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SOME CONDITIONS ON THE FORM OF THIRD ELEMENT FROM DIOPHANTINE PAIRS AND ITS APPLICATION

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Abstract. A set $\{a_1, a_2, \ldots, a_m\}$ of positive integers is called a Diophantine m-tuple if $a_i a_j + 1$ is a perfect square for all $1 \le i < j \le m$. In this paper, we show that the form of third element in Diophantine pairs and develop some results which are needed to prove the extendibility of the Diophantine pair $\{a, b\}$ with some conditions. By using this result, we prove the extendibility of Diophantine pairs $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$ and $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$, where F_n is the n-th Fibonacci number.

1. Introduction

A Diophantine m-tuple is a set which consists of m distinct positive integers satisfying the property that the product of any two of them is one less than a perfect square. If the set which consists of rational numbers satisfy the same property, then we called rational Diophantine m-tuple. Diophantus found the first rational Diophantine quadruple $\{1/16, 33/16, 17/4, 105/16\}$. However, the first Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Many famous mathematicians made lots of results related to the problems of Diophantine m-tuple, but still there are many open problems. Especially, the most famous problem is the extendibility of Diophantine m-tuple.

For any Diophantine triple $\{a, b, c\}$ with $a < b < c$, the set $\{a, b, c, d_\pm\}$ is a Diophantine quadruple, where

$$d_\pm = a + b + c + 2abc \pm 2rst$$

and r, s, t are the positive integers satisfying

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

A folklore conjecture is that there does not exist a Diophantine quintuple. The stronger version of this conjecture states that if $\{a, b, c, d\}$ is a Diophantine

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quadruple and $d > \max\{a, b, c\}$, then $d = d_+$. These Diophantine quadruples are called regular.

The reason why the extendibility is important is related to the elliptic curves. To extend the Diophantine triple $\{a, b, c\}$ to Diophantine quadruple, we have to solve the equations

$$ax + 1 = \Box,\ bx + 1 = \Box,\ cx + 1 = \Box.$$

Hence, we have the equation

$$E : y^2 = (ax + 1)(bx + 1)(cx + 1),$$

which is the elliptic curve by the product of three equations. Then we have always the integer points

$$(0, \pm 1),\ (d_+, \pm(at + rs)(bs + rt)(cr + st)),\ (d_-, \pm((at - rs)(bs - rt)(cr - st))),$$

and also $(-1, 0)$ if $1 \in \{a, b, c\}$. The conjecture means it is possible to prove that there are no other integer points on E for some family of Diophantine triples. For example, A. Dujella [5] proved that the elliptic curve

$$E_k : y^2 = ((k - 1)x + 1)((k + 1)x + 1)(4kx + 1)$$

has four integer points

$$(0, \pm 1),\ (16k^3 - 4k, \pm(128k^6 - 112k^4 - 20k^2 - 1))$$

under assumption that $\mathrm{rank}(E_k(\mathbb{Q})) = 1$. Similar results [7] and [13] were proved for the equation

$$y^2 = (F_{2k}x + 1)(F_{2k+2}x + 1)(F_{2k+4} + 1)$$

and

$$y^2 = (F_{2k+1}x + 1)(F_{2k+3}x + 1)(F_{2k+5} + 1),$$

respectively, where F_n is the n-th Fibonacci number, defined by $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Therefore, it is important not only how to prove the extendibility of Diophantine m-tuple but also which case of Diophantine m-tuple is proved.

In 1969, A. Baker and H. Davenport [1] proved that the Diophantine triple $\{1, 3, 8\}$ can not be extended to a Diophantine quintuple, namely, if the set $\{1, 3, 8, d\}$ is a Diophantine quadruple, then $d = 120$. This means the Diophantine triple $\{1, 3, 8\}$ is regular.

Let us consider the Diophantine pair $\{1, 3\}$. If the set $\{1, 3, c\}$ is a Diophantine triple, then we have the equation

$$y^2 - 3x^2 = -2,$$

since $c + 1 = x^2,\ 3c + 1 = y^2$. We easily find the form of solutions of the Diophantine equation above is

$$(y + x\sqrt{3}) = (y_0 + x_0\sqrt{3})(2 + \sqrt{3})^k.$$

If (y_0, x_0) belongs to the same class as either of the solutions $(\pm 1, 1)$, then we have the form of third element

$$c = c_k = \frac{1}{6}[(2+\sqrt{3})(7+4\sqrt{3})^k + (2-\sqrt{3})(7-4\sqrt{3})^k - 4].$$

In 1998, Dujella and A. Pethö [9] developed this result, that is, if the set $\{1, 3, c_k, d\}$ is a Diophantine quadruple, then $d = d_- = c_{k-1}$ or $d_+ = c_{k+1}$. This result shows how important the form of third element in Diophantine pair is. Then we have a question such that how can we find the form of third element in Diophantine pairs. The form of third element c_k in the Diophantine pair $\{1, 3\}$ is found by using Theorem 8 in [15]. In fact, for a fixed Diophantine pair $\{a, b\}$ with $a < b$, the theorem shows the form of third element, where $b < 4a$. This theorem was also used in [3], [4], [12].

The recent research of third element in Diophantine m-tuple is Lemma 4.1 in [10] which shows the form of third element in the Diophantine pair $\{a, b\}$, where $a < b \le 8a$. Unfortunately however, the general form of third elements are unknown until now.

In this paper, we use the following identity

$$F_{k-2}F_{k-1}F_{k+1}F_{k+2} + 1 = F_k^4,$$

which was proved by E. Cesàro. We can make many Diophantine pairs by using above identity. However, we consider only two sets

$$\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\} \quad \text{and} \quad \{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\},$$

since the first set satisfy the condition of Lemma 4.1 in [10], namely, $F_{k-1}F_{k+2} < 8F_{k-2}F_{k+1}$, but the second set does not satisfy the condition. Hence, we need to generalize the Lemma 4.1 for the extendibility of Diophantine pairs which do not satisfy the condition $b \le 8a$. Also, we develop some results which need to prove the extendibility of Diophantine pair $\{a, b\}$ with some conditions. Through these generalized result, we prove the extendibility of $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$ and $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$.

2. Preliminaries

2.1. The bounds of each elements of Diophantine triple

We can find the lower bounds of second element of the Diophantine triple $\{a, b, c\}$ with $a < b$ using the following lemma.

