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EFFECT OF PERTURBATION IN THE SOLUTION OF FRACTIONAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we study the initial value problem for neutral functional differential equations involving Caputo fractional derivative of order $\alpha \in (0,1)$ with infinite delay. Some sufficient conditions for the uniqueness and continuous dependence of solutions are established by virtue of fractional calculus and Banach fixed point theorem. Some results obtained showed that the solution was closely related to the conditions of delays and minor changes in the problem. An example is provided to illustrate the main results.

1. INTRODUCTION

This paper is concerned with the uniqueness and continuous dependence of solutions for neutral functional differential equations with fractional order and infinite delay that described by

$$^cD^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], \tag{1.1}$$

$$x_0 = \varphi \in \mathcal{B}, \tag{1.2}$$

where $0 < \alpha < 1$, $^cD^\alpha$ is the standard Caputo fractional derivative of order α, $f, g : [0, b] \times \mathcal{B} \to \mathbb{R}$ $(b > 0)$ are given functions satisfying some assumptions that will be specified in Section 3, and $\mathcal{B}$ the phase space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, which will be specified in Section 2.

Recently, the fractional differential equations became an important branch in mathematics and its applications such as mechanics, physics, chemistry, engineering. So a lot of authors and researchers have contributed in preparing books and papers about this field, for more details,

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see the monographs [1, 15, 17, 19, 21], and the papers [7, 8, 11, 16, 20], and the references therein.

The fractional delay of neutral functional differential equations appear frequently in applications as the model of equations, and for this reason, these equations have been extensively studied. Especially, the results dealing with infinite delay has received great attention in the last few years, see for instance. [2, 3, 4, 5, 6, 10, 13, 18, 22, 23, 25]. For example in [5], the authors used the Leray-Schauder type nonlinear alternative and contraction mapping principle to investigate the existence and uniqueness of solutions for the following problem

$$D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b],$$

$$x(t) = \varphi(t), \qquad\qquad t \in (-\infty, 0],$$

where $0 < \alpha < 1$, D^{α} is the Riemann-Liouville fractional derivative, $\varphi \in \mathcal{B}$, $\varphi(0) = 0$, $\mathcal{B}$ is a phase space and $f, g : [0, b] \times \mathcal{B} \to \mathbb{R}$ are suitable functions satisfying some assumptions with $g(0, \varphi) = 0$. In [4], the following problem was considered

$$^{c}D^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [t_0, \infty), \tag{1.3}$$

$$x_{t_0} = \varphi \in \mathcal{C}, \tag{1.4}$$

where $0 < \alpha < 1$, $^{C}D^{\alpha}$ is the Caputo fractional operator, $f, g : [t_0, \infty) \times \mathcal{C} \to \mathbb{R}^n$ are appropriate functions satisfying some hypotheses, and $\mathcal{C}$ is called a space of continuous functions on $[-\tau, 0]$. The authors employed the Krasnoselskii's fixed point theorem to study the existence result of the problem (1.3)-(1.4).

The main purpose of this paper is to discuss the uniqueness of solutions and effect of the perturbed data on the solutions by means of the Banach fixed point theorem.

The rest of this paper is organized as follows, In Section 2, we present some preliminary facts and make a list of the hypotheses that will be used throughout this paper. Section 3 is devoted to investigating the uniqueness of solutions of the problem (1.1)-(1.2). In Section 4, we introduce the continuous dependence of solutions to problem (1.1)-(1.2) in the space $C([a, b])$. Finally, an example to illustrate our results is given in Section 5.

2. PRELIMINARIES

In this section, we present some required notations, definitions and some hypotheses which are used throughout this paper. Let us denote by $C([0, b], \mathbb{R})$ the Banach spaces of continuous real functions $h : [0, b] \to \mathbb{R}$, with the norm $\|h\|_{\infty} = \sup\{|h(t)| : t \in [0, b]\}$, $C^1[0, b]$ the space of all continuously differentiable real functions defined on $[0, b]$, and by $L^1[0, b]$ the space of all real functions $h(t)$ such that $|h(t)|$ is Lebesgue integrable on $[0, b]$. For any continuous function $x : (-\infty, b] \to \mathbb{R}$ and for any $t \in [0, b]$, we denote by x_t the element of $\mathcal{B}$ defined by $x_t(s) = x(t + s)$, for $-\infty < s \leq 0$, we also consider the following space

$$\Upsilon = \{x : (-\infty, b] \to \mathbb{R};\ x\ |_{(-\infty,0]} \in \mathcal{B},\ x\ |_{[0,b]} \in C([0, b], \mathbb{R})\},$$

where $x\ |_{[0,b]}$ is the restriction of x to $[0, b]$.

Definition 2.1. *(*[15]*). Let $\alpha > 0$ and $h \in C([0,b])$. The Riemann-Liouville fractional integral of order α for a function h is determined as*

$$I_0^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{h(s)}{(t-s)^{1-\alpha}} ds, \ \ t \in [0,b],$$

where $\Gamma(.)$ is the gamma function. Moreover, $I_0^{\delta} I_0^{\gamma} h(t) = I_0^{\delta+\gamma} h(t)$, for $\delta, \gamma \geq 0$.

