*International Journal of Advanced Smart Convergence Vol.7 No.1 7-14 (2018)*
*http://dx.doi.org/10.7236/IJASC.2018.7.1.2*

IJASC 18-1-2

# Derivation of Transfer Function for the Cross-Coupled Filter Systems Using Chain Matrices

Kee-Hong Um

*IT Department, Hansei University, Korea*
*um@hansei.ac.kr*

***Abstract***

*In this paper, we derive a transfer function of cross-coupled microwave filter systems by using a characteristics of chain matrices. Depending on the lumped element of capacitor or inductor, the cross-coupled system is negatively- or positively system. We used a ladder network as a starting system composed of several subsystems connected in chain. Each subsystem is descrived by Laplace impedance. By solving the transmission zero characteristic equation derived from the cascaded subsystems, we can find the zeros of filter system with externally cross-coupled lumped elements. With the cross-coupled elements of capacitors, the numerator polynomial of system transfer function is used to locate the quadruplet zeros in complex plane. We show the polynomoials of numerator and denominator of cascaded transfer function, obtaining the zeros of the cross-coupled system.*

***Keywords:*** *Ladder network, Negative cross-coupling (NCC), transmission zeros, Cross-coupling, Bridged-T subsystem*

## 1. Introduction

The transfer function in the s-domain is the ratio of the Laplace transform of output signal (response) to the Laplace transform of the input signal (source). The zeros of the numerator polynomial in the transfer function is transmission zeros, which blocks the flow of signals from input to output terminals. To define the transfer function, the linear system is assumed to be a circuit where all initial conditions are zero. In this paper we show the procedures to derive the transfer function of cross-coupled system.

## 2. Transmission zeros from chain matrix

The entry $A$ can be calculated from the entry (1, 1) of the $(2 \times 2)$ chain matrix $\bar{T}$,

$$A = \bar{T}(1,1) \quad (1)$$

A voltage transfer function can be expressed as

Manuscript Received: Jan. 16, 2018 / Revised: Jan. 25, 2018 / Accepted: Feb. 11, 2018
Corresponding Author: um@hansei.ac.kr
Tel: +82-31-450-5308, Fax: +82-31-450-5172
IT Department, Hansei University, Korea

$$\mathbf{H}(\mathrm{s}) = \frac{\mathbf{V}_0(\mathrm{s})}{\mathbf{V}_\mathrm{i}(\mathrm{s})} = \frac{N(\mathrm{s})}{D(\mathrm{s})} \tag{2}$$

The voltage transfer function can be expressed in terms of the entry $A$

$$\mathrm{H}(\mathrm{s}) = \frac{1}{A} = \frac{1}{T(1,1)} = \frac{N(\mathrm{s})}{D(s)} \tag{3}$$

Equation (3) tells that if the entry (1.1) of the chain matrix is known the transfer function can be obtained [1, 2]. The chain matrix of $n$ cascaded networks can be represented as the product of each of the chain matrix

by [3]

$$\bar{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \prod_{\mathrm{i}=1}^{\mathrm{n}} \begin{bmatrix} Ai & Bi \\ Ci & Di \end{bmatrix} = \prod_{\mathrm{i}=1}^{\mathrm{n}} \bar{T}i \tag{4}$$

## 3. The Ladder Network

A ladder network is composed of series-connected and parallel-connected elements. The pattern is that every other element is alternatively in series-connected and shunt-connected as a signal travels from the source to the load. The important types of filters based on the resonators skipped are the three cases [3, 4]:

1) Without skipping any resonators (adjacent resonators),
2) Skipping one resonator,
3) Skipping two resonators.

