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THE SEQUENTIAL ATTAINABILITY AND ATTAINABLE ACE

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Abstract. For any non-negative real number ϵ_0, we shall introduce a concept of the ϵ_0-dense subset of R^m. Applying this concept, for any sequence $\{\epsilon_n\}$ of positive real numbers, we also introduce the concept of the $\{\epsilon_n\}$-attainable sequence and of the points of $\{\epsilon_n\}$-attainable ace in the open subset of R^m. We also study the characteristics of those sequences and of the points of $\{\epsilon_n\}$-dense ace. And we research the conditions that an $\{\epsilon_n\}$-attainable sequence has no $\{\epsilon_n\}$-attainable ace. We hope to reconsider the social consideration on the ace in social life by referring to these concepts about the aces.

1. Introduction

In this section, we briefly introduce the concept of the ϵ_0-dense subset in an open subset of R^m which we studied in [5]. Let's denote by $B(x,\epsilon)$ (resp. $\overline{B}(x,\epsilon)$) the open (resp. closed) ball in R^m with radius ϵ and center at x.

Definition 1.1. Let $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. If D is a non-empty subset of R^m then a point $a \in R^m$ is an ϵ_0-accumulation point of D if and only if $B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ for all positive real number $\epsilon > \epsilon_0$. And we denote by $D'_{(\epsilon_0)}$ the set of all the ϵ_0-accumulation points of D in R^m.

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Definition 1.2. Let $\epsilon_0 \geq 0$ and E be a non-empty open subset of R^m. A subset $D \subseteq E$ is called an ϵ_0-dense subset of E if and only if $E \subseteq D'_{(\epsilon_0)} \cup D$. In this case, we call that D is ϵ_0-dense in E.

Proposition 1.3. *Let D be a subset of a non-empty open subset E in R^m and $\epsilon_0 \geq 0$ be any, but fixed, non-negative real number. Then D is ϵ_0-dense in E if and only if $E \subseteq \underset{b\in D}{\cup}\overline{B}(b,\epsilon)$ for each positive real number $\epsilon > \epsilon_0$.*

Proof. $(\Rightarrow)$ Suppose that D is ϵ_0-dense in E and let any positive real number $\epsilon > \epsilon_0$ be given. For any vector $a \in E$, if $a \in D$ then we are done since $a \in \overline{B}(a,\epsilon)$. On the other hand, suppose that $a \in E - D$. Since D is ϵ_0-dense in E and $\epsilon > \epsilon_0$, we must have $B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$. Thus there exists an element $b \in D$ such that $b \in B(a,\epsilon)$. This immediately implies that $a \in B(b,\epsilon)$. Hence we have

$$a \in B(b,\epsilon) \subseteq \overline{B}(b,\epsilon) \subseteq \underset{b\in D}{\cup}\overline{B}(b,\epsilon).$$

$(\Leftarrow)$ Let any member $a \in E$ be given. And let any $\epsilon > \epsilon_0$ be given. If $a \in D$ then we are done since $a \in D'_{(\epsilon_0)} \cup D$. Suppose that $a \in E - D$. Since $E \subseteq \underset{b\in D}{\cup}\overline{B}(b,\epsilon_0 + \frac{\epsilon-\epsilon_0}{2})$ and $\epsilon_0 + \frac{\epsilon-\epsilon_0}{2} > \epsilon_0$, we have $a \in \overline{B}(b_\epsilon, \epsilon_0 + \frac{\epsilon-\epsilon_0}{2})$ for some element $b_\epsilon \in D$. Thus we have $b_\epsilon \in \overline{B}(a, \epsilon_0 + \frac{\epsilon-\epsilon_0}{2})$. Since $\epsilon_0 + \frac{\epsilon-\epsilon_0}{2} < \epsilon_0 + \epsilon - \epsilon_0 = \epsilon$, we have $b_\epsilon \in \overline{B}(a,\epsilon)$ which implies that $B(a,\epsilon) \cap (D - \{a\}) \neq \emptyset$ since this set contains the element $b_\epsilon \in D$ and $b_\epsilon \neq a$. Therefore, we must have $a \in D'_{(\epsilon_0)}$ which completes the proof. □

We have so far considered about the fixed value of ϵ_0. From now on, we will think about changing values of ϵ_0.

2. The sequentially attainable set in R^m

Now let's study about the concepts of the sequentially attainable (or dense) sequence and the sequentially attainable (or dense) subsets in R^m and investigate the shape of those sequences and sets. Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in R^m$ and ϵ be any non-negative real number. Let's denote by $C(\alpha,\epsilon) = \{x \in R^m : |x_k - \alpha_k| < \epsilon, k = 1,2,3,\ldots,m\}$ and $\overline{C}(\alpha,\epsilon) = \{x \in R^m : |x_k - \alpha_k| \leq \epsilon, k = 1,2,3,\ldots,m\}$ the open and closed m-dimensional cube in R^m.

Definition 2.1. Let a non-negative real number ϵ_1 be given. For a given point $a \in R^m$, a point $b \in R^m$ is an ϵ_1-adherent point of a if and only if $b \in C(a, \epsilon)$ for all $\epsilon > \epsilon_1$. And a point $b \in R^m$ is an ϵ_1-isolated point of a if and only if $b \notin C(a, \epsilon')$ for some positive real number $\epsilon' > \epsilon_1$.

Note that a point $b \in R^m$ is an ϵ_1-adherent point of a if and only if $b \in \overline{C}(a, \epsilon_1)$.

Definition 2.2. Let $\{\epsilon_n\}$ be any, but fixed, sequence of non-negative real numbers. For a given sequence $\{a_n\}$ in R^m, a point $b \in R^m$ is an $\{\epsilon_n\}-$adherent point of $\{a_n\}$ if and only if there exists a natural number $n_0 \in N$ such that b is an $\epsilon_{n_0}-$adherent point of a_{n_0}. And a point $b \in R^m$ is an $\{\epsilon_n\}-$isolated point of the sequence $\{a_n\}$ if and only if b is an ϵ_n-isolated point of a_n for each natural number $n \in N$.

Let's denote by $ADH(\{a_n\}, \{\epsilon_n\})$ the set of all the $\{\epsilon_n\}-$adherent points of $\{a_n\}$.

Definition 2.3. Let $\{\epsilon_n\}$ be any, but fixed, sequence of positive real numbers and E be any non-empty and open subset of R^m. We define that a sequence $\{a_n\}$ of the elements of E is an $\{\epsilon_n\}-$ attainable sequence in E if and only if $E \subseteq ADH(\{a_n\}, \{\epsilon_n\})$, i.e., every point of E is an $\{\epsilon_n\}-$adherent point of the sequence $\{a_n\}$. In this case, the ordered pair $(\{a_n\}, \{\epsilon_n\})$ is called a sequentially attainable pair of E.

Note that E can be a proper subset of $ADH(\{a_n\}, \{\epsilon_n\})$ in the definition just above.

Definition 2.4. Let $\{\epsilon_n\}$ be any, but fixed, sequence of positive real numbers and E be any non-empty and open subset of R^m. We define that E is an $\{\epsilon_n\}-$ sequentially attainable set if and only if there is a sequence $\{a_n\}$ of the elements of E such that $\{a_n\}$ is an $\{\epsilon_n\}-$ attainable sequence in E.