Definition 2.2. *(*[9]*) Let $0 < \alpha < 1$, and $h \in C([0,b], \mathbb{R})$. The Riemann-Liouville fractional derivative of order α for a function h is defined by*

$$D_0^{\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{h(s)}{(t-s)^{\alpha}} ds \ \ t \in [0,b].$$

Further, if $h \in L^1[0,b]$. Then $D_0^{\alpha} I_0^{\alpha} h(t) = h(t)$, for $t \in [0,b]$.

Definition 2.3. *(*[24]*). Let $0 < \alpha < 1$ and $h \in C^1([0,b], \mathbb{R})$. The Caputo fractional derivative of order α for a function h is described as*

$${}^{c}D_0^{\alpha} h(t) = D_0^{\alpha} \left(h(t) - h(0) \right).$$

Moreover, if ${}^{c}D_0^{\alpha} h(t) \in L^1([0,b])$, then

$$I_0^{\alpha} \, {}^{c}D_0^{\alpha} h(t) = h(t) - h(0).$$

Also, one has

$${}^{c}D_0^{\alpha} h(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{h'(s)}{(t-s)^{\alpha}} ds, \ \ t \in [0,b].$$

Obviously, the Caputo derivative of a constant is equal to zero.

Definition 2.4. *A function $x \in \Upsilon$ is said to be a solution of (1.1)–(1.2) if x satisfies the equation ${}^{c}D_0^{\alpha}[x(t) - g(t, x_t)] = f(t, x_t)$, $t \in [0,b]$, with initial condition $x_0 = \varphi$ and $[x(t) - g(t, x_t)]$ is absolutely continuous on $[0,b]$.*

Lemma 2.5. *(*[24]*) (Banach fixed point theorem). Let K be a non-empty closed subset of a Banach space X, then each contraction mapping $T : K \to K$ has a unique fixed point.*

In this paper, we employ an axiomatic definition for the phase space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ that is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$ and satisfying the following fundamental axioms which is similar to that introduced by Hale and Kato in [12] and exceedingly discussed in [14]:

(H1): If $x : (-\infty, b] \to \mathbb{R}$, such that x is a continuous on $[0,b]$ and $x_0 \in \mathcal{B}$, then for every $t \in [0,b]$ the following conditions are satisfied:

(i): $x_t \in \mathcal{B}$;

(ii): $|x(t)| \leq H \|x_t\|_{\mathcal{B}}$ for some $H > 0$;

(iii): $\|x_t\|_{\mathcal{B}} \leq K(t) \sup_{0 \leq s \leq t} |x(s)| + M(t) \|x_0\|_{\mathcal{B}}$, where $M, K : [0, +\infty) \to [0, +\infty)$ with M locally bounded and K continuous, such that M, K are independent of $x(.)$.

(H2): For the function $x(.)$ in (H1), the function $t \to x_t$ is continuous from $[0,b]$ into $\mathcal{B}$.

(H3): The space $\mathcal{B}$ is complete.

Remark 2.6. *Note that, the condition (ii) in (H1) is equivalent to* $|\varphi(0)| \leq H \|\varphi\|_{\mathcal{B}}$ *for every* $\varphi \in \mathcal{B}$.

3. Uniqueness results

In this section, we prove the uniqueness results for the problem (1.1)-(1.2) by means of the Banach fixed point theorem. Before starting and proving this one, we will display the following lemma

Lemma 3.1. *A function* $x \in \Upsilon$ *is a solution of the problem (1.1)-(1.2) if and only if* x *satisfies*

$$x(t) = \begin{cases} \varphi(0) - g(0,\varphi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s)ds, & t \in [0,b], \\ \varphi(t), & t \in (-\infty, 0]. \end{cases} \tag{3.1}$$

provided that the integral in (3.1) exists.

Proof. The proof is very simple and we can get it according to Definitions 2.1 and 2.3. □

Theorem 3.2. *Assume that* $f, g : [0,b] \times \mathcal{B} \to \mathbb{R}$ *are continuous functions. If the following conditions are satisfied*

(A1): There exist two functions $\delta_f(t) \in L^1([0,b],\mathbb{R})$ and $\delta_g(t) \in C([0,b],\mathbb{R})$ such that

$$|f(t,u) - f(t,v)| \leq \delta_f(t) \|u - v\|_{\mathcal{B}}, \text{ and } |g(t,u) - g(t,v)| \leq \delta_g(t) \|u - v\|_{\mathcal{B}},$$

for any $t \in [0,b]$ and for each $u, v \in \mathcal{B}$;

(A2): $\left(\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta \right) K_b < 1$, where $K_b = \sup_{t \in [0,b]} |K(t)|$ and $\eta = \int_0^t \delta_f(s)ds$.

Then there exists a unique solution to (1.1)-(1.2) on $(-\infty, b]$.

