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Making Sense of Drawn Models for Operations of Fractions Involving Mixed Numbers

Jihwa Noh[1)]

ABSTRACT. This study examined preservice elementary teachers' patterns and tendencies in thinking of drawn models of multiplication with fractions. In particular, it investigated preservice elementary teachers' work in a context where they were asked to select among drawn models for symbolic expressions illustrating multiplication with non-whole number fractions including a mixed number. Preservice teachers' interpretations of fraction multiplication used in interpreting different types of drawn models were analysed- both quantitatively and qualitatively. Findings and implications are discussed and further research is suggested.

I. What Constitutes Understanding of Fraction Operations

Understanding the multiplication [and division] of fractions involves understanding ideas about fractions and understanding ideas about multiplication [and division] (Mack, 2008). In order to understand the multiplication and division of fractions, one must understand what fractions are, what multiplication and division mean, and the connections between these two ideas. Images are the mental visualization of the concepts and operations of mathematics. These images can enhance our ability to work with fractions and fraction operations. The images used in fraction work can help us to reason about what fractions are and what the operations mean. The images we have of fractions and fraction operations may limit or enhance our ability to expand our understanding

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of fraction and fraction operations.

In the National Council of Teachers of Mathematics (NCTM) 2002 yearbook, Smith discusses the development of students' knowledge of fractions. Smith states there are two broad phases of development: the first is to make meaning for fractions by linking quotients to divided quantities and the second is to explore the mathematical properties of fractions as numbers. Thus students first learn what fractions are and then learn how to perform arithmetic operations on them.

1. Understanding Fractions: Partitioning and Iterating

In the first stage of understanding, Newton (2008) suggests that the learning of what fractions are is not difficult once students can partition. Partitioning is the idea of subdividing a unit (the whole) into subunits of equal size. (For example, a cookie that is cut into four equal size pieces has been subdivided into four subunits.) The students can then take a collection of the subdivided pieces (by iterating one of the pieces) and express this as a fraction (i.e. three of the four pieces of the cookie is "three-fourths" of the cookie, written as $\frac{3}{4}$). Even though partitioning helps in the understanding of fractions, there may be some challenges to understand partitioning. The key is to grasp the idea that fractions name the relationship between the collections of parts and the whole, not the size of the whole or its parts (Crespo, 2003, Toluk-Uçar, 2009). Smith suggests that students need practice with partitioning of wholes into many different sized pieces in order to bring understanding of partitioning.

Siebert and Gaskin (2006) discuss the power that comes from learning partitioning and iterating of the whole in understanding fractions. They claim that the images of partitioning and iterating are powerful because first, they make explicit the actions children can perform on quantities to produce, compare, and operate on fractional parts, second, these images provide ways for students to justify their fraction reasoning (p. 3). This process of partitioning and iterating keeps the referent whole for the fractions relevant. The fraction amount is based upon the referent whole, not on the number of pieces or parts they comprise. Thus the understanding of fractions is made more complete through the practice of partitioning the whole to find a "unit" fraction (i.e., subdividing the whole into six pieces and one of the pieces is “one-sixth” the whole and is a unit fraction) and then iterating to create other parts of the whole (i.e. iterating the “one-sixth” five times to produce “five-sixths”).

The referent whole is a necessary link for fraction understanding, because it allows for reasoning about what the fraction means. This helps students to understand fractions that are less than one and fractions of size greater than one. Because the students know the referent whole, eight- fifths becomes understandable, and the students are able to connect the idea of the fraction to their prior knowledge of quantities (Siebert and Gaskin, 2006). So, in essence understanding of fractions comes as the concepts of iterating, partitioning, and understanding what the fraction means in relation to the referent whole are learned and strengthened.

Once students have made meaning for fractions, they are then ready to move to the second stage which explores the mathematical properties of fractions as numbers (Smith, 2002). In this second stage the exploration of multiplication and division of fractions occurs. Students have learned what the fractional quantity means and then are able to combine two or more quantities to make new quantities. Acquiring understanding of multiplication and division of fractions involves at least two aspects. The first aspect (after understanding of fractions is attained) in understanding how to multiply and divide fractions is first to understand what it means to multiply and divide.

