

SIMPLIFYING AND FINDING ORDINARY DIFFERENTIAL EQUATIONS IN TERMS OF THE STIRLING NUMBERS

FENG QI*, JING-LIN WANG, AND BAI-NI GUO

ABSTRACT. In the paper, by virtue of techniques in combinatorial analysis, the authors simplify three families of nonlinear ordinary differential equations in terms of the Stirling numbers of the first kind and establish a new family of nonlinear ordinary differential equations in terms of the Stirling numbers of the second kind.

1. Motivation and main results

In [3, Theorem 1], it was acquired inductively and recursively that the nonlinear differential equations

$$G^{(n)}(t) = \frac{(-1)^n(n-1)!}{(1+t)^n} \sum_{j=2}^{n+1} (j-1)! H_{n-1,j-2} G^j(t) \quad (1)$$

for $n \in \mathbb{N}$ have a solution $G(t) = \frac{1}{\ln(1+t)}$, where $H_{n,0} = 1$ for $n \in \mathbb{N}$, $H_{n,1} = H_n = \sum_{k=1}^n \frac{1}{k}$ for $n \in \mathbb{N}$, and

$$H_{n,j} = \sum_{k=j}^n \frac{H_{k-1,j-1}}{k}, \quad 2 \leq j \leq n. \quad (2)$$

Received March 3, 2018. Revised November 27, 2018. Accepted December 3, 2018.

2010 Mathematics Subject Classification: 11B37, 34A05, 34A34.

Key words and phrases: nonlinear ordinary differential equation, Stirling number of the first kind, Stirling number of the second kind, combinatorial analysis.

*Corresponding author.

© The Kangwon-Kyungki Mathematical Society, 2018.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

In [4, Theorem 2.1], it was inductively and recursively obtained that the family of differential equations

$$F^{(n)}(t) = \frac{(-1)^{n-1}}{(1 + \lambda t)^n} \sum_{k=1}^n (k - 1)! \lambda^{n-k} (n - 1)! H_{n-1, k-1} e^{-kF(t)}$$

for $n \in \mathbb{N}$ have a solution

$$F(t) = \ln \left[1 + \frac{\ln(1 + \lambda t)}{\lambda} \right]. \tag{3}$$

In [2, Theorem 2.1], as did in [3, 4], it was obtained inductively and recursively that the differential equations

$$(n - 1)! e^{-nF(t)} = (-1)^{n-1} \sum_{k=1}^n \lambda^{n-k} (1 + \lambda t)^k a_k(n) F^{(k)}(t), \quad n \geq 1$$

have a solution $F(t)$ defined by (3), where $a_n(n) = 1$, $a_1(n) = 1$, and

$$a_k(n) = \sum_{i_{k-1}=0}^{n-k} \sum_{i_{k-2}=0}^{n-k-i_{k-1}} \cdots \sum_{i_1=0}^{n-k-i_{k-1}-\cdots-i_2} k^{i_{k-1}} \cdots 2^{i_1}.$$

We notice that, although the same method used in proofs of [2, Theorem 2.1], [3, Theorem 1], and [4, Theorem 2.1] is effectual, however, proofs of those main results in [2, Theorem 2.1], [3, Theorem 1], and [4, Theorem 2.1] are much long and tedious, and formulations of those main results in those papers [2–4] are less meaningful and significant.

The aim of this paper is, by virtue of techniques in combinatorial analysis, to simply and concisely verify and extend the above-mentioned [2, Theorem 2.1], [3, Theorem 1] and [4, Theorem 2.1].

Our main results can be stated as the following two theorems.

THEOREM 1.1. *For $n \in \mathbb{N}$, the n th derivative of the function $G(t)$ satisfies*

$$G^{(n)}(t) = \frac{1}{(1 + t)^n} \sum_{k=1}^n (-1)^k k! s(n, k) G^{k+1}(t) \tag{4}$$

and

$$\sum_{k=1}^n S(n, k) (1 + t)^k G^{(k)}(t) = \frac{(-1)^n n!}{[\ln(1 + t)]^{n+1}}, \tag{5}$$

where the Stirling numbers of the first and second kinds $s(n, k)$ and $S(n, k)$ can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!} \quad \text{and} \quad \frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}.$$

THEOREM 1.2. For $n \in \mathbb{N}$, the n th derivative of the function $F(t)$ defined in (3) satisfies

$$F^{(n)}(t) = \left(\frac{\lambda}{1 + \lambda t} \right)^n \sum_{k=1}^n \frac{(-1)^{k-1} (k-1)!}{\lambda^k} s(n, k) e^{-kF(t)} \quad (6)$$

and

$$\sum_{k=1}^n S(n, k) \left(\frac{1}{\lambda} + t \right)^k F^{(k)}(t) = \frac{(-1)^{n-1} (n-1)!}{[\lambda + \ln(1 + \lambda t)]^n}. \quad (7)$$

2. Simple proofs of main results

Now we are in a position to prove our main results.