Lemma 2.5. *Let $\{\epsilon_n\}$ be any, but fixed, sequence of positive real numbers and let $\{a_n\}$ be a given sequence in R^m. Then a point $b \in R^m$ is an $\{\epsilon_n\}-$adherent point of $\{a_n\}$ if and only if $b \in \underset{n \in N}{\cup} \overline{C}(a_n, \epsilon_n)$. Hence*

$$ADH(\{a_n\}, \{\epsilon_n\}) = \underset{n \in N}{\cup} \overline{C}(a_n, \epsilon_n).$$

Proof. For each natural number $n \in N$, b is an ϵ_n-adherent point of a_n if and only if $b \in C(a_n, \epsilon)$ for each positive real number $\epsilon > \epsilon_n$.

Since the last statement holds if and only if $b \in \overline{C}(a_n, \epsilon_n)$, the result follows. □

PROPOSITION 2.6. *Let $\{\epsilon_n\}$ be any, but fixed, sequence of positive real numbers and let $\{a_n\}$ be a given sequence in an open subset E of R^m. The sequence $\{a_n\}$ is $\{\epsilon_n\}-$attainable in E if and only if $E \subseteq \bigcup_{n\in N} \overline{C}(a_n, \epsilon_n)$.*

Proof. This follows immediately from the lemma 2.5. □

Note that the volume of the closed m-dimensional cube $\overline{C}(a_n, \epsilon_n)$ is given by

$$Vol\left(\overline{C}(a_n, \epsilon_n)\right) = 2^m \epsilon_n^m.$$

LEMMA 2.7. *Let E be a nonempty open subset of R^m. Then E is the union of a countable disjoint collection of half-open m-dimensional cubes, each of which is of the form*

$$\{(x_1, \cdots, x_m) : j_i 2^{-k} \leq x_i < (j_i + 1)2^{-k}, i = 1, 2, \cdots, m\}$$

for some integers $j_1, j_2, \cdots, j_m$ and some natural number k.

Proof. For each natural number k, let C_k be the set of all the m-dimensional cubes of the form

$$\{(x_1, \cdots, x_m) : j_i 2^{-k} \leq x_i < (j_i + 1)2^{-k}, i = 1, 2, \cdots, m\}$$

with arbitrary integers $j_1, j_2, \cdots, j_m$. It is clear that each C_k is countable and a partition of R^m. Moreover, if $k_1 < k_2$ then each m-dimensional cube in C_{k_2} is contained in some member of C_{k_1}. Now, for the given open subset E of R^m, let's construct another collection D of m-dimensional cubes inductively as follows. Let D be the empty set at the first step. At the k-th step, let's add to D those m-dimensional cubes in C_k that are included in E but are disjoint from all the m-dimensional cubes contained in D at earlier steps. Then D is clearly a countable disjoint collection of m-dimensional cubes whose union is included in E. Hence we need only to verify that E is a subset of the union $\cup D$. Let x be any element of E. Since E is an open subset of R^m, the m-dimensional cube in C_k which contains x is included in E if k is sufficiently large. Let k_0 be the smallest number of such natural numbers k. Then the m-dimensional cube in C_{k_0} that contains x belongs to D. Therefore, x belongs to the union of the cubes in D. □

Theorem 2.8. *Let $\{\epsilon_n\}$ be any, but fixed, sequence of positive real numbers and let E be a nonempty open subset of R^m. If Vol$(E) > 2^m \sum_{n=1}^{\infty} \epsilon_n^m$, then there exists no sequence $\{a_n\}$ in E such that $\{a_n\}$ is $\{\epsilon_n\}-$ attainable in E. Or equivalently, if E is an $\{\epsilon_n\}-$ sequentially attainable set then Vol$(E) \leq 2^m \sum_{n=1}^{\infty} \epsilon_n^m$. And the converse is not true in general.*

Proof. Since the volume of the closed m-dimensional cube $\overline{C}(a_n, \epsilon_n)$ is $2^m \epsilon_n^m$, if Vol$(E) > 2^m \sum_{n=1}^{\infty} \epsilon_n^m$ then no form of the union $\overset{\infty}{\underset{n=1}{\cup}} \overline{C}(a_n, \epsilon_n)$ shall contain the set E. On the other hand, in order to prove that the converse is not true in general, let $\{\epsilon_n\}$ be a sequence of positive real numbers such that $2^m \sum_{n=1}^{\infty} \epsilon_n^m < \infty$. Since $lim_{n\to\infty}\epsilon_n = 0$, the maximum $\epsilon_M = \max\{\epsilon_n : n \in N\}$ exists. Let's choose a natural number $K_0 \in N$ so large that $K_0 > 3\epsilon_M + 3$. And choose a sequence $\{b_n\}$ of vectors in R^m such that $b_n = ((n-1)K_0, 0, \cdots, 0) \in R^m$ for each natural number $n \in N$. Let E be the open subset given by

$$E = \overset{\infty}{\underset{n=1}{\cup}} C(b_n, \epsilon_n) - \{b_M\}.$$

Then we have Vol$\{E\} = 2^m \sum_{n=1}^{\infty} \epsilon_n^m < \infty$. But suppose that there exists a sequence $\{a_n\}$ in E such that $\{a_n\}$ is an $\{\epsilon_n\}-$ attainable sequence in E. Then we have $E \subseteq \overset{\infty}{\underset{n=1}{\cup}} \overline{C}(a_n, \epsilon_n)$. Since $b_M \notin E$, we have $a_n \neq b_M$ for all natural number $n \in N$. Hence there are at least two closed cubes, say $\overline{C}(a_p, \epsilon_p)$ and $\overline{C}(a_q, \epsilon_q)$, which have the non-empty intersections with the cube $C(b_M, \epsilon_M)$ since ϵ_M is the maximum. If $\epsilon_p = \epsilon_M$ or $\epsilon_q = \epsilon_M$ then $\underset{n\neq p,q}{\cup} \overline{C}(a_n, \epsilon_n)$ must contain the set $E - C(b_M, \epsilon_M)$. But this is impossible since

$$Vol\{\underset{n\neq p,q}{\cup} \overline{C}(a_n, \epsilon_n)\} = \sum_{n\neq p,q} 2^m \epsilon_n^m < \sum_{n\neq M} 2^m \epsilon_n^m = Vol\{E - C(b_M, \epsilon_M)\}.$$

And if $\epsilon_p \neq \epsilon_M$ for all ϵ_p such that $\overline{C}(a_p, \epsilon_p) \cap C(b_M, \epsilon_M) \neq \emptyset$ then there is a term ϵ_r such that $\epsilon_r < \epsilon_M$ and $C(b_r, \epsilon_r) \subseteq \overline{C}(a_M, \epsilon_M)$ in the best situations since the cube $\overline{C}(a_M, \epsilon_M)$ can not contain more than one cube in E. Hence $\underset{n\neq M}{\cup} \overline{C}(a_n, \epsilon_n)$ must contain the set $E - C(b_r, \epsilon_r)$ which is also impossible since

$$Vol\{E\} = \sum_{n\in N} Vol\{\overline{C}(a_n, \epsilon_n)\} \text{ and } \quad Vol\{\overline{C}(a_r, \epsilon_r)\} < Vol\{\overline{C}(a_M, \epsilon_M)\}.$$