Lemma 2.1 ([11, Lemma 1.3]). *Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$.*

(1) *If $b < 2a$, then $b > 2.1 \cdot 10^4$.*
(2) *If $2a \le b \le 8a$, then $b > 1.3 \cdot 10^5$.*
(3) *If $b > 8a$, then $b > 2 \cdot 10^3$.*

Let $\{a, b, c\}$ be a Diophantine triple, and r, s, t be the positive integers satisfying $ab+1=r^2,\ ac+1=s^2,\ bc+1=t^2$. Then we have

$$at^2 - bs^2 = a - b.$$

We easily find the form of solutions of the equation above is

$$(t\sqrt{a} + s\sqrt{b}) = (t_0\sqrt{a} + s_0\sqrt{b})(r + \sqrt{bc})^\nu.$$

If (t_0, s_0) belongs to the same class as either of the solutions $(\pm 1, 1)$, then s can be expressed as $s = s_\nu^\tau$, where $\tau \in \{\pm 1\}$ and

$$s_0 = s_0^\tau = 1,\ s_1^\tau = r + \tau a,\ s_{\nu+2}^\tau = 2rs_{\nu+1}^\tau - s_\nu^\tau.$$

Define $c_\nu^\tau = ((s_\nu^\tau)^2 - 1)/a$. Then, we obtain

$$\begin{aligned} c = c_\nu^\tau = \frac{1}{4ab}&[(a+b\pm 2\sqrt{ab})(2ab+1+2r\sqrt{ab})^\nu \\ &+ (a+b\mp 2\sqrt{ab})(2ab+1-2r\sqrt{ab})^\nu - 2(a+b)]. \end{aligned}$$

First, we have the form of third element c in the Diophantine triple $\{a, b, c\}$ by the following theorem.

Theorem 2.2 ([15, Theorem 8])**.** *If $a < b < 4a$, and b are in $\mathbb{Z}^+$, and*

$$ab+1=r^2, \quad ac+1=s^2, \quad bc+1=t^2$$

holds, then $c = c_k^+ = c_k^+(a, b)$ for some k or $c = c_j^- = c_j^-(a, b)$ for some j. The set c_j^- is omitted if $b = a + 2$.

However, it is difficult to apply this theorem in many cases, since the upper bound of b is small. Hence, there were a lot of researches to generalize the result. The following lemma generalizes of Theorem 2.2.

Lemma 2.3 ([10, Lemma 4.1])**.** *Let $\{a, b, c\}$ be a Diophantine triple. Assume that $a < b \le 8a$. Then $c = c_\nu^\tau$ for some ν and τ.*

Next, the following theorem gives us the bound of third element c in the Diophantine triple $\{a, b, c\}$.

Theorem 2.4 ([10, Theorem 1.2])**.** *Let $\{a, b, c\}$ be a Diophantine triple with $a < b$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d > d_+$ and that $\{a, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < d_-$, where d_+ and d_- are defined by*

$$d_\pm = a + b + c + 2abc \pm 2rst,$$

respectively.

(1) *If $b < 2a$, then $c < b^6$.*
(2) *If $2a \le b \le 8a$, then $c < 9.5b^4$.*
(3) *If $b > 8a$, then $c < b^5$.*

If $c = c_\nu^\tau$, then we can find the upper bound of c more specific by the following theorem.

Theorem 2.5 ([11, Theorem 1.4]). *Suppose that $\{a, b, c_\nu^\tau, d\}$ is a Diophantine quadruple with $d > c_{\nu+1}^\tau$ and that $\{a, b, c', c_\nu^\tau\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{\nu-1}^\tau$.*

(1) *If $b < 2a$, then $c \le c_3^+$.*
(2) *If $2a \le b \le 8a$, then $c \le c_2^+$.*

2.2. The properties of solutions of Pell equation

We have to solve the system

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2$$

to extend the Diophantine triple $\{a, b, c\}$ to the Diophantine quadruple $\{a, b, c, d\}$. One can eliminate d to obtain the following system of Pell equations:

$$ay^2 - bx^2 = a - b, \tag{1}$$

$$az^2 - cx^2 = a - c, \tag{2}$$

$$bz^2 - cy^2 = b - c. \tag{3}$$

Lemma 2.6 ([6, Lemma 1]). *There exist positive integers i_0, j_0 and integers $z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}, i = 1, \ldots, i_0, j = 1, \ldots, j_0$, with the following properties*:

(1) *$(z_0^{(i)}, x_0^{(i)})$ and $(z_1^{(j)}, y_1^{(j)})$ are solutions of (2) and (3), respectively.*
(2) *$z_0^{(i)}, x_0^{(i)}, z_1^{(j)}, y_1^{(j)}$ satisfy the following inequalities*

$$0 < x_0^{(i)} \le \sqrt{\frac{a(c-a)}{2(s-1)}} < \sqrt{\frac{s+1}{2}} < \sqrt[4]{ac},$$

$$0 \le |z_0^{(i)}| \le \sqrt{\frac{(s-1)(c-a)}{2a}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}} < \frac{c}{2},$$

$$0 < y_1^{(j)} \le \sqrt{\frac{b(c-b)}{2(t-1)}} < \sqrt{\frac{t+1}{2}} < \sqrt[4]{bc},$$

$$0 \le |z_1^{(j)}| \le \sqrt{\frac{(t-1)(c-b)}{2b}} < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}} < \frac{c}{3}.$$

(3) *If (z, x) and (z, y) are positive integers of (2) and (3), respectively, then there exist $i \in \{1, \ldots, i_0\}$, $j \in \{1, \ldots, j_0\}$ and integers $m, n \ge 0$ such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{c})(s + \sqrt{ac})^m, \tag{4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1^{(j)}\sqrt{b} + y_1^{(j)}\sqrt{c})(t + \sqrt{bc})^n. \tag{5}$$

From now on, we omit the superscripts (i) and (j). By (4), we may write $z = v_m$, where

$$v_0 = z_0, \; v_1 = sz_0 + cx_0, \; v_{m+2} = 2sv_{m+1} - v_m, \tag{6}$$

and by (5), we may write $z = w_n$, where

$$w_0 = z_1,\ w_1 = tz_1 + cy_1,\ w_{n+2} = 2tw_{n+1} - w_n. \tag{7}$$

Lemma 2.7 ([8, Lemma 3]). *If $v_m = w_n$, then $n-1 \le m \le 2n+1$.*

2.3. Congruence relation between solutions of Pell equations

In this section, we give the congruence relations between v_m and w_n, and properties of initial terms of (6) and (7).