2. Fraction Multiplication

Multiplication is most simply described as “fancy”, or efficient, counting. For example, three multiplied by four (written 3×4) means the total number in three groups of size four. The first number (or the second number) in the problem is the number of groups, while the second (or the first) is the size of the groups. The first number can be seen as an operator telling how many copies of the second number to combine. So to multiply $3\times\frac{1}{4}$ means to find how much there is in three groups (or copies) of size one-fourth and the answer is three-fourths. In multiplying two fractional quantities like $\frac{2}{3}\times\frac{4}{5}$ the question asked is how much is two-thirds a group of size four-fifths. Here again, the first number can be seen as an operator telling how many copies of the second number to combine, but in this case we are taking a fractional quantity of the group instead of a whole number quantity. This idea of fraction multiplication, i.e., $\frac{a}{b}\times\frac{c}{d}$ as being "*a*-*b*ths" of a group of size "*c*-*d*ths", is an extension of the concept of whole number multiplication. Having the understanding of whole number multiplication and what fractions are makes it possible to make a bridge between whole number multiplication and fraction multiplication, because students first have knowledge of what fractions are in relation to the referent whole.

The ideas of partitioning and iterating and understanding what fractions are in relation to the referent whole allow the students to find the solution to $\frac{2}{3}\times\frac{4}{5}$ (Izsák, 2008). The students' knowledge of what a fraction is in relation to the referent whole makes it clear that the two-thirds of a whole are two of the unit fraction of one-third of the same whole. In fraction multiplication, the operation $\frac{2}{3}\times\frac{4}{5}$ is performed by first identifying four-fifths. Next the four-fifths is partitioned into thirds, or three equal pieces, to identify one-third of four-fifths (four-fifteenths). Then, after identifying one-third of four-fifths, the one-third is iterated twice to obtain two-thirds of four-fifths. This gives a solution of eight-fifteenths of the whole, the same referent whole for four-fifths. The solution of $\frac{2}{3}\times\frac{4}{5}$ as $\frac{8}{15}$ means eight pieces of size one-fifteenth of the whole is two-thirds of a group of size four-fifths. It is important that the referent whole is kept in mind in order to make sense of what the answer means.

3. Fraction Division

Understanding the multiplication of fractions requires that students understand the concept of what fractions are and the concept of what it means to multiply. Division of fractions can be thought of in the same manner. Students must first understand the concept of what fractions are and the concept of what it means to divide. The concept of division, "at its foundation, has to do with forming groups [with] two kinds of groupings" (Ball, 1990b, p. 452). These two types of groupings formed from division are measurement and sharing division.

In the problem of $a \div b$, measurement division asks the question of how many groups of size b are in a group of size a. Sharing division interprets the problem as how large will the group be if a things are shared equally among b groups (Sinicrope, Mick, & Kolb, 2002; Ball, 1990). This understanding of the two types of division for whole numbers and fractions provide support for understanding of fraction division.

As in whole number division, there are two types of groupings formed in fraction division: measurement and sharing. However, to understand fraction division, extensions of whole number division must be made. Looking at measurement division, as described above where $a \div b$ means how many group of size b are in a groups of size a, an adjustment for $\frac{a}{b} \div \frac{c}{d}$ must be made. Now the division is determining how many groups of size $\frac{c}{d}$ are in a group of size $\frac{a}{b}$. In order to make sense of the division, it is necessary to understand what the fraction $\frac{a}{b}$ means in reference to the whole and how to interpret $\frac{c}{d}$ and its referent whole. The referent whole here is the same for both fractions.

However, the solution to $\frac{a}{b} \div \frac{c}{d}$ has a different referent whole, which is the group size. For example, in the problem of $\frac{5}{8} \div \frac{2}{3}$, measurement division would interpret this as how many groups of size two-thirds of the whole are in five-eighths of the same whole or how many groups of size two-thirds of the whole will cover a group of size five-eighths of the whole. The answer is there are fifteen-sixteenths groups of size two-thirds (where the referent whole is groups of size two-thirds) or it will take fifteen-sixteenths of the whole to cover five-eighths of the whole. An example of a story problem using $\frac{5}{8} \div \frac{2}{3}$ is: Derek has $\frac{5}{8}$ cups of tropical punch concentrate; it takes $\frac{2}{3}$ cups of concentrate to make one pitcher of tropical punch; how many pitchers of tropical punch can he make? In the measurement case of division, the referent whole for the answer is the divisor (the second number in the operation). The extension of whole number measurement division to measurement division for fractions can be made by expanding the meaning for whole number division to include what the referent whole is for each fraction in the problem, including the solution.