Hence there is no $\{\epsilon_n\}$-attainable sequence in E. □

THEOREM 2.9. *Let $\{\epsilon_n\}$ be a sequence of positive real numbers which satisfies the condition $\overline{\lim_{n\to\infty}}\, \epsilon_n = \epsilon_0 > 0$. Then any non-empty open subset E of R^m is an $\{\epsilon_n\}$-sequentially attainable set.*

Proof. Since $\overline{\lim_{n\to\infty}}\, \epsilon_n = \epsilon_0$ and $\frac{\epsilon_0}{2} > 0$, there are infinitely many natural numbers $n_1 < n_2 < n_3 < \cdots < n_k < \ldots$ such that $\forall k \in N \Rightarrow \epsilon_0 - \frac{\epsilon_0}{2} < \epsilon_{n_k}$. Since E is an open subset of R^m and $E \cap Q^m$ is countable, there is a sequence $\{b_k\}$ in E such that $E \cap Q^m = \{b_1, b_2, b_3, \ldots, b_k, \ldots\}$. Then we have $E \subseteq \overset{\infty}{\underset{k=1}{\cup}} C(b_k, \frac{\epsilon_0}{2})$. Set $n_0 = 0$. Then, for each natural number $n \in N$, there is a unique non-negative integer k such that $n_{k-1} + 1 \leq n \leq n_k$. Now, for each natural number $k \in N$, choose a sequence $\{a_n\}$ in E such that $a_n = b_k$ whenever $n_{k-1} + 1 \leq n \leq n_k$. Then $\{a_n\}_{n=1}^{\infty}$ is an infinite sequence in E and $a_{n_k} = b_k$ for each natural number $k \in N$. Thus we have

$$\begin{aligned} E \subseteq \overset{\infty}{\underset{k=1}{\cup}} C(b_k, \frac{\epsilon_0}{2}) &= \overset{\infty}{\underset{k=1}{\cup}} C(a_{n_k}, \frac{\epsilon_0}{2}) \\ &\subseteq \overset{\infty}{\underset{k=1}{\cup}} C(a_{n_k}, \epsilon_{n_k}) \\ &\subseteq \overset{\infty}{\underset{n=1}{\cup}} C(a_n, \epsilon_n) \\ &\subseteq \overset{\infty}{\underset{n=1}{\cup}} \overline{C}(a_n, \epsilon_n). \end{aligned}$$

□

When the dimension $m = 1$, we have the following proposition.

PROPOSITION 2.10. *Let $\{\epsilon_n\}$ be a sequence of positive real numbers. If $\sum_{n=1}^{\infty} \epsilon_n = \infty$ then any non-empty open subset E of the real number system R is an $\{\epsilon_n\}$-sequentially attainable set. And the converse is also true.*

Proof. Let any non-empty open subset E of the real number system R be given. By lemma 2.7, E can be represented as the union $E = \overset{\infty}{\underset{n=1}{\cup}} (c_n, d_n]$ of a disjoint collection of the half-open intervals $(c_n, d_n]$. For the interval $(c_1, d_1]$, choose a real number $b_1 = c_1 + \epsilon_1$. Now choose a sequence $\{b_n\}$ such that $b_{n+1} = b_n + \epsilon_n + \epsilon_{n+1}$ for each natural number

$n \in N$. Then we have $(c_1, d_1] \subseteq \bigcup_{i=1}^{\infty} \overline{C}(b_i, \epsilon_i)$ and $\mathrm{Vol}\left\{\bigcup_{i=1}^{\infty} \overline{C}(b_i, \epsilon_i)\right\} = 2\sum_{i=1}^{\infty} \epsilon_i$. Since $\sum_{i=1}^{\infty} \epsilon_i = \infty$, there is a natural number n_1 such that $(c_1, d_1] \subseteq \bigcup_{i=1}^{n_1} [b_i, \epsilon_i]$. Moreover, the minimal natural number, say m_1, of such $n_1's$ must exist since $(c_1, d_1]$ is bounded. Now choose a sequence $\{a_i\}$ in E such that $a_i = b_i$ for each natural number $i = 1, 2, \cdots, m_1 - 1$ and

$$a_{m_1} = \begin{cases} b_{m_1} & \text{if } \ b_{m_1} \in E \\ d_1 & \text{if } \ b_{m_1} \notin E. \end{cases}$$

Then we have $(c_1, d_1] \subseteq \bigcup_{i=1}^{m_1} \overline{C}(a_i, \epsilon_i)$ with $\{a_i\}_{i=1}^{m_1} \subseteq E$. Since we also have $\sum_{i=m_1+1}^{\infty} \epsilon_i = \infty$, we can prove by the same manner as the above that $(c_2, d_2] \subseteq \bigcup_{i=m_1+1}^{m_1+m_2} \overline{C}(a_i, \epsilon_i)$ for some finite sequence $\{a_i\}$ in E and some natural number m_2. Continuing this process, we can prove that $E = \bigcup_{n=1}^{\infty} (c_n, d_n] \subseteq \bigcup_{n=1}^{\infty} \overline{C}(a_n, \epsilon_n)$ for some infinite sequence $\{a_n\}$ in E. Hence E is an $\{\epsilon_n\}$-sequentially attainable set. And the converse is obviously true since the set R is $\{\epsilon_n\}$-sequentially attainable. $\square$

On the other hand, we have the following results.

LEMMA 2.11. *If E is any non-empty open subset of R^m then E is $\{\frac{1}{n^{1/m}}\}$-sequentially attainable.*

Proof. Since E is an open subset of R^m, E can be represented as the union of the countable disjoint collection of the half-open cubes $C_1, C_2, \cdots, C_n, \cdots$ in R^m. Let's choose a natural number $n_1 > 2^m$ so large that $\frac{1}{(n_1)^{1/m}}$ is less than the length of the edge of the cube C_1. Then the closure $\overline{C_1}$ can be written as the union of a finite collection, say $D_1, \cdots, D_k$, of closed cubes whose common size is $\frac{1}{(n_1)^{1/m}} \times \frac{1}{(n_1)^{1/m}} \times \cdots \times \frac{1}{(n_1)^{1/m}}$ (m terms) and with centers at C_1. But D_1 is the union of the two m-dimensional rectangles whose common size is $\left(\frac{1}{2}\frac{1}{(n_1)^{1/m}}\right) \times \frac{1}{(n_1)^{1/m}} \times \cdots \times \frac{1}{(n_1)^{1/m}}$. And the m-dimensional rectangle of this size consists of 2^{m-1} m-dimensional cubes of the size $\frac{1}{2}\frac{1}{(n_1)^{1/m}} \times \frac{1}{2}\frac{1}{(n_1)^{1/m}} \times \cdots \times \frac{1}{2}\frac{1}{(n_1)^{1/m}}$.