In this paper the first case is considered. When more than three resonators are skipped, they can be simplified to no. 2 or no. 3 above. Then the analysis follows the same procedure. In each case of filter configurations, coupling can be achieved in two different types: one is *negative cross-coupling* (NCC); the other is *positive cross-coupling* (PCC). Negative cross coupling means that the sign of cross coupling opposes the sign of the main line coupling (i.e. capacitive cross coupling in an inductively coupled circuit, or inductive cross coupling in a capacitive coupled main line). In a negatively cross-coupled implementation, the series-connected elements are all inductors (or capacitors) and the cross-coupled element is a capacitor (or inductor). These two filters have the same locations for the finite frequency TZ's (but not for infinite frequency or DC TZ's, and not necessarily the same TP's. In a positively cross-coupled implementation, the series-connected elements are all inductors (or capacitors) and the cross-coupled element is an inductor (or a capacitor). These two filters have the same TZ locations. A cross-coupled filter network skipping one resonator is first analyzed, for negative cross coupling.

## 4. Cross-coupled (CC) Filter Configuration

A cross-coupled filter of Figure 1 is considered. The cascaded chain matrices of five subsystems sectioned is used to conveniently represent the system. For the cross-coupled subsystem an equivalent system in the form of bridged-T network can be used to determine chain matrices. The analysis on the cross-coupled microwave filters also will show the sectioning the whole filter system into several subsystems. The chain-parameters for each subsystem are derived. Since the cross-coupled circuit is the bridged-T structure, the chain parameters of the structure are first found. With all the chain parameters, the transfer function is found [5, 6].

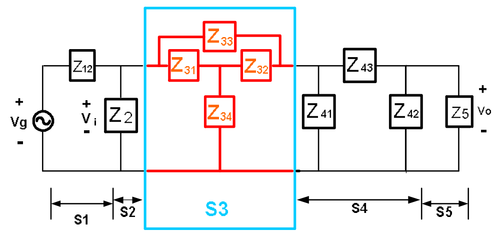


**Figure 1. A block diagram of cross-coupled filter network.**

From the transfer function, the locations of TZ's are found from the canonical form of the numerator polynomial of the transfer function. The whole filter network is considered to be composed of five subsystems cascaded. Since the cross-coupled subsystem S3 is the bridged-T structure, the chain parameters of this structure are first to be determined. With all the chain parameters determined for the five subsystems, the transfer function is found. As stated above, from the transfer function, the locations of TZ's are found from the canonical form of the numerator polynomial of the transfer function. The overall filter network is sectioned into five subsystems (*Si*, *i* =1-5) as shown in Figure 2.4. Each system is characterized by its own chain matrix of size $2 \times 2$.

$$\bar{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = T = \bar{T}_1 \cdot \bar{T}_2 \cdot \bar{T}_3 \cdot \bar{T}_4 \cdot \bar{T}_5 \qquad (5)$$

In Equation (5), each entry of five chain matrices must be expressed in terms of Laplace impedance shown in the Figure 1. In the figure above, *Zm* and *Zmn* as used herein are defined by

$Z_m$ : The Laplace impedance of the *m*-th subsystem with only one element.

$Z_{mn}$ : The Laplace impedance of the *n*-th element of the *m*-th subsystem, with more than one element.

For example, $Z_2$ means the Laplace impedance of the element of the 2nd subsystem, and $Z_{32}$ means the Laplace impedance of the 2nd element of 3rd subsystem. Following the definitions above, the Z2, Z34, Z41 and Z42 represent the impedances due to the shunt-connected tank circuits composed of (L2, C2), (L34, C34), (L41, C41), and (L42, C42), respectively. In the figure above, all impedances are consisted of inductors (capacitors) and all shunt impedances are consisted of parallel *LC*'s. Impedances Z31, Z32, and Z43 are due to series-connected inductors L31, L32, and L43, or capacitors C31, C32, and C43, respectively. For a negatively cross-coupled network, impedance Z33 is due to a single cross-coupled capacitor (*or* inductor) C33 (*or* L33), while for a positively cross-coupled network, impedance Z33 is due to a single cross-coupled inductor (*or* capacitor) L33 (*or* C33), respectively. The impedances Z1 and Z5 represent source and load impedances of 50 Ohms.