Proof. According to Lemma 3.1, the problem (1.1)-(1.2) is equivalent to the integral equation

$$x(t) = \varphi(0) - g(0,\varphi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s)ds, \quad t \in [0,b], \tag{3.2}$$

with $x_0 = \varphi$. Now, we must transform the problem (1.1)-(1.2) to be applicable to fixed point problem. Let the operator $\Pi : \Upsilon \to \Upsilon$ defined by

$$(\Pi x)(t) = \varphi(0) - g(0,\varphi) + g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x_s)ds, \quad t \in [0,b],$$

and $(\Pi x)(t) = \varphi(t)$, for $t \in (-\infty, 0]$.

For any continuos function $\varphi \in \mathcal{B}$, let $\widetilde{\varphi} : (-\infty, b] \to \mathbb{R}$ be the extension of φ such that

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(0), & t \in [0,b], \\ \varphi(t), & t \in (-\infty, 0]. \end{cases}$$

So, we get $\widetilde{\varphi}_0 = \varphi$. For every function $z \in C([0,b],\mathbb{R})$ with $z(0) = 0$, let $\widetilde{z} : (-\infty, b] \to \mathbb{R}$ be the extension of z to $(-\infty, b]$ such that

$$\widetilde{z}(t) = \begin{cases} z(t), & t \in [0,b], \\ 0, & t \in (-\infty, 0]. \end{cases}$$

If $x(.)$ satisfies the integral equation (3.2) then, we can analyze $x(.)$ as follows $x(t) = \widetilde{\varphi}(t) + \widetilde{z}(t)$, $t \in (-\infty, b]$, which allude to $x_t = \widetilde{\varphi}_t + \widetilde{z}_t$, for each $t \in [0,b]$ and the function $z(.)$ satisfies

$$z(t) = -g(0,\varphi) + g(t, \widetilde{\varphi}_t + \widetilde{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\varphi}_s + \widetilde{z}_s) ds, \ \ t \in [0,b], \quad (3.3)$$

with $\widetilde{z}_0 = 0$. Setting $\Upsilon_0 = \{z \in \Upsilon$, such that $z_0 = 0\}$. For any $z \in \Upsilon_0$ and let $\|.\|_{\Upsilon_0}$ be semi-norm in Υ_0 determined by

$$\|z\|_{\Upsilon_0} = \|z_0\|_{\mathcal{B}} + \|z\|_C = \sup_{t\in[0,b]} |z(t)| \, .$$

Then $(\Upsilon_0, \|z\|_{\Upsilon_0})$ is a Banach space. Consider the operator $\Pi_0 : \Upsilon_0 \to \Upsilon_0$ be defined by

$$(\Pi_0 z)(t) = -g(0,\varphi) + g(t, \widetilde{\varphi}_t + \widetilde{z}_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\varphi}_s + \widetilde{z}_s) ds, \ \ t \in [0,b], \quad (3.4)$$

and $(\Pi_0 z)(t) = 0$, $t \in (-\infty, 0]$. Hence, we get $(\Pi_0 z)_0 = 0$.

Certainly, Π has a fixed point equivalent to Π_0 that has a fixed point too. So, we go ahead to show that Π_0 has a fixed point in Υ_0 by the Banach fixed point theorem. In order that, we need to prove that the operator $\Pi_0 : \Upsilon_0 \to \Upsilon_0$ is a contraction map. In fact, consider $z_1, z_2 \in \Upsilon_0$ and $t \in [0,b]$, then by (3.4), Definition 2.1 and (A1), we have

$$\begin{aligned} |(\Pi_0 z_1)(t) - (\Pi_0 z_2)(t)| &\leq |g(t, \widetilde{\varphi}_t + \widetilde{z}_{1t}) - g(t, \widetilde{\varphi}_t + \widetilde{z}_{2t})| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, \widetilde{\varphi}_s + \widetilde{z}_{1s}) - f(s, \widetilde{\varphi}_s + \widetilde{z}_{2s})| \, ds \\ &\leq \|\delta_g\|_\infty \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} + \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \delta_f(s) ds \\ &= \left(\|\delta_g\|_\infty + I_0^{\alpha-1} I_0^1 \delta_f(t) \right) \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} \\ &= \left[\|\delta_g\|_\infty + \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-2} \int_0^s \delta_f(\tau) d\tau ds \right] \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} \\ &= \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta \right] \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} \, . \end{aligned}$$

Since

$$\begin{aligned} \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} &\leq K(t) \sup_{0\leq\tau\leq t} |\widetilde{z}_1(\tau) - \widetilde{z}_2(\tau)| + M(t) \|\widetilde{z}_{1_0} - \widetilde{z}_{2_0}\|_{\mathcal{B}} \\ &\leq K_b \sup_{0\leq\tau\leq t} |z_1(\tau) - z_2(\tau)| \end{aligned}$$

$$= K_b \left\| z_1 - z_2 \right\|_{\Upsilon_0}, \tag{3.5}$$

we get

$$\left|(\Pi_0 z_1)(t) - (\Pi_0 z_2)(t)\right| \leq \left[\left\|\delta_g\right\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b \left\|z_1 - z_2\right\|_{\Upsilon_0}.$$