Hence we have

$$\begin{aligned} D_1 &= \overline{C}(a_1, \frac{1}{2\times 2}\frac{1}{(n_1)^{1/m}}) \cup \cdots \cup \overline{C}(a_{2^{m-1}}, \frac{1}{2\times 2}\frac{1}{(n_1)^{1/m}}) \\ &\subseteq \overline{C}(a_1, \frac{1}{[4^m(n_1 - 2^{m-1}+1)]^{1/m}}) \cup \cdots \cup \overline{C}(a_{2^{m-1}}, \frac{1}{(4^m n_1)^{1/m}}) \end{aligned}$$

for some elements $a_1, a_2, \cdots, a_{2^{m-1}} \in E$. Note that the last inclusion is meaningful since $4^m n_1 - 4^m(n_1 - 2^{m-1}+1) \geq 1$. On the other hand, the m-dimensional rectangle D_2 is the union of $2^{m-1}\times 2^m$ numbers of the m-dimensional cubes of the size $\frac{1}{2\times 2}\frac{1}{(n_1)^{1/m}} \times \cdots \times \frac{1}{2\times 2}\frac{1}{(n_1)^{1/m}}$ (m terms). Hence

$$\begin{aligned} D_2 &= \overline{C}(a_{(2^{m-1}+1)}, \frac{1}{2\times 2\times 2}\frac{1}{(n_1)^{1/m}}) \cup \cdots \cup \\ &\quad \overline{C}(a_{(2^{m-1}+2^{m-1}\times 2^m)}, \frac{1}{2\times 2\times 2}\frac{1}{(n_1)^{1/m}}) (2^{m-1}\times 2^m \text{terms}) \\ &\subseteq \overline{C}(a_{(2^{m-1}+1)}, \frac{1}{(2^{3m}n_1 - 2^{m-1}\times 2^m + 1)^{1/m}}) \cup \cdots \cup \\ &\quad \overline{C}(a_{(2^{m-1}+2^{m-1}\times 2^m)}, \frac{1}{(2^{3m}n_1)^{1/m}}) \end{aligned}$$

for some $2^{m-1}\times 2^m$ elements $a_{(2^{m-1}+1)}, \cdots, a_{(2^{m-1}+2^{m-1}\times 2^m)} \in E$. Note that the last inclusion makes sense since $2^{3m}n_1 - 2^{m-1}\times 2^m + 1 \geq 4^m n_1$. Continuing this process, we can show that the m-dimensional rectangle D_k is the union of $2^{m-1}\times (2^m)^{k-1}$ numbers of the m-dimensional cubes of the size $\frac{1}{2^k}\frac{1}{(n_1)^{1/m}} \times \cdots \times \frac{1}{2^k}\frac{1}{(n_1)^{1/m}}$ (m terms). Hence

$$\begin{aligned} D_k &= \overline{C}(a_., \frac{1}{2\times 2^k}\frac{1}{(n_1)^{1/m}}) \cup \cdots \cup \\ &\quad \overline{C}(a_., \frac{1}{2\times 2^k}\frac{1}{(n_1)^{1/m}}) (2^{m-1}\times (2^m)^{k-1} \text{terms}) \\ &\subseteq \overline{C}(a_., \frac{1}{(2^{(k+1)m}n_1 - 2^{m-1}\times (2^m)^{k-1} + 1)^{1/m}}) \cup \cdots \cup \\ &\quad \overline{C}(a_., \frac{1}{(2^{(k+1)m}n_1)^{1/m}}) \end{aligned}$$

for some elements $a_.$'s in E. Note that the last inclusion is meaningful since $2^{(k+1)m}n_1 - 2^{m-1}\times (2^m)^{k-1} + 1 \geq 2^{km}n_1$. Hence the m-dimensional closed cube $\overline{C_1}$ can be contained in the union of a finite collection of m-dimensional cubes of the form $\overline{C}(\cdot, \frac{1}{n^{1/m}})$ with centers at E. Now we have proved that there is a natural number M_1 such that $\overline{C_1} \subseteq \bigcup_{n=1}^{M_1} \overline{C}(b_n, \frac{1}{n^{1/m}})$ for some sequence $\{b_n\}$ in E. On the other hand, if we choose a natural number n_2 so large that $\frac{1}{(n_2)^{1/m}}$ is less than the length of the edge of the cube C_2 and $n_2 > M_1$, then we can prove by the similar manner as the above that there is a natural number M_2 such that $\overline{C_2} \subseteq \bigcup_{n=M_1+1}^{M_2} \overline{C}(b_n, \frac{1}{n^{1/m}})$ for some sequence $\{b_n\}$ in E. Continuing this process, we have a sequence $\{b_n\}$ in E such that $E \subseteq \bigcup_{n=1}^{\infty} \overline{C}(b_n, \frac{1}{n^{1/m}})$. Hence E is $\{\frac{1}{n^{1/m}}\}$-sequentially attainable. □

Thus we have the following proposition.

Proposition 2.12. *Let $\{\epsilon_n\}$ be a sequence of positive real numbers which satisfies the condition $\varliminf_{n\to\infty} \left(n^{1/m}\epsilon_n\right) > 0$. Then any non-empty open subset E of R^m is $\{\epsilon_n\}$-sequentially attainable.*

Proof. Since $\varliminf_{n\to\infty} \left(n^{1/m}\epsilon_n\right) = \alpha > 0$, there is a natural number $K \in N$ such that $n \geq K \Rightarrow n^{1/m}\epsilon_n \geq \frac{\alpha}{2}$. Hence we have

$$\exists K \in N \text{ such that } n \geq K \Rightarrow \epsilon_n \geq \frac{\alpha/2}{n^{1/m}}.$$

By the proof of the lemma just above, any non-empty open subset E of R^m is also $\{\frac{\alpha/2}{n^{1/m}}\}_{n=K}^{\infty}$-sequentially attainable. Thus any non-empty open subset E of R^m is $\{\epsilon_n\}$-sequentially attainable since the cube of radius $\frac{\alpha/2}{n^{1/m}}$ is contained in the cube of radius ϵ_n for each natural number $n \geq K$. □

Note that we have the following remark when the dimension $m > 1$.

Remark 2.13. It is an open problem that every open subset E is $\{\epsilon_n\}$-sequentially attainable if $\sum_{n=1}^{\infty} \epsilon_n^m = \infty$ when the dimension $m > 1$.

But we have the following theorem.

Theorem 2.14. *Let $\{\epsilon_n\}$ be an infinite and bounded sequence of positive real numbers. Suppose that any non-empty open subset E of R^m is an $\{\epsilon_n\}-$sequentially attainable set. Then, for each sequence $\{a_p\}$*

of elements of R^m *and each sequence* $\{\delta_p\}$ *of positive real numbers, there is a partition*

$$\{\{(n_1)_k\}_{k=1}^{N_1}, \{(n_2)_k\}_{k=1}^{N_2}, \cdots, \{(n_p)_k\}_{k=1}^{N_p}, \cdots\}$$

of the set N *of all the natural numbers for some* $N_p \in N \cup \{\infty\}, p = 1, 2, 3, \cdots$ *and there are sequences* $\{d_{(n_p)_k}\}_{k=1}^{N_p}$ *of elements of the cube* $C(a_p, \delta_p)$ *for every* $p \in N$ *such that*

$$C(a_p, \delta_p) \subseteq \bigcup_{k=1}^{N_p} \overline{C}(d_{(n_p)_k}, \epsilon_{(n_p)_k})$$

for each natural numbers $p \in N$*. And the converse is also true.*

Proof. Let any sequence $\{a_p\}$ of elements of R^m and any sequence $\{\delta_p\}$ of positive real numbers be given. And set $\epsilon_M = \sup\{\epsilon_p | p \in N\}$ and let's denote by $e_1 = (1, 0, \cdots, 0)$ the unit vector of R^m. Now choose a sequence $\{D_p\}$ of cubes in R^m as follows.