## 5. Negatively Cross-coupled (NCC) Filter Network

In Figure 2, the series-connected elements are all inductors. A negatively cross-coupled filter network is obtained by using capacitor impedance for Z33 connected between the 1st and the 3rd resonators, as shown in Figure. If the series-connected elements are all capacitors, the cross-coupled (CC) elements should be an inductor to result in the same locations for the TZ's. Here is the first case to be considered. A cross-coupled circuit, or a bridge-T circuit, is installed from the 1st resonator (Z2) and the 3rd resonator (Z41). The whole system is considered to be composed of five subsystems (S1, S2, S3, S4, and S5) connected in cascade. Therefore, the *chain* (ABCD) matrix of the whole system is used. Each entry of five chain matrices must be expressed in terms of Laplace impedance shown in the Figure 2 [6].

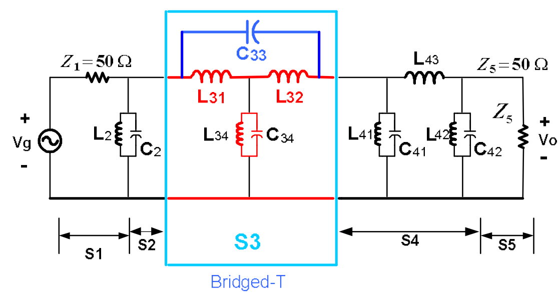


**Figure 2. A negatively cross-coupled filter network.**

In the figure above, the impedances (i.e. Laplace impedances) of the elements are expressed as:

$$\overline{Z}_1 = 50;$$

$$\overline{Z}_2 = \frac{s L_2}{L_2 C_2 s^2 + 1};$$

$$\overline{Z}_{31} = sL_{31},\ \overline{Z}_{32} = sL_{32},\ \overline{Z}_{33} = 1/sC_{33},\ \overline{Z}_{34} = \frac{sL_{34}}{L_{34}C_{34}s^2+1}; \quad (6)$$

$$\overline{Z}_{41} = \frac{sL_{41}}{L_{41}C_{41}s^2+1},\ \overline{Z}_{42} = \frac{sL_{42}}{L_{42}C_{42}s^2+1},\ \overline{Z}_{43} = sL_{43};$$

$$\overline{Z}_5 = 50.$$

The chain matrices of the subsystem in the network are given by

$$\overline{T}_1 = \begin{bmatrix} 1 & 50 \\ 0 & 1 \end{bmatrix} \quad (7)$$

$$\overline{T}_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad i = 2, 3, \&4. \tag{8}$$

$$\overline{T}_5 = \begin{bmatrix} 1 & 0 \\ 1/50 & 1 \end{bmatrix} \tag{9}$$

These matrices are due to the series source impedance *R*, shunt resonator #1, bridged-T subsystem, $\pi$ -network, and the load impedance, respectively. In Equation (7), matrix entry $\overline{T}(1,1)$ is dependent on each of the cascaded five networks. In Equation (8), all of the 12 entries of three matrices should be expressed in terms of Laplace impedances given in Equation (6). From Equation (3), the voltage transfer function *H(s)* has the numerator polynomial *N(s)* and denominator polynomial and *D(s)*, respectively. Using a MATLAB program, the chain matrices in Equation (4) are obtained based on the following detailed procedures.

## 6. Rational polynomial expressions of matrix entries

In Equation (6), to ensure that the conditions of the realizations of Hurwitz polynomial and /or polynomial of even degree for the complex conjugate roots is imposed in the numerator and denominator of a rational polynomial function, the rational expressions of any matrix entries are defined in this dissertation. The *i*-th chain matrix $\overline{T}_i$ of the *i*-th subsystem of a filter network is a $2\times 2$ matrix with four entry $A_i$, $B_i$, $C_i$, and $D_i$, since these are defined from the two-port systems. Any matrix obtained by mathematically manipulating any numbers of $2\times 2$ matrices is also $2\times 2$ matrix. Let the entry $Xi$ of the chain matrix $\overline{T}_i$ represent any of the matrix entry $Ai$, $Bi$, $Ci$, or $Di$. Four of these entries are meant by

$$Ai \equiv Entry\ (1,1)\ \ of\ \ the\ \ \overline{T}_i, \quad Bi \equiv Entry\ (1,2)\ \ of\ \ the\ \ \overline{T}_i,$$
$$Ci \equiv Entry\ (2,1)\ \ of\ \ the\ \ \overline{T}_i, \quad Di \equiv Entry(2,2)\ \ of\ \ the\ \ \overline{T}_i.$$