Proof of Theorem 1.1. In combinatorial analysis, the Faà di Bruno formula plays an important role and can be described in terms of the Bell polynomials of the second kind

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n-k+1 \\ \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^{n-k+1} i \ell_i = n \\ \sum_{i=1}^{n-k+1} \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!} \right)^{\ell_i}$$

for $n \geq k \geq 0$, see [1, p. 134, Theorem A], by

$$\frac{d^n}{dt^n} [f \circ h(t)] = \sum_{k=0}^n f^{(k)}(h(t)) B_{n,k}(h'(t), h''(t), \dots, h^{(n-k+1)}(t)) \quad (8)$$

for $n \geq 0$, see [1, p. 139, Theorem C]. Setting $f(u) = \frac{1}{u}$ and $u = h(t) = \ln(1+t)$ in (8) gives

$$G^{(n)}(t) = \sum_{k=0}^n \left(\frac{1}{u} \right)^{(k)} B_{n,k} \left(\frac{1}{1+t}, -\frac{1}{(1+t)^2}, \dots, (-1)^{n-k} \frac{(n-k)!}{(1+t)^{n-k+1}} \right)$$

$$\begin{aligned}
 &= \sum_{k=0}^n \frac{(-1)^k k!}{u^{k+1}} \left(\frac{1}{1+t}\right)^n (-1)^{n+k} B_{n,k}(0!, 1!, 2!, \dots, (n-k)!) \\
 &= \frac{1}{(1+t)^n} \sum_{k=0}^n \frac{(-1)^k k!}{\ln^{k+1}(1+t)} s(n, k) \\
 &= \frac{1}{(1+t)^n} \sum_{k=0}^n (-1)^k k! s(n, k) G^{k+1}(t) \\
 &= \frac{1}{(1+t)^n} \sum_{k=1}^n (-1)^k k! s(n, k) G^{k+1}(t)
 \end{aligned}$$

for $n \in \mathbb{N}$, where we used in the above lines the formula $s(n, 0) = 0$ for $n \in \mathbb{N}$ and the identities

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \tag{9}$$

and

$$B_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n, k) \tag{10}$$

listed in [1, p. 135, Theorem B]. This means that the nonlinear differential equations (4) have a solution $G(t)$.

The inversion theorem [10, Theorem 12.1, p. 171] reads that

$$a_n = \sum_{\alpha=0}^n S(n, \alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^n s(n, k) a_k. \tag{11}$$

Applying this inversion theorem to the equality (4) results in

$$(-1)^n n! G^{n+1}(t) = \sum_{k=1}^n S(n, k) (1+t)^k G^{(k)}(t), \quad n \in \mathbb{N}.$$

The equation (5) is thus proved. The proof of Theorem 1.1 is complete. □

Second proof of the equation (4) in Theorem 1.1. In [6, Corollary 2.3], it was inductively and recursively procured that the Stirling numbers of the first kind $s(n, k)$ for $1 \leq k \leq n$ can be expressed as

$$s(n, k) = (-1)^{n+k} (n-1)! \sum_{\ell_1=1}^{n-1} \frac{1}{\ell_1} \sum_{\ell_2=1}^{\ell_1-1} \frac{1}{\ell_2} \cdots \sum_{\ell_{k-2}=1}^{\ell_{k-3}-1} \frac{1}{\ell_{k-2}} \sum_{\ell_{k-1}=1}^{\ell_{k-2}-1} \frac{1}{\ell_{k-1}}. \tag{12}$$

See also [5, p. 27, Remark 3.3]. Comparing (2) with (12), we observe that

$$(-1)^{n+k}s(n, k) = (n-1)!H_{n-1, k-1}, \quad n \geq k \geq 1. \quad (13)$$

See also [7, p. 9, Section 2.5]. Then substituting $(j-2)$ by $(k-1)$ and $(n-1)!H_{n-1, k-1}$ by $(-1)^{n+k}s(n, k)$ in (1) leads to (4). The second proof of the equation (4) is complete. \square

Proof of Theorem 1.2. Setting $f(u) = \ln(1+u)$ and $u = h(t) = \frac{\ln(1+\lambda t)}{\lambda}$ in (8) gives

$$\begin{aligned} F^{(n)}(t) &= \sum_{k=0}^n [\ln(1+u)]^{(k)} \\ &\quad \times B_{n,k} \left(\frac{1}{1+\lambda t}, -\frac{\lambda}{(1+\lambda t)^2}, \dots, (-1)^{n-k} \frac{(n-k)! \lambda^{n-k}}{(1+\lambda t)^{n-k+1}} \right) \\ &= \sum_{k=0}^n [\ln(1+u)]^{(k)} \left(\frac{1}{1+\lambda t} \right)^n (-1)^{n+k} \lambda^{n-k} B_{n,k}(0!, 1!, 2!, \dots, (n-k)!) \\ &= \left(\frac{1}{1+\lambda t} \right)^n \sum_{k=0}^n [\ln(1+u)]^{(k)} (-1)^{n+k} \lambda^{n-k} (-1)^{n-k} s(n, k) \\ &= \frac{1}{(1+\lambda t)^n} \left[\ln(1+u) \lambda^n s(n, 0) + \sum_{k=1}^n \frac{(-1)^{k-1} (k-1)!}{(1+u)^k} \lambda^{n-k} s(n, k) \right] \\ &= \frac{1}{(1+\lambda t)^n} \sum_{k=1}^n \frac{(-1)^{k-1} (k-1)!}{\left[1 + \frac{\ln(1+\lambda t)}{\lambda} \right]^k} \lambda^{n-k} s(n, k) \\ &= \frac{1}{(1+\lambda t)^n} \sum_{k=1}^n (-1)^{k-1} (k-1)! \lambda^{n-k} s(n, k) e^{-kF(t)} \end{aligned}$$