$$\begin{aligned}
D_1 &= C(0, \delta_1) \\
D_2 &= C([3\epsilon_M + \delta_1 + \delta_2]\, e_1, \delta_2) \\
D_3 &= C([6\epsilon_M + \delta_1 + 2\delta_2 + \delta_3]\, e_1, \delta_3) \\
&\cdots \\
D_p &= C([3(p-1)\epsilon_M + \delta_1 + 2(\delta_2 + \delta_3 + \cdots + \delta_{p-1}) + \delta_p]\, e_1, \delta_p) \\
&\cdots
\end{aligned}$$

Then $E = \bigcup_{p=1}^{\infty} D_p$ is a non-empty open subset of R^m since it is the union of the set of a countable collection of the open cubes. Hence E is $\{\epsilon_p\}-$sequentially attainable. Thus there is a sequence $\{b_p\}$ of elements of E such that $E \subseteq \bigcup_{p=1}^{\infty} \overline{C}(b_p, \epsilon_p)$. Hence there is a finite or infinite subsequence $\{b_{(n_p)_k}\}_{k=1}^{N_p}$ of $\{b_p\}$ such that

$$D_p \cap \{b_p : p \in N\} = \{b_{(n_p)_k} | k \in \{1, 2, 3, \cdots, N_p\}\}.$$

Here $N_p = \infty$ if it is an infinite subsequence of $\{b_p\}$. Since $\{D_p : p \in N\}$ is a collection of the mutually disjoint open cubes, the set

$$\{\{(n_p)_k | k \in \{1, 2, 3, \cdots, N_p\}\} : p \in N\}$$

is a collection of mutually disjoint subsets of N. Since if there is a natural number $q \in N$ such that $q \notin \bigcup_{p=1}^{\infty} \{(n_p)_k | k \in \{1, 2, 3, \cdots, N_p\}\}$ then we need only to add the cube $C(b_q, \epsilon_q)$, we may assume that the set

$$\{\{(n_p)_k | k \in \{1, 2, 3, \cdots, N_p\}\} : p \in N\}$$

is a countable partition of the set N of all the natural numbers. Moreover, we have $D_p \subseteq \bigcup_{k=1}^{N_p} \overline{C}(b_{(n_p)_k}, \epsilon_{(n_p)_k})$ for each $p \in N$. Now put

$$d_{(n_p)_k} = b_{(n_p)_k} - [3(p-1)\epsilon_M + \delta_1 + 2(\delta_2 + \cdots + \delta_{p-1}) + \delta_p]\, e_1 + a_{(n_p)_k}$$

for each $p \in N$ and $k \in N$. Then we have

$$C(a_p, \delta_p) \subseteq \bigcup_{k=1}^{N_p} \overline{C}(d_{(n_p)_k}, \epsilon_{(n_p)_k})$$

for each $p \in N$ since it is the translation of D_p by the vector

$$a_p - [3(p-1)\epsilon_M + \delta_1 + 2(\delta_2 + \cdots + \delta_{p-1}) + \delta_p]\, e_1$$

for each $p \in N$.

In order to prove the statement of the converse in this theorem, suppose that the sequence $\{\epsilon_p\}$ satisfies the conclusion in this theorem. Let any non-empty open subset E of R^m be given. Since R^m is a second countable space and the set of all the open cubes in R^m forms a basis for the usual topology on R^m, E may be written as the union of a countable collection $\{C_p\}$ of the open cubes. Hence there is a sequence $\{a_p\}$ of the elements of R^m and there is another sequence $\{\delta_p\}$ of positive real numbers such that $C_p = C(a_p, \delta_p)$ for each $p \in N$. Hence, by assumption, there is a partition

$$\{\{(n_1)_k\}_{k=1}^{N_1}, \{(n_2)_k\}_{k=1}^{N_2}, \cdots, \{(n_p)_k\}_{k=1}^{N_p}, \cdots\}$$

of the set N of all the natural numbers for some $N_p \in N \cup \{\infty\}, p = 1, 2, 3, \cdots$ and there are sequences $\{d_{(n_p)_k}\}_{k=1}^{N_p}$ of elements of the cube $C(a_p, \delta_p)$ for all $p \in N$ such that

$$C(a_p, \delta_p) \subseteq \bigcup_{k=1}^{N_p} \overline{C}\left(d_{(n_p)_k}, \epsilon_{(n_p)_k}\right)$$

for every natural numbers $p \in N$. Thus we have

$$E = \bigcup_{p=1}^{\infty} C(a_p, \delta_p) \subseteq \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{N_p} \overline{C}\left(d_{(n_p)_k}, \epsilon_{(n_p)_k}\right) = \bigcup_{p=1}^{\infty} \overline{C}\left(d_p, \epsilon_p\right).$$

Since each d_p is an element of E for each $p \in N$, this implies that E is an $\{\epsilon_p\}$-sequentially attainable set. $\square$

Note that if $\{\epsilon_p\}$ is a sequence such that $\varliminf_{p\to\infty}\left(p^{1/m}\epsilon_p\right) = \alpha > 0$ then all the numbers of terms N_p in the theorem above can be chosen as the natural numbers in view of the proposition 2.12.

COROLLARY 2.15. *Let $\{\epsilon_p\}$ be an infinite and bounded sequence of positive real numbers. (1) If any non-empty open subset E of R^m is an $\{\epsilon_p\}_{p=1}^{\infty}-$sequentially attainable set then any non-empty open subset E of R^m is an $\{\epsilon_p\}_{p=K}^{\infty}-$sequentially attainable set for each natural number $K \in N$. (2) If there is a natural number $K \in N$ such that any non-empty open subset E of R^m is an $\{\epsilon_p\}_{p=K}^{\infty}-$sequentially attainable set then any non-empty open subset E of R^m is an $\{\epsilon_p\}_{p=1}^{\infty}-$sequentially attainable set.*

Proof. (1) Suppose that any non-empty open subset E of R^m is an $\{\epsilon_p\}_{p=1}^{\infty}-$sequentially attainable set and let any natural number $K \in N$ be given. And let any non-empty open subset E of R^m be given. Since R^m is a second countable space and the set of all the open cubes in R^m forms a basis for the usual topology on R^m, E may be written as the union of a countable collection $\{C_p\}$ of the open cubes. Hence there is a sequence $\{a_p\}$ of the elements of R^m and there is another sequence $\{\delta_p\}$ of positive real numbers such that $C_p = C(a_p, \delta_p)$ for each $p \in N$. Now consider a sequence $\{D_p\}$ of cubes defined by the relation

$$D_p = C_q \text{ if } \ (q-1)K + 1 \le p \le qK$$

for each natural number $q = 1, 2, \cdots$. Then the centers of $\{D_p\}$ forms an infinite sequence of vectors in R^m and the radii of $\{D_p\}$ forms an infinite sequence of positive real numbers. Hence by the theorem above, there is a partition

$$\{\{(n_1)_k\}_{k=1}^{N_1}, \{(n_2)_k\}_{k=1}^{N_2}, \cdots, \{(n_p)_k\}_{k=1}^{N_p}, \cdots\}$$

of the set N for some $N_p \in N \cup \{\infty\}, p = 1, 2, 3, \cdots$ and there are sequences $\{d_{(n_p)_k}\}_{k=1}^{N_p}$ of elements of the cube D_p for all $p \in N$ such that