Consequently,

$$\left\|\Pi_0 z_1 - \Pi_0 z_2\right\|_{\Upsilon_0} \leq \left[\left\|\delta_g\right\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b \left\|z_1 - z_2\right\|_{\Upsilon_0}.$$

From the condition (A2), we infer that

$$\left\|\Pi_0 z_1 - \Pi_0 z_2\right\|_{\Upsilon_0} \leq \left\|z_1 - z_2\right\|_{\Upsilon_0}.$$

This implies that Π_0 is contraction map. So applying the Lemma 2.5, we can conclude that Π_0 has a fixed point z in Υ_0 which is the unique solution to the equation (3.3) on $[0, b]$, what means that the operator Π has a fixed point $x = \widetilde{\varphi} + \widetilde{z}$ in Υ that is the unique solution to the fractional differential equation (1.1)-(1.2) on $(-\infty, b]$. The proof is completed.

Remark 3.3. *Note that, in Theorem 3.2, if the functions $\delta_g(t)$ and $\delta_f(t)$ are replaced by two positive constants L_g and L_f respectively, the second result follows.*

□

Corollary 3.4. *Assume that $f, g : [0, b] \times \mathcal{B} \to \mathbb{R}$ are continuous functions and if*

(B1): There exist two constants $L_f, L_g > 0$ such that

$$|f(t, u) - f(t, v)| \leq L_f \left\|u - v\right\|_{\mathcal{B}}, \text{ and } |g(t, u) - g(t, v)| \leq L_f \left\|u - v\right\|_{\mathcal{B}},$$

for any $t \in [0, b]$ and for each $u, v \in \mathcal{B}$;

(B2): $\left[L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f\right] K_b < 1.$

Then there exists a unique solution to (1.1)-(1.2) on $(-\infty, b]$.

Proof. see ([2]). □

4. Continuous dependence

In this section, we discuss the continuity dependence with respect to parameters φ, α.

Definition 4.1. ([9]). *The solution $x \in C([0, b])$ of the problem (1.1)-(1.2) is continuously dependence on initial data if for every $\varphi, \psi \in \mathcal{B}$,*

$$\left\|x_\varphi(.) - x_\psi(.)\right\|_C \leq O(\left\|\varphi - \psi\right\|_{\mathcal{B}}),$$

where $x_\varphi(.)$ is solution of the problem (1.1)-(1.2) and $x_\psi(.)$ is solution of the following problem

$$^cD_0^\alpha[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0, b], \tag{4.1}$$

$$x_0 = \psi \in \mathcal{B}. \tag{4.2}$$

Firstly, in the following theorem, we investigate the continuous dependence of the solution for the problem (1.1)-(1.2) on the initial value φ.

Theorem 4.2. *Assume that the hypotheses of Theorem 3.2 are satisfied. Let x_φ and x_ψ are solutions of (1.1)-(1.2) and (4.1)-(4.2) for $\varphi, \psi \in \mathcal{B}$, respectively. Then there exists a constant Λ such that*

$$\|x_\varphi(.) - x_\psi(.)\|_C \leq \Lambda \|\varphi - \psi\|_{\mathcal{B}}, \ \ \forall \varphi, \psi \in \mathcal{B}.$$

Proof. In view of Theorem 3.2, we know that for every $\varphi, \psi \in \mathcal{B}$, the equation (1.1) has solutions x_φ and x_ψ on $(-\infty, b]$, respectively. Further, there are $z_1, z_2 \in C([0,b])$ such that for all $t \in [0,b]$, $x_\varphi(t) = \varphi(0) + z_1(t)$ and $x_\psi(t) = \psi(0) + z_2(t)$ together with

$$z_1(t) = -g(0,\varphi) + g(t, \widetilde{\varphi}_t + (\widetilde{z_1})_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\varphi}_s + (\widetilde{z_1})_s) ds,$$

$$z_2(t) = -g(0,\psi) + g(t, \widetilde{\psi}_t + (\widetilde{z_2})_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\psi}_s + (\widetilde{z_2})_s) ds.$$