Each of the entry $Xi$ of matrix $\overline{T}_i$ has a numerator polynomial $f_i(s)$ and a denominator polynomial $g_i(s)$. Therefore, entry $Xi$ can be expressed in terms of two quantities as

$$Xi = \frac{f_i(s)}{g_i(s)}. \tag{10}$$

The numerator function $f_i(s)$ has its own numerator $n(f_i(s))$ and denominator $d(f_i(s))$.

The denominator function $g_i(s)$ has its own numerator $n(g_i(s))$ and denominator $d(g_i(s))$. Therefore, $Xi$ can be expressed in terms of the four quantities as

$$Xi = \frac{f_i(s)}{g_i(s)} = \frac{\dfrac{n(f_i(s))}{d(f_i(s))}}{\dfrac{n(g_i(s))}{d(g_i(s))}} . \quad (11)$$

To get a rational polynomial function for the entry $Xi$, the following expression is used.

$$Xi = \frac{f_i(s)}{g_i(s)} = \frac{\dfrac{n(f_i(s))}{d(f_i(s))}}{\dfrac{n(g_i(s))}{d(g_i(s))}} = \frac{n(f_i(s)) \cdot d(g_i(s))}{n(g_i(s)) \cdot d(f_i(s))} . \quad (12)$$

The resultant numerator is a polynomial, and the resultant denominator is also a polynomial. Two notations $NXi$ and $DXi$ are introduced as

$$NXi \equiv n(f_i(s)) \cdot d(g_i(s))$$

and

$$DXi \equiv n(g_i(s)) \cdot d(f_i(s)) .$$

Matrix entry $Xi$ is given by a rational polynomial function as

$$Xi \equiv \frac{NXi}{DXi} . \quad (13)$$

The Equation (13) is used to represent a rational polynomial. The numerator and denominator may or may not have common terms.

## 7. General Form of Transfer Function

By the *chain* (ABCD) matrices of subsystems, the transfer function is obtained. The transfer function of the whole system is written as [7, 8]

$$H(s) = \frac{N(s)}{D(s)} . \quad (14)$$

In Equation (14), $N(s)$ is the numerator of polynomial of $H(s)$, and $D(s)$ is the denominator polynomial of $H(s)$ [9, 10]. Each has the following expressions, respectively.

$$N(s) = 50 \cdot \mathrm{DC2} \cdot (\mathrm{DA3} \cdot \mathrm{DB3} \cdot \mathrm{DC3} \cdot \mathrm{DD3}) \cdot (\mathrm{DA4} \cdot \mathrm{DC4} \cdot \mathrm{DD4}) \quad (15)$$

$$\begin{aligned} D(s) = {} & (\,50\ \mathrm{NA4\ DB3\ DD3\ DC4\ DD4\ DC3\ DC2} \\ & + 2500\ \mathrm{NA4\ DB3\ DD3\ DC4\ DD4\ DC3\ NC2} \\ & + \mathrm{NB4\ DA4\ DB3\ DD3\ DC4\ DD4\ DC3\ DC2} \\ & + 50\ \mathrm{NB4\ DA4\ DB3\ DD3\ DC4\ DD4\ DC3\ NC2}\,) \cdot \mathrm{NA3} \\ & + 2500\ \mathrm{NC4\ DA3\ DC3\ DA4\ DD4\ NB3\ DD3\ NC2} \\ & + 2500\ \mathrm{NC4\ DA3\ DC3\ DA4\ DD4\ ND3\ DC2\ DB3} \\ & + 2500\ \mathrm{NA4\ DB3\ DD3\ DC4\ DD4\ NC3\ DC2\ DA3} \\ & + 50\ \mathrm{NC4\ DA3\ DC3\ DA4\ DD4\ NB3\ DD3\ DC2} \\ & + 50\ \mathrm{NB4\ DA4\ DB3\ DD3\ DC4\ DD4\ NC3\ DC2\ DA3} \\ & + \mathrm{ND4\ DA3\ DC3\ DA4\ DC4\ NB3\ DD3\ DC2} \\ & + 50\ \mathrm{ND4\ DA3\ DC3\ DA4\ DC4\ NB3\ DD3\ NC2} \\ & + 50\ \mathrm{ND4\ DA3\ DC3\ DA4\ DC4\ ND3\ DC2\ DB3} \end{aligned} \quad (16)$$