$$D_p \subseteq \bigcup_{k=1}^{N_p} \overline{C}(d_{(n_p)_k}, \epsilon_{(n_p)_k})$$

for every natural numbers $p \in N$. Since the subscripts $(n_p)_k$ form a partition of N and

$$D_1 = D_2 = \cdots = D_K \subseteq \bigcup_{k=1}^{N_p} \overline{C}(d_{(n_p)_k}, \epsilon_{(n_p)_k})$$

for all $p = 1, 2, \cdots, K$, there must exist a natural number $1 \le p_1 \le K$ such that $(n_{p_1})_k \ge K$ for all $k \in N$. Similarly, since the subscripts form a partition of N and, for each $q \in N$,

$$D_{(q-1)K+1} = D_{(q-1)K+2} = \cdots = D_{qK} \subseteq \bigcup_{k=1}^{N_p} \overline{C}(d_{(n_p)_k}, \epsilon_{(n_p)_k})$$

for all $p = (q-1)K+1, (q-1)K+2, \cdots, qK$, there must exist a natural number $(q-1)K+1 \le p_q \le qK$ such that $(n_{p_q})_k \ge K$ for all $k \in N$ and for each $q \in N$. Now we have

$$E = \cup C_p = \bigcup_{q=1}^{\infty} D_{p_q} \subseteq \bigcup_{q=1}^{\infty} \bigcup_{k=1}^{N_{pq}} \overline{C}(d_{(n_{pq})_k}, \epsilon_{(n_{pq})_k}).$$

Therefore, E is an $\{\epsilon_p\}_{p=K}^{\infty}-$sequentially attainable set. (2) It is obvious since we need only to add the remaining terms. □

3. The sequential dense-ace in R^m

Let's denote by $\{a_n\}_{n=1}^{K}$ a finite or infinite sequence with $K \in N \cup \{\infty\}$. For each natural number $n_0 \in N$, let's denote by $\{a_n\}_{n \neq n_0}$ the finite or infinite sequence which is obtained from $\{a_n\}_{n=1}^{K}$ by removing the term a_{n_0}. Note that the $(n_0+1)st$ term a_{n_0+1} in $\{a_n\}_{n=1}^{K}$ is the n_0-th term in $\{a_n\}_{n \neq n_0}$. Moreover, let's denote the maximum norm of a vector $x \in R^m$ by $\|x\|_\infty = max\{|x_i| : i = 1, 2, \cdots, m\}$. In this section, we study some properties of the attainable (or dense) sequence and introduce a concept of the sequentially attainable (or dense) ace .

Definition 3.1. Let $\{\epsilon_n\}_{n=1}^{K}$ be any finite or infinite sequence of positive real numbers with $K \in N \cup \{\infty\}$. And let E be a non-empty open subset of R^m. A finite or infinite sequence $\{a_n\}_{n=1}^{K}$ in E is called an $\{\epsilon_n\}$-attainable (or dense) sequence in E if and only if $E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n, \epsilon_n)$.

Definition 3.2. Let $\{\epsilon_n\}_{n=1}^{K}$ be any finite or infinite sequence of positive real numbers with $K \in N \cup \{\infty\}$ and E be a non-empty open subset of R^m. Suppose that a finite or infinite sequence $\{a_n\}_{n=1}^{K}$ in E is an $\{\epsilon_n\}$-attainable sequence in E. A term a_{n_0} is called an $\{\epsilon_n\}$-attainable ace of the sequence $\{a_n\}_{n=1}^{K}$ in E if and only if $E \not\subseteq \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n)$. In this case, we call the ordered pair $(a_{n_0}, \epsilon_{n_0})$ the pair of the $\{\epsilon_n\}$-attainable ace of $\{a_n\}_{n=1}^{K}$ in E.

Let's denote by $Aaop_E(\{a_n\},\{\epsilon_n\})$ the set of all the pair (a_{n_0},ϵ_{n_0}) of the $\{\epsilon_n\}$-attainable ace of $\{a_n\}_{n=1}^K$ in E.

Lemma 3.3. *Let $\{\epsilon_n\}_{n=1}^K$ be any sequence of positive real numbers with $K \in N \cup \{\infty\}$, E be a non-empty open subset of R^m and $\{a_n\}_{n=1}^K$ be an $\{\epsilon_n\}_{n=1}^K$-attainable sequence in E. Then a term a_{n_0} is an $\{\epsilon_n\}$-attainable ace of the sequence $\{a_n\}_{n=1}^K$ in E if and only if there exists $x \in E$ such that $x \in \overline{C}(a_{n_0},\epsilon_{n_0})$ and $\|x-a_n\|_\infty > \epsilon_n$ for all $n \in N-\{n_0\}$.*

Proof. Since $E \subseteq \bigcup\limits_{n=1}^{K} \overline{C}(a_n,\epsilon_n)$, we have the following equivalent statements:

$$\begin{aligned}
& a_{n_0} \text{ is an } \{\epsilon_n\}_{n=1}^K - \text{attainable ace of } \{a_n\}_{n=1}^K. \\
\Leftrightarrow\ & E \not\subseteq \bigcup_{n\neq n_0} \overline{C}(a_n,\epsilon_n) \\
\Leftrightarrow\ & \exists x \in E \ s.t.\ x \notin \bigcup_{n\neq n_0} \overline{C}(a_n,\epsilon_n) \text{ and } x \in \overline{C}(a_{n_0},\epsilon_{n_0}) \\
\Leftrightarrow\ & \exists x \in E \ s.t.\ \big[\forall n \in \{1,2,\cdots,K\}-\{n_0\} \Rightarrow x \notin \overline{C}(a_n,\epsilon_n)\big] \wedge x \in \overline{C}(a_{n_0},\epsilon_{n_0}) \\
\Leftrightarrow\ & \exists x \in E \ s.t.\ x \in \overline{C}(a_{n_0},\epsilon_{n_0}) \wedge [\forall n \neq n_0 \Rightarrow \|x-a_n\|_\infty > \epsilon_n]
\end{aligned}$$

This completes the proof. □

Note that if $\{a_n\}_{n=1}^K$ is an $\{\epsilon_n\}_{n=1}^K$-attainable sequence in E then a_{n_0} is not an $\{\epsilon_n\}$-attainable ace of the sequence $\{a_n\}_{n=1}^K$ in E if and only if $\forall x \in E \cap \overline{C}(a_{n_0},\epsilon_{n_0}), \exists n \neq n_0$ s.t. $\|x-a_n\|_\infty \leq \epsilon_n$. Moreover, we have the following lemma.