Therefore, by hypotheses of Theorem 3.2, for $t \in [0,b]$, we have

$$\begin{aligned} |x_\varphi(t) - x_\psi(t)| &\leq |\varphi(0) - \psi(0)| + |z_1(t) - z_2(t)| \\ &\leq |\varphi(0) - \psi(0)| + |g(0,\varphi) - g(0,\psi)| \\ &\quad + \left| g(t, \widetilde{\varphi}_t + (\widetilde{z_1})_t) - g(t, \widetilde{\psi}_t + (\widetilde{z_2})_t) \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left| f(s, \widetilde{\varphi}_s + (\widetilde{z_1})_s) - f(s, \widetilde{\psi}_s + (\widetilde{z_2})_s) \right| ds \\ &\leq H \|\varphi - \psi\|_{\mathcal{B}} + \delta_g(0) \|\varphi - \psi\|_{\mathcal{B}} \\ &\quad + \delta_g(t) \left[\left\| \widetilde{\varphi}_t - \widetilde{\psi}_t \right\|_{\mathcal{B}} + \|(\widetilde{z_1})_t - (\widetilde{z_2})_t\|_{\mathcal{B}} \right] \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \delta_f(s) \left[\left\| \widetilde{\varphi}_s - \widetilde{\psi}_s \right\|_{\mathcal{B}} + \|(\widetilde{z_1})_s - (\widetilde{z_2})_s\|_{\mathcal{B}} \right] ds \\ &\leq H \|\varphi - \psi\|_{\mathcal{B}} + \|\delta_g\|_\infty \|\varphi - \psi\|_{\mathcal{B}} \\ &\quad + \|\delta_g\|_\infty \left[\left\| \widetilde{\varphi}_t - \widetilde{\psi}_t \right\|_{\mathcal{B}} + \|(\widetilde{z_1})_t - (\widetilde{z_2})_t\|_{\mathcal{B}} \right] \\ &\quad + \eta \left[\left\| \widetilde{\varphi}_t - \widetilde{\psi}_t \right\|_{\mathcal{B}} + \|(\widetilde{z_1})_t - (\widetilde{z_2})_t\|_{\mathcal{B}} \right] \frac{1}{\Gamma(\alpha-2)} \int_0^t (t-s)^{\alpha-2} ds. \end{aligned}$$

In a similar way to (3.5), we can deduce that

$$\|(\widetilde{z_1})_t - (\widetilde{z_2})_t\|_{\mathcal{B}} \leq K_b \|z_1 - z_2\|_C . \tag{4.3}$$

Also, we have

$$\begin{aligned} \left\| \widetilde{\varphi}_t - \widetilde{\psi}_t \right\|_{\mathcal{B}} &\leq K(t) \sup_{0 \leq \tau \leq t} \left| \widetilde{\varphi}(\tau) - \widetilde{\psi}(\tau) \right| + M(t) \left\| \widetilde{\varphi}_0 - \widetilde{\psi}_0 \right\|_{\mathcal{B}} \\ &\leq K_b |\varphi(0) - \psi(0)| + M_b \|\varphi - \psi\|_{\mathcal{B}} \\ &\leq (K_b H + M_b) \|\varphi - \psi\|_{\mathcal{B}}, \end{aligned}$$

where $M_b = \sup\{M(t) : t \in [0,b]\}$. From preceding processes, we can conclude that

$$
\begin{aligned}
& |x_\varphi(t) - x_\psi(t)| \\
\leq\ & \big(H + \|\delta_g\|_\infty\big) \|\varphi - \psi\|_{\mathcal{B}} + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] (K_b H + M_b) \|\varphi - \psi\|_{\mathcal{B}} \\
& + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b \|z_1 - z_2\|_C \\
\leq\ & \big(H + \|\delta_g\|_\infty\big) \|\varphi - \psi\|_{\mathcal{B}} + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] (K_b H + M_b) \|\varphi - \psi\|_{\mathcal{B}} \\
& + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b \left[\|x_\varphi(.) - x_\psi(.)\|_C + H\|\varphi - \psi\|_{\mathcal{B}}\right].
\end{aligned}
$$

Accordingly, and by (A2), we obtain

$$
\begin{aligned}
\|x_\varphi(.) - x_\psi(.)\|_C \leq\ & \big(H(1 + K_b) + \|\delta_g\|_\infty\big) \|\varphi - \psi\|_{\mathcal{B}} \\
& + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] (K_b H + M_b) \|\varphi - \psi\|_{\mathcal{B}} \\
& + \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b \|x_\varphi(.) - x_\psi(.)\|_C . \\
\leq\ & \frac{\theta}{1 - \Theta} \|\varphi - \psi\|_{\mathcal{B}} .
\end{aligned}
$$

where $\theta = \left[H(1 + K_b) + \|\delta_g\|_\infty + \frac{\Theta}{K_b}(K_b H + M_b)\right]$ and $\Theta = \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right] K_b$.
Take $\Lambda = \frac{\theta}{1-\Theta}$, we get

$$\|x_\varphi(.) - x_\psi(.)\|_C \leq \Lambda \|\varphi - \psi\|_{\mathcal{B}} .$$

This proves the desired. □

Next, we discuss the consequences of modification of the order to the fractional differential equation.