As defined before, the notations, for example, are used to mean the following;

DB3 means denominator polynomial of entry B, or (1, 2), of subsystem *S3*.
ND4 means numerator polynomial of entry D, or (2, 2), of subsystem *S4*.

Equations (15) and (16) represent the numerator and denominator polynomials of the transfer function of the whole filter system, respectively. To find out actual polynomials of complex variable $s$, the values of $L's$ and $C's$ of the each subsystem should be used. Depending on the existence of common terms in the numerator polynomial and the denominator polynomial, the relevant terms will be cancelled, so that $N(s)$ and $D(s)$ should be prime polynomials to determine the locations of transmission zeros. The two equations hold for any network composed of five cascaded subsystems.

## 8. Conclusion

In this paper, an investigation of a practical method to determine *quantitatively* the numerator and denominator polynomials of transfer function of cross-coupled microwave filter network.

To take advantage of chain matrices applied to cascaded subsystem, the cross-coupled subsystem was considered by calculating the four entries of 2 by two matrices. The subsystem was characterized by its own chain matrix. The cascaded chain matrices represent the whole filter network. The matrix entry (1, 1) was used to find the transfer function.

## Acknowledgement

The author wishes to express his thanks to Professor Dae-HeePark(Won Kwang University) and President Bo-Kyeong Kim(Osung Mega Power Co., Ltd.). Without their help and guidance, this work would have been impossible to complete.

**Remarks:**

This work is the extended and advanced from Ref. [3, 5, 6].

## References

[1] D. M. Pozar, *Microwave Engineering 2nd ed.*, John Wiley & Sons, Inc., pp.207, 1998.

[2] G. Gonzalez, *Microwave Transistor Amplifiers, Analysis and Design, 2nd Ed.,* Prentice Hall, Inc., pp.18, 1997.

[3] K. Um, Y. S. Im, G. K. Kim, J. J. Kang, *Determination of a Pair of Single Stationary Zeros in Cross-Coupled Systems*,pp.209-215, FGIT 2012., DOI: https://doi.org/10.1007/978-3-642-35585- 1_30.

[4] G. Hong, J. Sheng and J. M. Lancaster, *Cross-Coupled Microstrip Hairpin-Resonator Filters* IEEE Trans. on MTT, Vol. 46, No.1, Jan. 1998.

[5] K. Um., *Method for Theoretically Determining the Locus and Locations of the Transmission Zeros in Microwave Filter Networks*, Ph.D. Dissertation. New Jersey Institute of Technology, Newark, NJ, USA. 2003.

[6] K. Um., *Complex Quadruplet Zero Locations from the Perturbed Values of Cross-Coupled Lumped Ele ment* , International Journal of Advanced Smart Convergence Vol.5 No.4 33-40 (2017), http://dx.doi.org/ 10.7236/IJASC.2016.5.4.5.

[7] A. Papoulis, *Circuits and Systems: a modern approach*, New York: Holt, Rinehart, Winston, pp.187, 1980.

[8] G. Gonzalez, *Microwave Transistor Amplifiers, Analysis and Design, 2nd Ed.,* Prentice Hall, Inc., pp.18, 1997.

[9] G. Hong, J. Shengand J. M., Lancaster, *Microstrip Filters for FR/Microwave Applications*, John Wiley & Sons, Inc., pp.12,2001.

[10] C. Wang, K. A. Zaki , A. E. Atia , and T. G. Dolan, *Dielectric comb line resonators and filters*, *IEEE Transactions on Microwave Theory and Techniques*, Vol. 46, Issue 12 Part 2, pp. 2501 - 2506, Dec. 1998, DOI: 10.1109/22.739240.