Lemma 2.8 ([6, Lemma 4]). *We have the following properties of v_m and w_n.*

$$v_{2m} \equiv z_0 + 2c[az_0m^2 + sx_0m] \pmod{8c^2},$$
$$v_{2m+1} \equiv sz_0 + c[2asz_0m(m+1) + x_0(2m+1)] \pmod{4c^2},$$
$$w_{2n} \equiv z_1 + 2c[bz_1n^2 + ty_1n] \pmod{8c^2},$$
$$w_{2n+1} \equiv tz_1 + c[2btz_1n(n+1) + y_1(2n+1)] \pmod{4c^2}.$$

We have a question such that when does $v_m = w_n$ have a solution and if there exists a solution of $v_m = w_n$, then what are the values possible for the solution. The following lemma gives us the answer.

Lemma 2.9 ([8, Lemma 8]). *We have the following results.*

(1) *If the equation $v_{2m} = w_{2n}$ has a solution, then $z_0 = z_1$. Furthermore, $|z_0| = 1$ or $|z_0| = cr - st$ or $|z_0| < \min\{0.869a^{-5/14}c^{9/14}, 0.972b^{-0.3}c^{0.7}\}$.*
(2) *If the equation $v_{2m+1} = w_{2n}$ has a solution, then $|z_0| = t, |z_1| = cr - st$ and $z_0z_1 < 0$.*
(3) *If the equation $v_{2m} = w_{2n+1}$ has a solution, then $|z_0| = cr - st, |z_1| = s$ and $z_0z_1 < 0$.*
(4) *If the equation $v_{2m+1} = w_{2n+1}$ has a solution, then $|z_0| = t, |z_1| = s$ and $z_0z_1 > 0$.*

Furthermore, the solution of $v_m = w_n$ is more specific when $c = c_\nu^\tau \le c_3^+$ by the following lemma.

Lemma 2.10 ([11, Lemma 3.1]). *Assume that $a < b \le 8a$.*

(i) *Assume that $b < 3a$. In the case of $c = c_1^-$, we have $v_{2m+1} \ne w_{2n}$, $v_{2m} \ne w_{2n+1}$ and $v_{2m+1} \ne w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1 = 1$.*

(ii) *In the case of $c = c_1^+$, we have $v_{2m+1} \ne w_{2n}$, $v_{2m} \ne w_{2n+1}$ and $v_{2m+1} \ne w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*

(iii) *In the case of $c = c_2^-$, we have $v_{2m+1} \ne w_{2n}$ and $v_{2m+1} \ne w_{2n+1}$. Moreover, we have the following:*

(1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$ or $cr - st$.*
(2) *If $v_{2m} = w_{2n+1}$, then $|z_0| = cr - st$ and $|z_1| = s$ with $z_0z_1 < 0$.*

Furthermore, (2) occurs if and only if (1) with $|z_0| = cr - st$ occurs.

(iv) *In the case of $c \in \{c_2^+, c_3^-, c_3^+\}$, we have $v_{2m+1} \ne w_{2n}$ and $v_{2m} \ne w_{2n+1}$. Moreover, we get the following:*

(1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*
(2) *If $v_{2m+1} = w_{2n+1}$, then $|z_0| = t$ and $|z_1| = s$ with $z_0z_1 > 0$.*

2.4. Some theorems for applying the reduction method

From (4), (5) and sum of their conjugate, respectively, we get

$$v_m = \frac{1}{2\sqrt{a}}[(z_0\sqrt{a}+x_0\sqrt{c})(s+\sqrt{ac})^m + (z_0\sqrt{a}-x_0\sqrt{c})(s-\sqrt{ac})^m],$$

$$w_n = \frac{1}{2\sqrt{b}}[(z_1\sqrt{b}+y_1\sqrt{c})(t+\sqrt{bc})^n + (z_1\sqrt{b}-y_1\sqrt{c})(t-\sqrt{bc})^n].$$

Hence, we transform the equation $v_m = w_n$ into the following inequality.

Lemma 2.11 ([6, Lemma 5]). *Assume that $c > 4b$. If $v_m = w_n$ and $m, n \neq 0$, then*

$$0 < m\log(s+\sqrt{ac}) - n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} < \frac{8}{3}ac(s+\sqrt{ac})^{-2m}.$$

We use the following theorem and lemma to obtain the upper bound for m.

Theorem 2.12 ([2, p. 20]). *For a linear form*

$$\Lambda = \beta_1\log\alpha_1 + \cdots + \beta_l\log\alpha_l \neq 0$$

in logarithms of l algebraic numbers $\alpha_1, \alpha_2, \ldots, \alpha_l$ with rational coefficients $\beta_1, \beta_2, \ldots, \beta_l$, we have

$$\log|\Lambda| \geq -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log\beta,$$

where $\beta := \max\{|\beta_1|, \ldots, |\beta_l|\}$, $d := [\mathbb{Q}(\alpha_1, \ldots, \alpha_l) : \mathbb{Q}]$ and

$$h'(\alpha) = \frac{1}{d}\max\{h(\alpha), |\log\alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

Lemma 2.13 ([9, Lemma 5]). *Suppose that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$ and let $\epsilon = \|\mu q\| - M\cdot\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer.*

(1) *If $\epsilon > 0$, then there is no solution of the inequality*

$$0 < m\kappa - n + \mu < AB^{-m} \tag{8}$$

in integers m and n with

$$\frac{\log(Aq/\epsilon)}{\log B} \leq m \leq M.$$

(2) *Let $r = \lfloor \mu q + \frac{1}{2}\rfloor$. If $p - q + r = 0$, then there is no solution of inequality (8) in integers m and n with*

$$\max\left\{\frac{\log(3Aq)}{\log B}, 1\right\} < m \leq M.$$

3. The generalized bound for the second element

We can easily know that the form of third elements of Diophantine triple $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c\}$ by using Lemma 2.3. However, we can't apply the Lemma 2.3 in the case of $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$, since

$$8F_{k-2}F_{k-1} < F_{k+1}F_{k+2} < 21F_{k-2}F_{k-1}.$$

Hence, we need to generalize Lemma 2.3 to find the form of third elements of Diophantine triple $\{a, b, c\}$ when $b > 8a$.