For sharing division, as with measurement division, an adjustment must be made to transition from $a \div b$ (which for sharing means how large will each group be if a things are shared equally among b groups) to $\frac{a}{b} \div \frac{c}{d}$ in sharing division. Again this transition is made through understanding the division for whole numbers and identifying what each fraction in the process represents, by identifying its referent whole (Whitin, 2004). For $\frac{a}{b} \div \frac{c}{d}$ we want to know if a group of size $\frac{a}{b}$ was shared among $\frac{c}{d}$ of a group, how large is the group size. The referent whole for $\frac{a}{b}$ is the same as the referent whole for the solution, but the referent whole for $\frac{c}{d}$ is the group size.

For example the problem of $\frac{5}{8} \div \frac{2}{3}$ is how large is the group if two-thirds of the group is five-eighths of the whole, which is fifteen-sixteenths of the whole. Here the solution of fifteen-sixteenths has the same referent whole as five-eighths and the referent whole for two-thirds is the size of the group (Siebert, 2002, Toluk-Uçar, 2009). An example of a story problem using $\frac{5}{8} \div \frac{2}{3}$ is: Alex is printing out copies of his novel to give to friends to look over before he sends it to a publisher; he manages to get $\frac{5}{8}$ copies of his novel printed with the $\frac{2}{3}$ ream of paper left in his printer; how many copies of his novel can he print on one ream of paper? (Alex can print fifteen-sixteenths of his novel on one ream of paper). The referent whole for five-eighths and fifteen-sixteenths is the novel and the referent whole for two-thirds is the ream of paper, or the group size. The understanding of sharing division for fractions is built from the concepts of whole number division and fractions. Identifying the original number of objects, how many groups receive objects, and how many objects are in each group is what sharing division means. For fractions the number of objects is a portion of a whole number, the divisor is a fractional quantity of the number of groups, and the solution is the size of each group—a fraction with the same referent whole as the fractional quantity the problem began with.

The bridge between understanding the arithmetic operation on fractions can be built from the understanding of the arithmetic operation on whole numbers together with understanding of what fractions are. Building upon what it means to divide two whole numbers, from the sharing and measurement perspectives, and what fractions are, in relation to the referent whole, allows students to make meaning of the results of division of fractions (McAllister & Beaver, 2012).

4. Drawn Models

Knowing underlying meanings of and having flexibility with representations are characteristics of a competent problem solver (CCSSI, 2010; Goldin, 2002; NCTM, 2000). Teachers' understanding of representations boost or limit their ability to support the development of student understanding of rational numbers (Izsák, 2008). While literature has suggested the positive contribution of drawn representations, as an example, moving from concrete to abstract understandings of mathematical concepts (e.g., Post, Wachsmuth, Lesh, & Behr, 1985), many studies done on representations have focused on graphs (e.g., Gagatsis & Shiakalli, 2004); but few investigated the ability to interpret drawn diagrams such as area and number line models. Also, although the transitions from visual representations to symbolic representations are often addressed, the inverse (addressing the transitions from symbolic representations to visual representations) is seldom touched upon (Luo, Lo, & Leu, 2011). The present study attempted to address these gaps.

II. Method

1. Research Questions

To examine preservice teachers' understanding of drawn models for fraction multiplication, this study attempted to address the following questions:

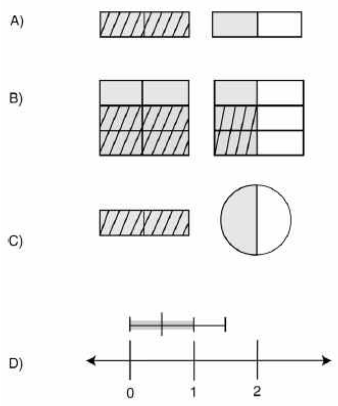
1. How do preservice teachers interpret drawn models for a given fraction multiplication problem?
2. What approach do preservice teachers use in their descriptions of fraction multiplication problem?

3. What relationship, if any, exists between the ability to interpret drawn models for fraction multiplication and the ability to interpret the meaning of fraction multiplication?