Lemma 3.4. *Let $\{\epsilon_n\}_{n=1}^K$ be any sequence of positive real numbers with $K \in N \cup \{\infty\}$, E be a non-empty open subset of R^m and $\{a_n\}_{n=1}^K$ be an $\{\epsilon_n\}_{n=1}^K$-attainable sequence in E. Then a term a_{n_0} is not an $\{\epsilon_n\}$-attainable ace of $\{a_n\}_{n=1}^K$ in E if and only if $E \cap \overline{C}(a_{n_0},\epsilon_{n_0}) \subseteq \bigcup\limits_{n\in A_{n_0}} \overline{C}(a_n,\epsilon_n)$. Here $A_{n_0} = \{n \in \{1,2,\cdots,K\}-\{n_0\} : \overline{C}(a_{n_0},\epsilon_{n_0}) \cap \overline{C}(a_n,\epsilon_n) \neq \emptyset\}$.*

Proof. We have the following equivalent statements:

$$a_{n_0} \text{ is not an } \{\epsilon_n\}_{n=1}^{K} - \text{attainable ace of } \{a_n\}_{n=1}^{K}.$$

$$\Leftrightarrow \quad E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n,\epsilon_n) \text{ and } E \subseteq \bigcup_{n \neq n_0} \overline{C}(a_n,\epsilon_n)$$

$$\Leftrightarrow \quad E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n,\epsilon_n) \text{ and } E \cap \overline{C}(a_{n_0},\epsilon_{n_0}) \subseteq \bigcup_{n \neq n_0} \overline{C}(a_n,\epsilon_n)$$

$$\Leftrightarrow \quad E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n,\epsilon_n) \text{ and } E \cap \overline{C}(a_{n_0},\epsilon_{n_0}) \subseteq \bigcup_{n \in A_{n_0}} \overline{C}(a_n,\epsilon_n)$$

Since $\{a_n\}_{n=1}^{K}$ is an $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence in E, this implies that a_{n_0} is not an $\{\epsilon_n\}_{n=1}^{K}$-attainable ace of $\{a_n\}_{n=1}^{K}$ if and only if

$$E \cap \overline{C}(a_{n_0},\epsilon_{n_0}) \subseteq \bigcup_{n \in A_{n_0}} \overline{C}(a_n,\epsilon_n).$$

□

Lemma 3.5. *Let $\{\epsilon_n\}_{n=1}^{K}$ be any sequence of positive real numbers with $K \in N \cup \{\infty\}$, E be a non-empty open subset of R^m and $\{a_n\}_{n=1}^{K}$ be an $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence in E.*

(1) *If $(a_{n_1},\epsilon_{n_1}), (a_{n_2},\epsilon_{n_2}) \in Aaop_E(\{a_n\},\{\epsilon_n\})$ and $a_{n_1} = a_{n_2}$ then $\epsilon_{n_1} = \epsilon_{n_2}$ and $n_1 = n_2$.*

(2) *If $(a_{n_1},\epsilon_{n_1}) \in Aaop_E(\{a_n\},\{\epsilon_n\})$ and $a_{n_1} = a_{n_2}$ for some $n_2 \neq n_1$ then $\epsilon_{n_1} > \epsilon_{n_2}$.*

Proof. (1) Assume that $\epsilon_{n_1} \neq \epsilon_{n_2}$. Then we first have $n_1 \neq n_2$. Now suppose that $\epsilon_{n_1} > \epsilon_{n_2}$. Since $(a_{n_2},\epsilon_{n_2}) \in Aaop_E(\{a_n\},\{\epsilon_n\})$, we have

$$E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n,\epsilon_n) \text{ and } E \not\subseteq \bigcup_{n \neq n_2} \overline{C}(a_n,\epsilon_n).$$

But this is impossible since $\overline{C}(a_{n_2},\epsilon_{n_2}) \subseteq \overline{C}(a_{n_1},\epsilon_{n_1})$ and $\overline{C}(a_{n_1},\epsilon_{n_1})$ is still a member of the collection in the last union. Similarly, suppose that $\epsilon_{n_1} < \epsilon_{n_2}$. Since $(a_{n_1},\epsilon_{n_1}) \in Aaop_E(\{a_n\},\{\epsilon_n\})$, we have

$$E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n,\epsilon_n) \text{ and } E \not\subseteq \bigcup_{n \neq n_1} \overline{C}(a_n,\epsilon_n).$$

But this is also impossible since $\overline{C}(a_{n_1},\epsilon_{n_1}) \subseteq \overline{C}(a_{n_2},\epsilon_{n_2})$ and $\overline{C}(a_{n_2},\epsilon_{n_2})$ is still a member of the collection in the last union. Hence we have $a_{n_1} = a_{n_2}$ and $\epsilon_{n_1} = \epsilon_{n_2}$. And such a proof just above also shows that

$n_1 = n_2$. (2) Suppose that $(a_{n_1}, \epsilon_{n_1}) \in Aaop_E(\{a_n\}, \{\epsilon_n\})$ and $a_{n_1} = a_{n_2}$ for some $n_2 \neq n_1$. Then we have

$$E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n, \epsilon_n) \text{ and } E \not\subseteq \bigcup_{n \neq n_1} \overline{C}(a_n, \epsilon_n).$$

Since $\overline{C}(a_{n_2}, \epsilon_{n_2})$ is still a member of the collection in the last union, it is impossible that $\epsilon_{n_1} \leq \epsilon_{n_2}$. Hence we have $\epsilon_{n_1} > \epsilon_{n_2}$. □

In view of the lemma just above, the set of all the points of the $\{\epsilon_n\}$-attainable ace of $\{a_n\}$ in E is well-defined and we denote it by $Aap_E(\{a_n\}, \{\epsilon_n\})$.

Definition 3.6. Let $\{\epsilon_n\}_{n=1}^{K}$ be a sequence of positive real numbers with $K \in N \cup \{\infty\}$, E be a non-empty open subset of R^m and $\{a_n\}_{n=1}^{K}$ be an $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence in E. If $a_{n_0} \in Aap_E(\{a_n\}, \{\epsilon_n\})$ then an element $b \in E$ is called an $\{\epsilon_n\}_{n=1}^{K}$-replaceable ace of a_{n_0} in E if and only if the sequence, denoted by $\{a_n\}_{(b_{n_0})}$, which is obtained from $\{a_n\}_{n=1}^{K}$ by replacing the term a_{n_0} by b is also an $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence in E. And we denote by $Rap_E(\{a_n\}, \{\epsilon_n\}; n_0)$ the set of all the points of $\{\epsilon_n\}_{n=1}^{K}$-replaceable ace of a_{n_0} in E.

Proposition 3.7. *Let π_k be the projection map from R^m onto R such that $\pi_k(x) = x_k$ for each $k = 1, 2, \cdots, m$. Let $\{\epsilon_n\}_{n=1}^{K}$ be a sequence of positive real numbers with $K \in N \cup \{\infty\}$ and E be a non-empty open subset of R^m. Suppose that a_{n_0} is an $\{\epsilon_n\}_{n=1}^{K}$-attainable ace of the $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence $\{a_n\}_{n=1}^{K}$. If we set*

$$S = E \cap \left[\overline{C}(a_{n_0}, \epsilon_{n_0}) - \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right]$$

then

$$\begin{aligned} & Rap_E(\{a_n\}, \{\epsilon_n\}; n_0) \\ = \; & E \cap \left\{ \prod_{k=1}^{m} [\sup \pi_k(S) - \epsilon_{n_0}, \inf \pi_k(S) + \epsilon_{n_0}] \right\}. \end{aligned}$$

Here $\prod_{k=1}^{m} [\epsilon_{n_0} - \sup \pi_k(S), \epsilon_{n_0} + \inf \pi_k(S)]$ denotes the cartesian product of the closed intervals.