Lemma 3.1. *Let $\{a, b, c\}$ be a Diophantine triple and $a < b \leq 24a$. Suppose that $\{1, 3, a, b\}$ is not a Diophantine quadruple. Then $c = c_\nu^\tau$ for some n and τ.*

Proof. The proof is same as the proof of Lemma 4.1 in [10], except the case of $c' < b$. If $c' = 0$, then $s' = rs - at = 1$, and $c = c_1^-$ or c_1^+. Let $r' = s'r - at'$ and $b' = ((r')^2 - 1)/a$. Then, $b' = a + b + c' + 2abc' - 2rs't'$, and thus, [15, Lemma 4] and $b \leq 24a$ together imply that

$$b' < \frac{b}{4ac'} \leq \frac{24a}{4ac'} = \frac{6}{c'}.$$

First, we consider the case of $c' = 1$. Since b' and c' satisfy the condition $b'c' + 1 = b' + 1 = \square$, we have $b' = 0$. For the other cases, we have the following results.

$$b' = \begin{cases} 0 & \text{if } c' \geq 6, \\ 0, 1 & \text{if } c' = 3, 4, 5, \\ 0, 1, 2 & \text{if } c' = 2. \end{cases}$$

We need to check only the case $(b', c') = (1, 3)$ except $b' = 0$ using the condition $b'c' + 1 = \square$. However, this means the set $\{1, 3, a, b\}$ satisfies Diophantine quadruple, which is a contradiction by assumption. In all cases, we obtain $b' = 0$ which yields $c' = a + b - 2r = c_1^-$ and $c = c_2^-$. Hence, we proved the lemma. $\square$

Assume that $a < b \leq 24a$ and $\{1, 3, a, b\}$ is not a Diophantine quadruple. Then, Lemma 3.1 implies that the positive solutions of the Diophantine equation

$$ay^2 - bx^2 = a - b$$

are given by

$$y\sqrt{a} + x\sqrt{b} = (\pm\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^l$$

with some non-negative integer l. Thus, we may write $x = p_l$, $y = V_l$, where

$$p_0 = 1, \quad p_1 = r \pm a, \quad p_{l+2} = 2rp_{l+1} - p_l, \tag{9}$$

$$V_0 = \pm 1, \quad V_1 = b \pm r, \quad V_{l+2} = 2rV_{l+1} - V_l. \tag{10}$$

From equations (4) and (5), we may also write $x = q_m$, $y = W_n$, where

$$q_0 = x_0, \quad q_1 = sx_0 + az_0, \quad q_{m+2} = 2sq_{m+1} - q_m, \tag{11}$$

$$W_0 = y_1, \quad W_1 = ty_1 + bz_1, \quad W_{n+2} = 2tW_{n+1} - W_n. \tag{12}$$

The result of next lemma is similar to Lemma 2.10, but the bound of b is further extended.

Lemma 3.2. *Assume that $a < b \leq 24a$ and $\{1, 3, a, b\}$ is not a Diophantine quadruple.*

(i) *In the case of c_1^+, we have $v_{2m+1} \neq w_{2n}$, $v_{2m} \neq w_{2n+1}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, if $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*

(ii) *In the case of $c = c_2^-$, we have $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$. Moreover, we have the following*:

(1) *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$ or $cr - st$.*
(2) *If $v_{2m} = w_{2n+1}$, then $|z_0| = cr - st$ and $|z_1| = s$ with $z_0 z_1 < 0$.*

Furthermore, (2) occurs if and only if (1) with $|z_0| = cr - st$ occurs.

(iii) *In the case of $c \in \{c_2^+, c_3^-, c_3^+\}$, then we have $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$. Moreover, we get the following*:

- *If $v_{2m} = w_{2n}$, then $z_0 = z_1$ and $|z_0| = 1$.*
- *If $v_{2m+1} = w_{2n+1}$, then $|z_0| = t$ and $|z_1| = s$ with $z_0 z_1 > 0$.*

Proof. The proof proceeds along the same lines as that of [11, Lemma 3.1].

(i) From (10) and (12), we have $y_1^2 \equiv 1 \pmod{b}$ either n is even or odd in W_n. Since $a < b - 2$ and $r < b - 1$, so $c = c_1^+ = a + b + 2r \leq 4b - 7$. Hence

$$0 < y_1^2 \leq \frac{t+1}{2} \leq \frac{\sqrt{b(4b-7)+1}}{2} < b.$$

This means $y_1^2 = 1$, that is, $y_1 = 1$ and $z_1 = \pm 1$. Hence, the case $v_{2m} = w_{2n}$ only occur and $z_0 = z_1$ from Lemma 2.9.

(ii) Suppose that $|z_0| = t$. Then we have $c > 4ab^2$ by upper bound of y. However, this is a contradiction, since

$$c = c_2^- = 4ab\left\{b - \left(2r - a - \frac{1}{a}\right) - \frac{r-a}{ab}\right\} < 4ab^2.$$

Therefore, we have $|z_0| \neq t$ and this means $v_{2m+1} \neq w_{2n}$ and $v_{2m+1} \neq w_{2n+1}$ by Lemma 2.9. Also, the last case of $|z_0|$ in $v_{2m} = w_{2n}$ occurs when the Diophantine quadruple $\{a, b, c, d\}$ is only irregular. Hence, it suffices to show that any Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ is regular when the cases of $c \leq c_3^+$. Assume that the set $\{a, b, c, d_0\}$ with $d_0 < d_-$ is a Diophantine quadruple. First, since $r = \sqrt{ab+1} \leq \sqrt{24a^2+1} < 5a$ and together with Lemma 2.1, we have following inequality

$$c \leq c_3^+ < a^{2.5}b^{3.5}\left(\frac{176}{b} + \frac{16}{b^{0.5}} + \frac{184}{b^{2.5}} + \frac{24}{b^{1.5}} + \frac{39}{b^{3.5}} + \frac{9}{b^{2.5}}\right) < a^{2.5}b^{3.5}.$$

However, we have $c > 16a^{2.5}b^{3.5}$ by Lemma 15 of [14]. Hence, Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ can not be irregular. It is easy to see that v_{2m} and w_{2n+1} in (2) are equal to v_{2m} and w_{2n} in (1) with $|z_0| = cr - st$.