2. Data Collection & Analysis

Data for this study were collected from 82 preservice elementary teachers at the exit of their teacher preparation program in Korea. Depending on what subject matter participants declared as their major, they had taken two or three mathematics content courses. The data reported in this study were gathered as part of a larger study of teacher knowledge in which an assessment of preservice elementary teachers' mathematical knowledge for teaching over a variety of topics including numbers and operations and proportions was developed and administered. Items on the assessment were a combination of multiple choice and constructed response items and included items taken from existing literature (Hill, Schilling, & Ball, 2004) as well as several items created by a group of mathematics educators, including the author of this study. The items, used in the analysis of the study reported here were:

1. Which model below cannot be used to show that $1\frac{1}{2}\times\frac{2}{3}=1$?



2. How can $\frac{2}{3}\times\frac{5}{7}$ be described in words?

III. Results

1. On Research Question 1

The drawn model task was scored as correct or incorrect. The suggested correct answer was option C. Options A, B, and D exhibited models that could be used to represent the given fraction expression. The diagram in option C was not a valid model because the two shapes (one rectangle and one circle) used to represent the unit whole would not guarantee the same area. 25 out of 82 preservice teachers chose the suggested answer (Option C) as the model that cannot be used to show the multiplication problem presented in the task. A majority of the remaining preservice teachers chose either option B (involving two identical rectangles with each being divided into 6 parts) or option D (showing a number line model instead of an area model). Option D was the most frequently chosen answer. Distribution of responses on the drawn model task is displayed in Table III-1.

<Table III-1> Distribution of responses on the drawn model task

Options	Total (*n*=82)
A	7 (9%)
B	20 (24%)
C (suggested answer)	25 (30%)
D	28 (34%)
no answer	2 (3%)

2. On Research Question 2

Regardless of whether preservice teachers could solve a fraction multiplication problem, the approaches to interpreting the fraction multiplication expression were analysed and presents in Table III-2. 23 out of 82 preservice teachers used an interpretation applicable for multiplication of fractions such as part-of-part, area of a rectangular shape, scaling, or multiplication rule of probability for independents events. Sample story problems from responses include the following: If she uses $\frac{5}{7}$ of a stick of butter per pie, how many sticks of butter are necessary for $\frac{2}{3}$ of a pie? (part-of-part, n=14); Bobby is planting a small garden in his backyard. The length of the garden is $\frac{2}{3}$ meters, while the width is $\frac{5}{7}$ meters. Bobby wants to know what area he has available to fill with plants (area of a rectangular shape, n=1); I have $\frac{2}{3}$ cup of [flour]. My grandma says she'll give me $\frac{5}{7}$ times more. How much do I have now? (scaling, n=7); I have wrecked 2 out of the 3 cars I have owned. I have also bought 7 new tires for these cars, and 5 out of 7 were junk. What are the chances of me wrecking a car and blowing a tire? (multiplication rule of probability, n=1). These examples contain errors: Some are subtle and others are fundamental. The remaining 59 preservice teachers used a different operation such as addition, subtraction, or division; or a combination involving more than one operation, provided insufficient information, or were left blank. Therefore, it made no sense to evaluate what types of approaches were utilized in those descriptions. To reveal any possible effect of a particular type of approaches on the ability to describe a meaning of fraction multiplication, the descriptions that illustrated a multiplication situation involving the two fractions given in the task, whether they were solved by the calculation that was asked to be represented (i.e., descriptions with no errors) or not (i.e., descriptions with errors), were examined to classify the types of approaches in them. Part-of-part and scaling were predominantly utilized in those problems, with part-of-part being the prevalent type of approach used in the descriptions with no errors.

<Table III-2> Types of approaches used in descriptions illustrating fraction multiplication

Approaches	Description with no errors (n=7)	Description with errors (n=16)
rectangle area	1	0
part-of-part	6	8
multiplication rule in probability	0	1
scaling	0	7

3. On Research Question 3

After the two tasks were analysed individually, their results were compared to reveal any relationship that might exist between the observed approaches in the description task and the ability to interpret drawn models for fraction multiplication. To do so, first, the ways preservice teachers who described the problem correctly answered the drawn model task were examined and presented in Table III-3. Of 7 that described the problem correctly, 4 preservice teachers chose the suggested answer for the drawn model task.