for $n \in \mathbb{N}$, where we used in the above lines the formula $s(n, 0) = 0$ for $n \in \mathbb{N}$ and the identities (9) and (10). This means that the nonlinear differential equations (6) have a solution $F(t)$.

Applying the inversion theorem stated in (11) to the equation (6) results in

$$\frac{(-1)^{n-1} (n-1)!}{\lambda^n} e^{-nF(t)} = \sum_{k=1}^n S(n, k) \left(\frac{1+\lambda t}{\lambda} \right)^k F^{(k)}(t).$$

The identity (7) is proved. The proof of Theorem 1.2 is complete. \square

3. Remarks

Finally we list several remarks on our main and the closely related results.

REMARK 3.1. The relation (13) shows that we should not call the quantities $H_{n,k}$ defined in (2) the generalized harmonic numbers, as did in [4, p. 745].

REMARK 3.2. Till now we can see that formulations and proofs of Theorems 1.1 and 1.2 in this paper are simpler, more concise, more meaningful, and more significant than those in the papers [2–4].

REMARK 3.3. This paper is an extended and simplified version of the preprints [8, 9].

Acknowledgements. The authors are grateful to the anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

References

- [1] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, Revised and Enlarged Edition, D. Reidel Publishing Co., Dordrecht and Boston, 1974; Available online at <https://doi.org/10.1007/978-94-010-2196-8>.
- [2] G.-W. Jang, J. Kwon, and J. G. Lee, *Some identities of degenerate Daehee numbers arising from nonlinear differential equation*, *Adv. Difference Equ.* (2017) **2017**:206, 10 pages; Available online at <https://doi.org/10.1186/s13662-017-1265-4>.
- [3] D. S. Kim and T. Kim, *Some identities for Bernoulli numbers of the second kind arising from a non-linear differential equation*, *Bull. Korean Math. Soc.* **52** (6) (2015), 2001–2010; Available online at <https://doi.org/10.4134/BKMS.2015.52.6.2001>.
- [4] T. Kim and D. S. Kim, *Some identities of degenerate Daehee numbers arising from certain differential equations*, *J. Nonlinear Sci. Appl.* **10** (2) (2017), 744–751; Available online at <https://doi.org/10.22436/jnsa.010.02.35>.
- [5] F. Qi, *Diagonal recurrence relations for the Stirling numbers of the first kind*, *Contrib. Discrete Math.* **11** (2016), 22–30; Available online at <http://hdl.handle.net/10515/sy5wh2dx6> and <https://doi.org/10515/sy5wh2dx6>.
- [6] F. Qi, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, *Filomat* **28** (2) (2014), 319–327; Available online at <https://doi.org/10.2298/FIL14023190>.

- [7] F. Qi and B.-N. Guo, *Viewing some ordinary differential equations from the angle of derivative polynomials*, Iran. J. Math. Sci. Inform. **14** (2) (2019), in press; Available online at <https://doi.org/10.20944/preprints201610.0043.v1>.
- [8] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying and finding nonlinear ordinary differential equations*, ResearchGate Working Paper (2017), available online at <https://doi.org/10.13140/RG.2.2.28855.32166>.
- [9] F. Qi, J.-L. Wang, and B.-N. Guo, *Simplifying two families of nonlinear ordinary differential equations*, Preprints **2017**, 2017030192, 6 pages; Available online at <http://dx.doi.org/10.20944/preprints201703.0192.v1>.
- [10] J. Quaintance and H. W. Gould, *Combinatorial Identities for Stirling Numbers*. The unpublished notes of H. W. Gould. With a foreword by George E. Andrews. World Scientific Publishing Co. Pte. Ltd., Singapore, 2016.

Feng Qi

¹Institute of Mathematics, Henan Polytechnic University
Jiaozuo 454010, Henan, China

²College of Mathematics, Inner Mongolia University for Nationalities
Tongliao 028043, Inner Mongolia, China

³School of Mathematical Sciences, Tianjin Polytechnic University
Tianjin 300387, China
E-mail: qifeng618@gmail.com

Jing-Lin Wang

¹Teda No. 2 Primary School
Economic and Technological Development Zone
Tianjin 300457, China

²Department of Mathematics, College of Science
Tianjin Polytechnic University
Tianjin 300387, China
E-mail: jing-lin.wang@hotmail.com

Bai-Ni Guo

School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo 454010, Henan, China
E-mail: bai.ni.guo@gmail.com