Proof. Since $\{a_n\}_{n=1}^{K}$ is an $\{\epsilon_n\}_{n=1}^{K}$-attainable sequence in E, $E \subseteq \bigcup_{n=1}^{K} \overline{C}(a_n, \epsilon_n)$. But we have

$$\begin{aligned}
\bigcup_{n=1}^{K} \overline{C}(a_n, \epsilon_n) &= \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \cup \overline{C}(a_{n_0}, \epsilon_{n_0}) \\
&= \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \cup \left[\overline{C}(a_{n_0}, \epsilon_{n_0}) - \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \right].
\end{aligned}$$

Hence we have

$$\begin{aligned}
E &= \left[E \cap \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \right] \\
&\quad \cup \left(E \cap \left[\overline{C}(a_{n_0}, \epsilon_{n_0}) - \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \right] \right) \\
&= \left[E \cap \left\{ \bigcup_{n \neq n_0} \overline{C}(a_n, \epsilon_n) \right\} \right] \cup S.
\end{aligned}$$

Note that the set $S \neq \emptyset$ since a_{n_0} is an $\{\epsilon_n\}_{n=1}^{K}$-attainable ace. Since the last union just above is the disjoint union and $a_{n_0} \in S$, we have $b \in Rap_E(\{a_n\}, \{\epsilon_n\}; n_0)$ if and only if $b \in E$ and $S \subseteq \overline{C}(b, \epsilon_{n_0})$. And these hold if and only if

$$b \in E \cap \left\{ \prod_{k=1}^{m} [\sup \pi_k(S) - \epsilon_{n_0}, \inf \pi_k(S) + \epsilon_{n_0}] \right\}.$$

$\square$

Now we have our main theorem which provides a way to get rid of the ace.

THEOREM 3.8. *(No Aces) Let $\{\epsilon_n\}_{n=1}^{\infty}$ be an infinite sequence of positive real numbers and $\{a_n\}_{n=1}^{\infty}$ be an $\{\epsilon_n\}_{n=1}^{\infty}$-attainable sequence in R^m. Suppose that $M = \frac{\sup\{\epsilon_n : n \in N\}}{\inf\{\epsilon_n : n \in N\}}$ is finite. If $Aap_{R^m}(\{a_n\}, \{\epsilon_n\}) \neq \emptyset$ then $\{a_n\}_{n=1}^{\infty}$ is not an $\{\frac{1}{2M}\epsilon_n\}_{n=1}^{\infty}$-attainable sequence in R^m. Or equivalently, if $\{a_n\}_{n=1}^{\infty}$ is an $\{\epsilon_n\}_{n=1}^{\infty}$-attainable sequence in R^m, then we have $Aap_{R^m}(\{a_n\}, \{2M\epsilon_n\}) = \emptyset$.*

Proof. Let $a_{n_0} \in Aap_{R^m}(\{a_n\}, \{\epsilon_n\}) \neq \emptyset$. Then, by lemma 3.3, we have

$$\exists x \in R^m \ s.t. \ x \in \overline{C}(a_{n_0}, \epsilon_{n_0}) \wedge [\forall n \neq n_0 \Rightarrow \|x - a_n\|_\infty > \epsilon_n].$$

Now put $\alpha = \inf\{\epsilon_n : n \in N\}$ and $\beta = \sup\{\epsilon_n : n \in N\}$. Then $M = \frac{\beta}{\alpha}$. Now the following two cases occur since $M \geq 1$.

Case 1. The case where $M = 1$.

In this case, there exists $\epsilon_0 > 0$ such that $\epsilon_n = \epsilon_0$ for all $n \in N$. Since

$$\forall n \neq n_0 \Rightarrow \|x - a_n\|_\infty > \epsilon_0,$$

there is a subset $F \subseteq R^m$ such that $\overline{C}(x, \frac{\epsilon_0}{2}) \neq F$ and

$$\overline{C}(x, \frac{\epsilon_0}{2}) \subseteq F \text{ and } F \cap \left(\underset{n \neq n_0}{\cup} \overline{C}(a_n, \frac{\epsilon_0}{2}) \right) = \emptyset.$$

Since $\overline{C}(x, \frac{\epsilon_0}{2})$ which has the same size with $\overline{C}(a_{n_0}, \frac{\epsilon_0}{2})$ is a proper subset of F, this implies that $F - \overline{C}(a_{n_0}, \frac{\epsilon_0}{2}) \neq \emptyset$. Thus we have

$$R^m \not\subseteq \bigcup_{n=1}^{\infty} \overline{C}(a_n, \frac{\epsilon_0}{2}) = \bigcup_{n=1}^{\infty} \overline{C}(a_n, \frac{\epsilon_n}{2})$$

which implies that $\{a_n\}_{n=1}^{\infty}$ is not an $\{\frac{1}{2}\epsilon_n\}_{n=1}^{\infty}$-attainable sequence in R^m.

Case 2. The case where $M > 1$.

Since $\|x - a_n\|_\infty > \epsilon_n$ for all $n \neq n_0$, we have

$$C(x, \epsilon_n - \frac{\epsilon_n}{2M}) \cap \left(\underset{n \neq n_0}{\cup} \overline{C}(a_n, \frac{\epsilon_n}{2M}) \right) = \emptyset$$

for all $n \neq n_0$. Since $\alpha \leq \epsilon_n$ for all $n \neq n_0$, we have

$$C(x, \alpha - \frac{\alpha}{2M}) \cap \left(\underset{n \neq n_0}{\cup} \overline{C}(a_n, \frac{\epsilon_n}{2M}) \right) = \emptyset.$$

But we have

$$\alpha - \frac{\alpha}{2M} > \alpha - \frac{\alpha}{M+1} = \frac{M\alpha}{M+1} = \frac{\beta}{M+1} > \frac{\beta}{2M} \geq \frac{\epsilon_{n_0}}{2M}$$

since $M > 1$. Hence $\overline{C}(a_{n_0}, \frac{\epsilon_{n_0}}{2M})$ does not contain the cube $C(x, \alpha - \frac{\alpha}{2M})$. Therefore, we must have

$$R^m \not\subseteq \bigcup_{n=1}^{\infty} \overline{C}(a_n, \frac{\epsilon_n}{2M}).$$

Consequently, $\{a_n\}_{n=1}^{\infty}$ is not an $\{\frac{\epsilon_n}{2M}\}_{n=1}^{\infty}$-attainable sequence in R^m. Finally, the last statement of this theorem is induced from the contraposition of this statement. $\square$

The following example shows that the theorem above does not hold for an open subset E of R^m in general.

Example 3.9. Let's choose an open subset

$$E = C((0, \cdots, 0), 1) \cup C((6, 0, \cdots, 0), 1).$$

If we choose a sequence $\{a_1, a_2\}$ of vectors so that $a_1 = (0, \cdots, 0)$ and $a_2 = (6, 0, \cdots, 0)$ and a sequence $\{3, 3\}$ of positive real numbers then $\{a_1, a_2\}$ is a $\{3, 3\}$-attainable sequence in E and $Aap_E(\{a_1, a_2\}, \{3, 3\}) = \{a_1, a_2\}$. But $\{a_1, a_2\}$ is also a $\{1.5, 1.5\}$-attainable sequence in E and $Aap_E(\{a_1, a_2\}, \{6, 6\}) = \{a_1, a_2\} \neq \emptyset$.

We live in an age where the ace is everything. The ace is of course important, but the ace himself will never live a happy life because he will be tired. In some ways a society without aces might be a happier society.

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