<Table III-3> Distribution of the drawn model responses for correct description

Options	Correct description (*n*=7)
A	0
B	1
C (suggested answer)	4
D	2

Second, to reveal how useful a particular type of approach of fraction multiplication may have been to interpreting drawn models, types of approaches used in descriptions were examined among the 25 preservice teachers who correctly answered the drawn model task. The results are presented in Table III-4. Twenty three of the preservice teachers used an approach that is appropriate to reason about multiplication of fractions. Although the part-of-part approach can be seen as one of the most useful ways to reason about drawn models for fraction multiplication, only 9 used the part-of-part approach. There were two teachers who used the part-of-part approach; however, both were incorrect on the drawn model task. One selected option D as the answer while the other chose to leave it blank. The remaining 12 preservice teachers who were able to answer the drawn model task correctly were unsuccessful in utilizing a valid approach for fraction multiplication.

<Table III-4> Distribution of types of approaches for correct responses on the drawn model task

Types of Approaches	Drawn model responses that were

	correct (n=25)
rectangle area	1 (4%)
part-of-part	9 (36%)
multiplication rule in probability	0 (0%)
scaling	3 (12%)
unanalyzable	12 (48%)

IV. Discussion & Implications

A majority of preservice teachers did not demonstrate a robust understanding of what fraction multiplication entails: meaning of the numerator, denominator, and the unit whole; knowing that when a fraction multiplication problem involves a fraction that is less than one, then the product will be smaller rather than larger; and knowing the different types of approaches of fraction multiplication. They may know the procedural aspect of fraction multiplication, which was not assessed in this study, but their understanding of fraction multiplication assessed through the tasks used in this study suggests that their understanding of fraction multiplication is lacking. One common way in which the preservice teachers described the fraction multiplication was through the use of choosing an arbitrary whole number. It seemed to give them direction in illustrating the problem. Before sorting through the data, it was hypothesized that part-of-part would be the most useful approach to reason about drawn models for fraction multiplication. Of a total of 7 preservice teachers that used the part-of-part interpretation in their story problems, only 4 were able to answer which model could not be used to show the given fraction multiplication problem. Even though part-of-part seemed to be an appropriate and logical interpretation to use when interpreting drawn models for fraction multiplication, apparently it did not influence all 14 preservice teachers' reasoning about them. This suggests that while preservice teachers may recognize the part-of-part approach and know how to describe a fraction multiplication using the approach, they may not be fluent in applying that knowledge to models for fraction multiplication. Being able to translate this approach to a diagram such as drawn model would indicate a more complete understanding of fraction multiplication.

Although more information could have been gained by employing additional data sources such as an interview or more assessment items, the findings of the study suggest several implications for teacher education and future study. From this study, preservice teachers' understanding of fraction multiplication was not found to be satisfactory and their understandings of other operations might be similar. To help overcome this inadequacy, it may be beneficial to provide opportunities where preservice teachers create contextual problem situations and reason about models for operations with fractions (Toluk-Uçar, 2009). Preservice teachers' increased opportunities to explore these ideas would help deepen their knowledge of the meanings of fraction operations, which would help their future students learn them more meaningfully. Such opportunities would also enhance their understanding of and flexibility with multiple representations of fraction operations. Preservice teachers need to be proficient in moving among models (visual representations), story problems, mathematical sentences (symbolic representations), and algorithms. The ability to analyse and understand the mathematical meanings exhibited in these representations is imperative and is a very important form of mathematical knowledge needed for teaching. A thorough investigation of various textbooks and other instructional resources is also suggested (e.g., Charalambous, Hill, & Mitchell, 2012). Such an investigation would help preservice teachers gain the knowledge of available curriculum materials, different approaches those materials take, and how effective the approaches are.

Being familiar with the types of materials can help preservice teachers broaden their range of resources for their future classrooms. It is essential that preservice teachers understand how various topics regarding fraction operations are presented so that they can evaluate the different approaches and gauge which would be advantageous to use. This way, they will be able to teach fraction multiplication to their students so that the students can develop deep conceptual and procedural understandings of such an important mathematical topic.

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Noh, Jihwa
Department of Mathematics Education
Pusan National University
Pusan, 46241 Korea
E-mail address: nohjihwa@gmail.com