(iii) Since we show that if $c \le c_3^+$, then any Diophantine quadruple $\{a, b, c, d_0\}$ with $d_0 < c$ is regular in (ii), it suffices to show that $\max\{|z_0|, |z_1|\} < cr - st$ when $c \ge c_2^+$. Since $c \ge c_2^+ > 4ab^2 \ge 32ab$, we have

$$cr - st = \frac{c^2 - (a+b)c - 1}{cr + st} > \frac{(1 - a/c - b/c - 1/c^2)c}{2\sqrt{ab}\sqrt{1 + 1/(ab)}} > \frac{c}{2.1\sqrt{ab}}.$$

If $\max\{|z_0|, |z_1|\} \ge cr - st$, then the bounds of $|z_0|$ and $|z_1|$ show that

$$c < \left(\frac{2.1}{\sqrt{2}}\right)^4 ab^2 < 5ab^2.$$

However, we have the following inequalities

$$c \ge c_2^+ > 4ab(a + b + 2r) > 4ab^2\left(\frac{1}{24} + 1 + \frac{1}{\sqrt{6}}\right) > 5.79ab^2,$$

since $b \le 24a$ implies that $r = \sqrt{ab+1} > b/(2\sqrt{6})$. Hence, this is a contradiction. □

4. The extendibility of $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$

In this section, we prove the extendibility of the Diophantine pair $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}\}$. From Theorem 2.5, we only need to check the cases of $c \le c_2^+$.

Lemma 4.1. *Suppose that $m, n \ge 2$.*

(1) *If $v_{2m} = w_{2n}$, then*

$$m \ge \begin{cases} (\sqrt{2F_k + 1} - 1)/2, & \text{if } c_1^+ \text{ or } c_2^- \text{ with } |z_0| = cr - st, \\ \sqrt[4]{F_k}/2, & \text{if } c_2^- \text{ with } |z_0| = 1 \text{ or } c_2^+. \end{cases}$$

(2) *If $v_{2m} = w_{2n+1}$ with c_2^- or $v_{2m+1} = w_{2n+1}$ with c_2^+, then*

$$m \ge \frac{\sqrt{2F_k + 1} - 1}{2}.$$

Proof. Since $c_i^{\pm}$ is divisible by F_k for all $i = 1, 2$ and

- In the case of c_1^+,
  $s_1^+ \equiv F_{k-2}F_{k+1} = a \pmod{F_k}$ and
  $t_1^+ \equiv -F_{k-2}F_{k+1} = -a \pmod{F_k}$.
- In the case of c_2^-,
  $s_2^- = 2F_{k-1}^2F_k^2 - 1 \equiv -1 \pmod{F_k}$ and
  $t_2^- = 2F_k^2(F_{k-1}F_k \pm 1) + 1 \equiv 1 \pmod{F_k}$.
- In the case of c_2^+,
  $s_2^+ = 2F_{k-2}F_{k+1}^3 + 1 \equiv -1 \pmod{F_k}$ and
  $t_2^+ = 2F_k^2F_{k+1}^2 - 1 \equiv -1 \pmod{F_k}$.

Hence, we have the following lower bounds from Lemma 2.8.

(1) The case of $v_{2m} = w_{2n}$:
We easily find the relation between v_{2m} and w_{2n} such that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

- First, we consider the case of c_1^+.
Since $F_{k-1}F_{k+2} \equiv -F_{k-2}F_{k+1} \pmod{F_k}$ and $\gcd(F_{k-2}F_{k+1}, F_k) = 1$, so we have

$$\pm(m^2 + n^2 \pm m \pm n) \equiv 0 \pmod{F_k}.$$

This means $2(m^2+m) \geq m^2+n^2 \pm m \pm n \geq m^2+n^2-m-n > 0$. Hence $2(m^2 + m) \geq F_k$.
- In the case of c_2^- with $|z_0| = cr - st$, we get the equation

$$\pm am^2 + am \equiv \pm bn^2 + bn \pmod{F_k},$$

since $st \equiv -1 \pmod{F_k}$ and $r \equiv 0 \pmod{F_k}$. Hence, similar to above, we have the result.
- Let us consider the other case of c_2^-, that is, c_2^- with $|z_0| = 1$. We have the equation

$$\pm a(m^2 + n^2) \equiv m + n \pmod{F_k},$$

and squaring both sides, then $a^2(m^2+n^2)^2 \equiv (m+n)^2 \pmod{F_k}$. Since $a^2 \equiv 1 \pmod{F_k}$, we have

$$(m^2 + n^2)^2 - (m + n)^2 \geq 0.$$

This means $4m^4 \geq F_k$, so we have the desired result.
- Lastly, in the case of c_2^+ with $|z_0| = 1$, we have

$$\pm a(m^2 + n^2) \equiv m - n \pmod{F_k}.$$

Similar to above, squaring both sides and we have

$$(m^2 + n^2) - (m - n)^2 \equiv 0 \pmod{F_k}.$$

Hence, $4m^4 \geq F_k$, and the result is deduced.

(2) The cases of $v_{2m} = w_{2n+1}$ with c_2^- and $v_{2m+1} = w_{2n+1}$ with c_2^+:
From Lemma 2.8, we have

$$\pm a(-st)m^2 - astm \equiv \mp 2bstn(n + 1) \pmod{F_k}$$

and

$$\pm 2atsm(m + 1) \equiv \pm 2btsn(n + 1) \pmod{F_k},$$

respectively. So, they become

$$a(m^2 + n^2 + n \pm m) \equiv 0 \pmod{F_k}$$

and

$$\pm a(m^2 + n^2 + m + n) \equiv 0 \pmod{F_k},$$

respectively, and we have the desired result. $\square$

Let us find the logarithmic inequality for $c = c_1^+$.

Lemma 4.2. *If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$, then*

$$\begin{aligned} 0 &< 2m\log(s+\sqrt{ac}) - 2n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})} \\ &< 3.018(s+\sqrt{ac})^{-4m}. \end{aligned}$$

Proof. Put

$$P = \frac{1}{\sqrt{a}}(x_0\sqrt{c}+z_0\sqrt{a})(s+\sqrt{ac})^m, \quad Q = \frac{1}{\sqrt{b}}(y_1\sqrt{c}+z_1\sqrt{b})(t+\sqrt{bc})^n.$$

Then

$$P^{-1} = \frac{\sqrt{a}(x_0\sqrt{c}-z_0\sqrt{a})}{c-a}(s-\sqrt{ac})^m, \quad Q^{-1} = \frac{\sqrt{b}(y_1\sqrt{c}-z_1\sqrt{b})}{c-b}(t-\sqrt{bc})^n.$$

Therefore, the relation $v_m = w_n$ becomes

$$P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}.$$

Since $P > 0,\ Q > 0$ and

$$P - Q > \frac{c-a}{a}(Q-P)P^{-1}Q^{-1},$$

it follows that $P > Q$. Furthermore, we have

$$\frac{P-Q}{P} < \frac{c-a}{a}P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{170},$$

since the case of $\{3,5\}$ is proved by Filipin, Fujita and Togbé [11]. Hence,

$$\begin{aligned} 0 < \log\frac{P}{Q} = -\log(1-\frac{P-Q}{P}) &< \frac{171}{170}(\frac{c-a}{a})P^{-2} \\ &< \frac{171}{170}\frac{c-a}{(\sqrt{c}-\sqrt{a})^2}(s+\sqrt{ac})^{-2m}. \end{aligned}$$

Since $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 3$, we obtain the result. □

4.1. The theorem of Baker and Wüstholz

Now we apply theorem of Baker and Wüstholz.

