contribuci on al estudio de lasfunciones enteras conìndice exponencial finito*, Doctoral Dissertation, University of Seville, Spain, 1984.

[3] W. Bergweiler *On the Nevanlinna characteristic of a composite function*, Complex Variables. **10** (1988), 225–236.

[4] L. Debnath, S. K. Datta, T. Biswas and A. Kar *Growth of meromorphic functions depending on (p, q)-th relative order*, Facta Universititatis series: Mathematics and Informatics **31** (3) (2016), 691–705.

[5] S. K. Datta, T. Biswas and C. Biswas *Measure of growth ratios of composite entire and meromorphic functions with a focus on relative order*, International J. of Math. Sci. & Engg. Appls. (IJMSEA) **8** (4) (2014), 207–218.

[6] S. K. Datta, T. Biswas and C. Biswas: Relative type and relative weak type oriented growth analysis of composite entire and meromorphic functions, General Mathematics Notes (GMN) **25** (2) (December, 2014), pp. 41–49.

[7] W.K. Hayman : Meromorphic Functions, The Clarendon Press, Oxford (1964).

[8] O. P. Juneja, G. P. Kapoor and S. K. Bajpai : On the (p, q)-order and lower (p, q)-order of an entire function, J. Reine Angew. Math. **282** (1976), 53–67.

[9] O. P. Juneja, G. P. Kapoor and S. K. Bajpai : On the (p, q)-type and lower (p, q)-type of an entire function, J. Reine Angew. Math. **290** (1977), 180–190.

[10] L. M. S. Ruiz, S. K. Datta, T. Biswas and G. K. Mondal: On the (p, q)-th relative order oriented growth properties of entire functions, Abstract and Applied Analysis, Vol.2014, Article ID 826137, 8 pages, http://dx.doi.org/10.1155/2014/826137

[11] E.C. Titchmarsh : The theory of functions , 2nd ed. Oxford University Press, Oxford , 1968.

[12] G. Valiron : Lectures on the general theory of integral functions, Chelsea Publishing Company, 1949.

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