Definition 4.3. *The solution $x \in C([0,b])$ of the problem (1.1)-(1.2) is continuously dependence on order α to fractional differential equations if $0 < \alpha < \widetilde{\alpha} < 1$,*

$$\|x_\alpha(.) - x_{\widetilde{\alpha}}(.)\|_C \leq O\,|\widetilde{\alpha} - \alpha| ,$$

where $x_\alpha(.)$ is solution of the problem (1.1)-(1.2) and $x_{\widetilde{\alpha}}(.)$ is solution of the following problem

$$
\begin{aligned}
& {}^cD_0^{\widetilde{\alpha}}[x(t) - g(t, x_t)] = f(t, x_t), \quad t \in [0,b], \\
& x_0 = \varphi \in \mathcal{B}.
\end{aligned}
$$

Theorem 4.4. *Assume that the assumptions of Theorem 3.2 hold. Then there exists a constant λ such that*

$$\|x_\alpha(.) - x_{\widetilde{\alpha}}(.)\|_C \leq \lambda\,|\widetilde{\alpha} - \alpha| , \quad 0 < \alpha < \widetilde{\alpha} < 1.$$

Proof. The uniqueness of the solution can be concluded as above. Let $z_1, z_2 \in C([0,b])$ be such that $x_\alpha(t) = \varphi(0) + z_1(t)$ and $x_{\widetilde{\alpha}}(t) = \varphi(0) + z_2(t)$, for all $t \in [0,b]$. Then for $t \in [0,b]$, z_1 and z_2 satisfy

$$z_1(t) = -g(t,\varphi) + g(t, \widetilde{\varphi}_t + (\widetilde{z}_1)_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \widetilde{\varphi}_s + (\widetilde{z}_1)_s) ds,$$
$$z_2(t) = -g(t,\varphi) + g(t, \widetilde{\varphi}_t + (\widetilde{z}_2)_t) + \frac{1}{\Gamma(\widetilde{\alpha})} \int_0^t (t-s)^{\widetilde{\alpha}-1} f(s, \widetilde{\varphi}_s + (\widetilde{z}_2)_s) ds.$$

Consequently, by hypotheses of Theorem 3.2 and (4.3), we have

$$\begin{aligned}
|x_\alpha(t) - x_{\widetilde{\alpha}}(t)| &= |z_1(t) - z_2(t)| \\
&\le |g(t, \widetilde{\varphi}_t + (\widetilde{z}_1)_t) - g(t, \widetilde{\varphi}_t + (\widetilde{z}_2)_t)| \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, \widetilde{\varphi}_s + (\widetilde{z}_1)_s) - f(s, \widetilde{\varphi}_s + (\widetilde{z}_2)_s)| \, ds \\
&\quad + \int_0^t \left[\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right] |f(s, \widetilde{\varphi}_s + (\widetilde{z}_2)_s)| \, ds \\
&\le \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta \right] \|\widetilde{z}_{1t} - \widetilde{z}_{2t}\|_{\mathcal{B}} \\
&\quad + \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right| |f(s, \widetilde{\varphi}_s + (\widetilde{z}_2)_s)| \, ds \\
&\le \left[\|\delta_g\|_\infty + \frac{b^{\alpha-1}}{\Gamma(\alpha)} \eta \right] K_b \|z_1 - z_2\|_C \\
&\quad + \sup_{(t,u) \in [0,b] \times \mathcal{B}} |f(t,u)| \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right| ds.
\end{aligned}$$

Now, we estimate the integral in the right-hand side of the inequalities above

$$\begin{aligned}
&\int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right| ds \\
\le\ & \int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\alpha)} \right| ds + \int_0^t \left| \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right| ds \\
=\ & \frac{1}{\Gamma(\alpha)} \int_0^t \left| \tau^{\alpha-1} - \tau^{\widetilde{\alpha}-1} \right| d\tau + \left| \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\widetilde{\alpha})} \right| \int_0^t \tau^{\widetilde{\alpha}-1} d\tau \\
\le\ & \frac{1}{\Gamma(\alpha)} \left[\frac{b^\alpha}{\alpha} - \frac{b^{\widetilde{\alpha}}}{\widetilde{\alpha}} \right] + \left| \frac{\Gamma(\widetilde{\alpha}) - \Gamma(\alpha)}{\Gamma(\alpha)\Gamma(\widetilde{\alpha})} \right| \frac{b^{\widetilde{\alpha}}}{\widetilde{\alpha}}.
\end{aligned}$$

Since $\alpha < \widetilde{\alpha}$ and by Mean Value Theorem of classical differential calculus, we get

$$\int_0^t \left| \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\widetilde{\alpha}-1}}{\Gamma(\widetilde{\alpha})} \right| ds \le \frac{|\widetilde{\alpha} - \alpha| \, b^{\widetilde{\alpha}}}{\Gamma(\alpha+1)\widetilde{\alpha}} + \frac{(\widetilde{\alpha} - \alpha) b^{\widetilde{\alpha}}}{\Gamma(\widetilde{\alpha}+1)\alpha} \left| \frac{\Gamma(\widetilde{\alpha}) - \Gamma(\alpha)}{(\widetilde{\alpha} - \alpha)} \right|$$

$$\begin{aligned}
&\leq \frac{|\widetilde{\alpha}-\alpha|\, b}{\Gamma(\alpha+1)\alpha}\left[1+\left|\frac{\Gamma(\widetilde{\alpha})-\Gamma(\alpha)}{(\widetilde{\alpha}-\alpha)}\right|\right] \\
&= |\widetilde{\alpha}-\alpha|\left[\frac{b\,(1+|\Gamma'(\widetilde{\alpha}-\theta(\widetilde{\alpha}-\alpha))|)}{\Gamma(\alpha+1)\alpha}\right] \\
&= |\widetilde{\alpha}-\alpha|\, R,
\end{aligned}$$

for some θ such that $0<\alpha<\theta<\widetilde{\alpha}<1$ and $R=\left[\frac{b(1+|\Gamma'(\widetilde{\alpha}-\theta(\widetilde{\alpha}-\alpha))|)}{\Gamma(\alpha+1)\alpha}\right]$.