(i) First, we consider the equation $v_{2m} = w_{2n}$. We may assume that $k > 2$. Using Lemma 2.11 and Lemma 4.2, and apply Theorem 2.12, then we have $l = 3$, $d = 4$, $\beta = 2m$,

$$\alpha_1 = s+\sqrt{ac}, \quad \alpha_2 = t+\sqrt{bc}, \quad \alpha_3 = \frac{(\sqrt{c}\pm\sqrt{a})\sqrt{b}}{(\sqrt{c}\pm\sqrt{b})\sqrt{a}}.$$

Let α_3' and α_3'' be the conjugates of α_3 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t),$$

$$\begin{aligned} h'(\alpha_3) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_3\alpha_3'\alpha_3'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}(\sqrt{c}+\sqrt{a})(\sqrt{c}+\sqrt{b})(c-a))\} < \log(1.42c), \end{aligned}$$

and

$$\log|\Lambda| \geq -18\cdot 4!3^4(32\cdot 4)^5\frac{1}{2}\log(2s)\frac{1}{2}\log(2t)\log(1.42c)\cdot\log(24)\cdot\log(2m).$$

Since

$$\log(\frac{8}{3}ac(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac)$$

and

$$\log(3.018(s+\sqrt{ac})^{-4m}) < (-2m+1)\log(4ac),$$

we have

$$\frac{2m-1}{\log(2m)} < 9.556\cdot 10^{14}\log(2c)\log(1.42c). \tag{13}$$

Let $x = F_k$.

- If $c = c_1^+$, then

$$\sqrt{2x+1} - 2 < 4.778\cdot 10^{14}\log(3.17x)^2\cdot\log(2.67x)^2\cdot\log(2x+1).$$

  Hence, $x < 1.64\cdot 10^{42}$ and $c_1^+ < 1.35\cdot 10^{85}$.
- If $c = c_2^-$ with $|z_0| = cr - st$, then

$$\sqrt{2x+1} - 2 < 4.778\cdot 10^{14}\log(2x)^6\cdot\log(1.89x)^6\cdot\log(2x+1).$$

  Hence, $x < 1.72\cdot 10^{44}$ and $c_2^- < 6.22\cdot 10^{266}$.
- If $c = c_2^-$ with $|z_0| = \pm 1$, then

$$\sqrt[4]{x} - 1 < 2.389\cdot 10^{14}\log(2x)^6\cdot\log(1.89x)^6\cdot\log x.$$

  Hence, $x < 4.38\cdot 10^{91}$ and $c_2^- < 1.68\cdot 10^{551}$.
- If $c = c_2^+$, then

$$\sqrt[4]{x} - 1 < 2.389\cdot 10^{14}\log(2.55x)^6\cdot\log(2.41x)^6\cdot\log x.$$

  Hence, $x < 4.42\cdot 10^{91}$ and $c_2^+ < 8.83\cdot 10^{551}$.

Since $F_k = (\alpha^k - \bar{\alpha}^k)/\sqrt{5}$, where $\alpha = (1+\sqrt{5})/2 > 1.618$, we have the following inequality from Fibonacci numbers

$$(1.618)^k < (\alpha)^k = \bar{\alpha}^k + \sqrt{5}\cdot F_k. \tag{14}$$

We can find the upper bounds of k and m by using inequalities (13) and (14), respectively.

- If the case of c_1^+, then $k \leq 203$ and $m \leq 9.04\cdot 10^{20}$.
- If the case of c_2^-, then $k \leq 440$ and $m \leq 4.07\cdot 10^{22}$.
- If the case of c_2^+, then $k \leq 440$ and $m \leq 4.08\cdot 10^{22}$.

(ii) Let $v_{2m} = w_{2n+1}$ with $n \neq 0$. We have $l = 3$, $d = 4$, $\beta = 2m+1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_4 = \frac{((sr-ta)\sqrt{c} \pm (cr-st)\sqrt{a})\sqrt{b}}{(r\sqrt{c} \mp s\sqrt{b})\sqrt{a}}.$$

Let α_4' and α_4'' be the conjugates of α_4 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t),$$

$$\begin{aligned} h'(\alpha_4) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_4\alpha_4'\alpha_4'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}((sr-ta)\sqrt{c} + (cr-st)\sqrt{a})(r\sqrt{c}+s\sqrt{b})(c-a))\} \\ &< \log(1.42\sqrt{r}c), \end{aligned}$$

since $(sr-ta)\sqrt{c} > (cr-st)\sqrt{a}$ and $sr - ta < r$. Hence, we have

$$\frac{2m-1}{\log(2m+1)} < 9.556 \cdot 10^{14}\log(2c)\log(1.42\sqrt{r}c). \tag{15}$$

If $c = c_2^-$, then

$$\sqrt{x+1} - 2 < 4.778 \cdot 10^{14}\log(1.97x)^6 \cdot \log(1.7x)^7 \cdot \log(x+1),$$

where $x = F_k$. Hence, we have $x < 4.9 \cdot 10^{44}$ and $c_2^- < 3.98 \cdot 10^{269}$. Again, by using the inequality from (14) and (15), we get the bound of $k \leq 215$ and $m \leq 1.11 \cdot 10^{22}$.

(iii) Let $v_{2m+1} = w_{2n+1}$ with $n \neq 0$. We have $l = 3$, $d = 4$, $\beta = 2m+1$,

$$\alpha_1 = s + \sqrt{ac}, \quad \alpha_2 = t + \sqrt{bc}, \quad \alpha_5 = \frac{(r\sqrt{c} \pm t\sqrt{a})\sqrt{b}}{(r\sqrt{c} \pm s\sqrt{b})\sqrt{a}}.$$

Let α_5' and α_5'' be the conjugates of α_5 whose absolute values are greater than one. Then

$$h'(\alpha_1) = \frac{1}{2}\log(\alpha_1) < \frac{1}{2}\log(2s), \quad h'(\alpha_2) = \frac{1}{2}\log(\alpha_2) < \frac{1}{2}\log(2t),$$

$$\begin{aligned} h'(\alpha_5) &\leq \frac{1}{4}\{\log(a^2(c-b)^2) + \log(\alpha_5\alpha_5'\alpha_5'')\} \\ &= \frac{1}{4}\{\log(b\sqrt{ab}(r\sqrt{c}+t\sqrt{a})(r\sqrt{c}+s\sqrt{b})(c-a))\} \\ &< \log(1.42\sqrt{r}c). \end{aligned}$$

Hence, we have

$$\frac{2m}{\log(2m+1)} < 9.556 \cdot 10^{14}\log(2c)\log(1.42\sqrt{r}c). \tag{16}$$

If $c = c_2^+$, then

$$\sqrt{x+1} - 1 < 4.778 \cdot 10^{14}\log(2.53x)^6 \cdot \log(2.11x)^7 \cdot \log(x+1),$$

where $x = F_{2k+2}$. Hence, we get $x < 4.95 \cdot 10^{44}$ and $c_2^+ < 1.91 \cdot 10^{270}$. This means $k \leq 215$ and $m \leq 1.12 \cdot 10^{22}$.

4.2. The reduction method

Now dividing logarithmic inequalities from Lemma 2.11 and Lemma 4.2 by $\log \alpha_2$, respectively leads us to the inequalities

$$\begin{aligned}
0 &< m_1\kappa - n_1 + \mu_1 < A_1 B^{m_1},\\
0 &< m_1\kappa - n_1 + \mu_1 < A_2 B^{m_1},\\
0 &< m_1\kappa - n_2 + \mu_2 < A_1 B^{m_1},\\
0 &< m_2\kappa - n_2 + \mu_3 < A_1 B^{m_2},
\end{aligned}$$

where $m_1 := 2m,\ m_2 := 2m+1,\ n_1 := 2n,\ n_2 := 2n+1$ and

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2},\ \mu_1 = \frac{\log \alpha_3}{\log \alpha_2},\ \mu_2 = \frac{\log \alpha_4}{\log \alpha_2},\ \mu_3 = \frac{\log \alpha_5}{\log \alpha_2},$$

$$A_1 = \frac{(8/3)ac}{\log \alpha_2},\ A_2 = \frac{3.018}{\log \alpha_2},\ B = \alpha_1^2.$$

We apply Lemma 2.13 to the logarithmic inequalities with $M_1 := 2m \leq 8.16 \cdot 10^{22}$ and $M_2 := 2m+1 \leq 2.24 \cdot 10^{22}$. We have to examine $2\cdot 203+2\cdot 440+2\cdot 440+2\cdot 215+2\cdot 215 = 3026$ cases. The program was developed in **PARI/GP** running with 400 digits. For the computations, if the first convergent such that $q > 6M_i$ with $i = 1, 2$ does not satisfy the condition $\epsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Then we have the results as the following Table 1.

Table 1. Results from **PARI/GP** running

Case of c	Initial values	Use the next convergent
c_1^+	$z_0 = z_1 = 1$	0 case
	$z_0 = z_1 = -1$	99 cases
c_2^-	$z_0 = z_1 = 1$	330 cases
	$z_0 = z_1 = -1$	352 cases ($k = 89, \ldots, 440$)
c_2^-	$z_0 = cr - st, z_1 = -s$	0 case
	$z_0 = st - cr, z_1 = s$	0 case
c_2^+	$z_0 = z_1 = 1$	353 cases ($k = 88, \ldots, 440$)
	$z_0 = z_1 = -1$	353 cases ($k = 88, \ldots, 440$)
c_2^+	$z_0 = t, z_1 = s$	0 case
	$z_0 = -t, z_1 = -s$	172 cases ($k = 44, \ldots, 215$)

We have the upper bounds 13, 12 and 8 of m in the case of c_1^+, c_2^- with $|z_0| = 1$, and c_2^+ with $|z_0| = 1$, respectively. If we take $M = 2m$ and continue the program again, then we have $m \leq 1$. Other cases, that is, c_2^- with $|z_0| = cr - st, |z_1| = s$, and c_2^+ with $|z_0| = t, |z_1| = s$ have the upper bounds 7 and 8 of m, respectively. Hence, we take $M = 2m$ or $2m + 1$, and continue the program again, we also get $m \leq 1$. Therefore, we get the following theorem.

Theorem 4.3. *Let $k \geq 3$ be an integer and the set $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c, d\}$ be a Diophantine quadruple with $c < d$, then $d = d_+$.*

Corollary 4.4. *The set $\{F_{k-2}F_{k+1}, F_{k-1}F_{k+2}, c\}$ can be extended only to regular.*

5. The extendibility of $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$

In this section, we show that the set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ can be extended to only regular, where $c > \max\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}\}$. We can easily show that the Fibonacci numbers $F_{k-2}F_{k-1}$ and $F_{k+1}F_{k+2}$ do not satisfy the Diophantine quadruple with $\{1, 3\}$, by considering the period of Fibonacci numbers in modulo 4. Hence, we can find the form of third element in the Diophantine triple $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ by Lemma 3.1.

Lemma 5.1. *Let $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ be a Diophantine triple. Then $c = c_\nu^\tau$ for some ν and τ.*

The small values of c_ν^τ are listed below.

$$\begin{aligned}
c_1^+ &= 4F_kF_{k+1}, \\
c_2^- &= 4F_k^2(4F_k^3F_{k-1} + 1), \\
c_2^+ &= 4F_k^2(4F_k^3F_{k+1} - 1), \\
c_3^- &= F_k[64F_{k-2}^2F_{k-1}^3F_{k+1}^2F_{k+2}^2 + 16F_{k-2}F_{k-1}F_kF_{k+1}F_{k+2} \\
&\quad + 96F_{k-2}F_{k-1}^2F_{k+1}F_{k+2} + 12F_k + 36F_{k-1}], \\
c_3^+ &= F_k[64F_{k-2}^2F_{k-1}^2F_{k+1}^3F_{k+2}^2 + 96F_{k-2}F_{k-1}F_kF_{k+1}^2F_{k+2} \\
&\quad - 16F_{k-2}F_{k-1}^2F_{k+1}F_{k+2} + 24F_k + 36F_{k-1}].
\end{aligned}$$

By Theorem 2.4 and Lemma 3.1, we have the upper bound of third elements c in the Diophantine triple $\{a, b, c\}$ with $a < b \leq 21a$.