In the end, by (A2), we arrive at the following

$$\begin{aligned}
&\|x_\alpha(.)-x_{\widetilde{\alpha}}(.)\|_C \\
\leq\ &\left[\|\delta_g\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right]K_b\,\|z_1-z_2\|_C+|\widetilde{\alpha}-\alpha|\, R\sup_{(t,u)\in[0,b]\times\mathcal{B}}|f(t,u)| \\
\leq\ &\left[\|\delta_g\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right]K_b\,\|x_\alpha(.)-x_{\widetilde{\alpha}}(.)\|_C+|\widetilde{\alpha}-\alpha|\, R\sup_{(t,u)\in[0,b]\times\mathcal{B}}|f(t,u)| \\
\leq\ &|\widetilde{\alpha}-\alpha|\,\frac{R}{1-\left[\|\delta_g\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right]K_b}\sup_{(t,u)\in[0,b]\times\mathcal{B}}|f(t,u)|\,.
\end{aligned}$$

Take λ so that, $\lambda=\frac{R}{1-\left[\|\delta_g\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right]K_b}\sup_{(t,u)\in[0,b]\times\mathcal{B}}|f(t,u)|$. Then, we get

$$\|x_\alpha(.)-x_{\widetilde{\alpha}}(.)\|_C\leq\lambda\,|\widetilde{\alpha}-\alpha|\,.$$

This proves the required. □

5. AN EXAMPLE

Firstly, Let ρ be a positive real constant and we define the functional space $\mathbb{B}_\rho$ by

$$\mathbb{B}_\rho=\{x\in C((-\infty,0],\mathbb{R}):\lim_{s\to-\infty}e^{\rho s}x(s)\text{ exist in }\mathbb{R}\},$$

endowed with the following norm $\|x\|_\rho=\sup\{e^{\rho s}|x(s)|:-\infty<s\leq 0\}$.

Then $\mathbb{B}_\rho$ satisfies axioms (H1), (H2) and (H3) with $K(t)=M(t)=1$ and $H=1$. (see [5]).

Next, we consider the fractional neutral functional differential equation

$$^{c}D_0^\alpha\left[x(t)-\frac{e^{-\rho t}}{(2+e^{-2t})}\frac{\|x_t\|}{1+\|x_t\|}\right]=\frac{e^{-\rho t}}{(1+t)}\frac{\|x_t\|}{1+\|x_t\|},\quad t\in[0,b], \tag{5.1}$$

with initial condition

$$x_0=\varphi,\quad on\ (-\infty,0], \tag{5.2}$$

where $\varphi\in\mathbb{B}_\rho$.

Finally, we applying Theorem 3.2. Consider $f(t,u)=\frac{e^{-\rho t}}{(1+t)}\frac{u}{1+u}$ and $g(t,u)=\frac{e^{-\rho t}}{(2+e^{-2t})}\frac{u}{1+u}$, for $(t,u)\in[0,b]\times\mathbb{B}_\rho$. Let $x,y\in\mathbb{B}_\rho$, and $t\in[0,b]$. Then

$$|f(t,x)-f(t,y)| = \frac{e^{-\rho t}}{(1+t)}\left|\frac{x}{1+x}-\frac{y}{1+y}\right|$$

$$
\begin{aligned}
&= \frac{e^{-\rho t}}{(1+t)}\left|\frac{x-y}{(1+x)(1+y)}\right| \\
&\leq \frac{e^{-\rho t}}{(1+t)}|x-y| \\
&\leq \frac{1}{(1+t)}\|x-y\|_{\rho}
\end{aligned}
$$

and

$$
\begin{aligned}
|g(t,x)-g(t,y)| &= \frac{e^{-\rho t}}{(2+e^{-2t})}\left|\frac{x}{1+x}-\frac{y}{1+y}\right| \\
&= \frac{e^{-\rho t}}{(2+e^{-2t})}\left|\frac{x-y}{(1+x)(1+y)}\right| \\
&\leq \frac{e^{-\rho t}}{(2+e^{-2t})}|x-y| \\
&\leq \frac{1}{(2+e^{-2t})}\|x-y\|_{\rho}.
\end{aligned}
$$

Hence, the condition (A1) holds with $\delta_f(t)=\frac{1}{(1+t)}\in L^1([0,b],\mathbb{R})$ and $\delta_g(t)=\frac{1}{(2+e^{-2t})}\in C([0,b],\mathbb{R})$. It can be checked that condition (A2) is satisfied with $K_b=1$, $\alpha=1/4$ and $b=1$. Indeed, since $\|\delta_g\|_\infty=\sup\{|\delta_g(t)|:t\in[0,1]\}=\sup\{\left|1/(2+e^{-2t})\right|:t\in[0,1]\}=\frac{1}{2}$, $\eta=\int_0^1\delta_f(s)ds=\int_0^1\frac{1}{(1+s)}ds=\ln 2$ and $\frac{1}{\Gamma(\frac{1}{4})}\ln 2<\frac{1}{2}$. Then, $K_b\left[\|\delta_g\|_\infty+\frac{b^{\alpha-1}}{\Gamma(\alpha)}\eta\right]<1$.

According to Theorem 3.2, then problem (5.1)-(5.2) has a unique solution on $(-\infty,1]$.

6. CONCLUSIONS

This paper has investigated the uniqueness and continuous dependence of solutions for the fractional neutral functional differential equation (1.1)-(1.2) with infinite delay which includes the Caputo fractional derivative. Firstly, we introduced some properties of fractional calculus with mention useful definitions and lemmas related to the fixed point theory, phase space. Further, we offered the list of appropriate conditions for f and g. Secondly, the uniqueness results are investigated by utilizing the Banach fixed point theorem. Moreover, the continuous dependence of solutions to the problem (1.1)-(1.2) is discussed in the space $C[0,b]$. Finally, an example is provided to show the effectiveness of the proposed results.

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REFERENCES

[1] S. Abbas, *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electronic Journal of Differential Equations, **9** (2011) 1–11.

[2] M. S. Abdo and S. K. Panchal, *Existence and continuous dependence for fractional neutral functional differential equations*, Journal of Mathematical Modeling(JMM), **5** (2017) 153–170.
[3] M. S. Abdo and S. K. Panchal, *Some New Uniqueness Results of Solutions to Nonlinear Fractional Integro-Differential Equations*, Annals of Pure and Applied Mathematics, **16** (2018), 345–352.
[4] R. Agarwal, Y. Zhou and Y. He, *Existence of fractional neutral functional differential equations*, Computers and Mathematics with Applications, **59** (2010) 1095–1100.
[5] M. Benchohra, J. Henderson, S. K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, Journal of Mathematical Analysis and Applications, **338** (2008) 1340–1350.
[6] J. Cao, H. Chen and W. Yang, *Existence and continuous dependence of mild solutions for fractional neutral abstract evolution equations*, Advances in Difference Equations, **2015** (2015) 1–6.
[7] D. Delboso and L. Rodino, *Existence and uniqueness for a nonlinear fractional differential equation*, Journal of Mathematical Analysis and Applications, **204** (1996) 609–625.
[8] K. Diethelm and N. J. Ford, *Analysis of fractional differential equations*, Journal of Mathematical Analysis and Applications, **265** (2002) 229–248.
[9] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics 2004, Springer, Berlin, 2010.
[10] Q. Dong, *Existence and continuous dependence for weighted fractional differential equations with infinite delay*, Advance Difference Equations, **2014** (2014) 1–13.
[11] A. M. A. El-Sayed, F. Gaafar and M. El-Gendy, *Continuous dependence of the solution of Ito stochastic differential equation with nonlocal conditions*, Applied Mathematical Sciences, **10**(2016) 1971–1982.
[12] J. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkcial Ekvac, **21** (1978) 11–41.
[13] E. Hernández, *Existence results for partial neutral functional integrodifferential equations with unbounded delay*, Journal of Mathematical Analysis and Applications, **292** (2004). 194–210.
[14] Y. Hino, S. Murakami and T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Math 1473, Springer-Verlag, Berlin, 1991.
[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud. 204, Elsevier, Amsterdam, 2006.
[16] V. Lakshmikantham, *Theory of fractional functional differential equations*, Nonlinear Analysis: Theory, Methods & Applications, **69** (2008) 3337–3343.
[17] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[18] GM. N. Mophou, GM. Guérékata, *Existence of mild solutions of some semilinear neutral fractional functional evolution equations with infinite delay*, Applied Mathematics and Computation, **216** (2010) 61–69.
[19] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[20] H. A. H. Salem, *On the existence of continuous solutions for a singular system of non-linear fractional differential equations*, Applied Mathematics and Computation, **198** (2008) 445–452.
[21] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[22] M. Sharma, and S. Dubey, *Analysis of Fractional Functional Differential Equations of Neutral Type with Nonlocal Conditions*, Differential Equations and Dynamical Systems, **25** (2017) 499–517.
[23] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*,Computers & Mathematics with Applications, **59** (2010) 1063–1077.
[24] Y. Zhou, *Basic theory of fractional differential equations, Singapore*, World Scientific, 2014.
[25] Y. Zhou and F. Jiao and J. Li, *Existence and uniqueness for p-type fractional neutral differential equations*, Nonlinear Analysis: Theory, Methods & Applications, **71** (2009) 2724–2733.