Lemma 5.2. *Let $\{a, b\}$ be a Diophantine pair with $a < b \leq 21a$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d > c_{n+1}^\tau$ and that $\{a, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < c_{n-1}^\tau$. Then $c \leq c_3^+$.*

Proof. Since $c_4^- = 64a^3b^3(a+b-2r) + 64a^2b^2(2a+2b-3r) + 80ab(a+b-r) + 8(2a+2b-r)$, it suffices to show that $64a^3b^3(a+b-2r) > b^5$, by Theorem 2.4. We have

$$\frac{c_4^-}{b^5} > \frac{64a^3(a+b-2r)}{b^2} > \frac{64a^3(a+b-2r)}{(21a)^2} > 1.16 > 1.$$

Hence, this completes the proof of lemma. $\square$

Lemma 5.3. *Suppose that $m, n \geq 2$.*

(1) *If $v_{2m} = w_{2n}$, then*

$$m \geq \begin{cases} (\sqrt[4]{F_k}/2, & \text{if } c_2^- \text{ with } |z_0| = 1 \text{ or } c_2^+, \\ \sqrt{2F_k+1} - 1)/2, & \text{if the other cases.} \end{cases}$$

(2) *If $v_{2m} = w_{2n+1}$ with c_2^- or $v_{2m+1} = w_{2n+1}$ with c_2^+, c_3^- and c_3^+, then*

$$m \geq \frac{\sqrt{2F_k+1}-1}{2}.$$

Proof. We have the following relation between c_ν^τ and F_k.

- In the case of c_1^+: $s_1^+ \equiv a \pmod{F_k}$, $t_1^+ \equiv b \equiv -a \pmod{F_k}$.
- In the case of c_2^-: $s_2^- \equiv -1 \pmod{F_k}$, $t_2^- \equiv 1 \pmod{F_k}$.
- In the case of c_2^+: $s_2^+ \equiv -1 \pmod{F_k}$, $t_2^+ \equiv 1 \pmod{F_k}$.
- In the case of c_3^-: $s_3^- \equiv a \pmod{F_k}$, $t_3^- \equiv -b \equiv a \pmod{F_k}$.
- In the case of c_3^+: $s_3^+ \equiv -a \pmod{F_k}$, $t_1^+ \equiv -b \equiv a \pmod{F_k}$.

We can also easily find the relation between v_m and w_n.

(1) The case of $v_{2m} = w_{2n}$:

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{F_k}.$$

(2) The case of $v_{2m} = w_{2n+1}$:

$$\pm a(-st)m^2 - astm \equiv \mp 2bstn(n+1) \pmod{F_k}.$$

(3) The case of $v_{2m+1} = w_{2n+1}$:

$$\pm 2atsm(m+1) \equiv \pm 2btsn(n+1) \pmod{F_k}.$$

Since $F_{k+1}F_{k+2} \equiv -F_{k-2}F_{k-1} \pmod{F_k}$ and $\gcd(F_{k-2}F_{k-1}, F_k) = 1$, we can prove the lemma by the same procedure as in the proof of Lemma 4.1. □

Lemma 5.4. *If $v_{2m} = w_{2n}$ with c_1^+ and $m, n \neq 0$, then*

$$0 < 2m\log(s+\sqrt{ac}) - 2n\log(t+\sqrt{bc}) + \log\frac{\sqrt{b}(x_0\sqrt{c}+z_0\sqrt{a})}{\sqrt{a}(y_1\sqrt{c}+z_1\sqrt{b})}$$
$$< 2.09(s+\sqrt{ac})^{-4m}.$$

Proof. The proof follows along the same line as that of Lemma 4.2. In this case, we have

$$\frac{P-Q}{P} < \frac{c-a}{a}P^{-2} < \frac{1}{a(c-a)} \leq \frac{1}{23}.$$

Hence,

$$0 < \log\frac{P}{Q} = -\log(1-\frac{P-Q}{P}) < \frac{24}{23}(\frac{c-a}{a})P^{-2} < \frac{24}{23}\frac{c-a}{(\sqrt{c}-\sqrt{a})^2}(s+\sqrt{ac})^{-2m}.$$

Since $\frac{\sqrt{c}+\sqrt{a}}{\sqrt{c}-\sqrt{a}} < 2$, we obtain the result. □

5.1. The theorem of Baker and Wüstholz

By the theorem of Baker and Wüstholz again, we have the equations (13), (15) and (16) for $v_{2m} = w_{2n}$, $v_{2m} = w_{2n+1}$ and $v_{2m+1} = w_{2n+1}$, respectively, since $2.09(s+\sqrt{ac})^{-4m} < (-2m+1)\log(4ac)$.

Let $x = F_k$. First, let us consider the cases of the equation $v_{2m} = w_{2n}$. We have the following inequalities for each c_i^τ, $i = 1, 2, 3$.

- If $c = c_1^+$, then
$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(4x)^2 \cdot \log(3.38x)^2 \cdot \log(2x+1).$$
Hence, $x < 1.65 \cdot 10^{42}$ and $c_1^+ < 2.18 \cdot 10^{85}$.
- If $c = c_2^-$, then
$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(1.85x)^6 \cdot \log(1.75x)^6 \cdot \log x.$$
Hence, $x < 4.36 \cdot 10^{91}$ and $c_2^- < 1.1 \cdot 10^{551}$.
- If $c = c_2^+$, then
$$\sqrt[4]{x} - 1 < 2.389 \cdot 10^{14} \log(2.04x)^6 \cdot \log(1.93x)^6 \cdot \log x.$$
Hence, $x < 4.38 \cdot 10^{91}$ and $c_2^+ < 2.26 \cdot 10^{551}$.
- If $c = c_3^-$, then
$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(2.39x)^{10} \cdot \log(2.31x)^{10} \cdot \log(2x+1).$$
Hence, $x < 1.52 \cdot 10^{45}$ and $c_3^- < 2 \cdot 10^{455}$.
- If $c = c_3^+$, then
$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(2.56x)^{10} \cdot \log(2.47x)^{10} \cdot \log(2x+1).$$
Hence, $x < 1.52 \cdot 10^{45}$ and $c_3^+ < 3.84 \cdot 10^{455}$.

From (14), we get the upper bound of k for each c. Also from (13) and the upper bound of c, we get the upper bound of m.

- If $c = c_1^+$, then $k \le 203$ and $m \le 9.08 \cdot 10^{20}$.
- If $c = c_2^-$, then $k \le 440$ and $m \le 4.07 \cdot 10^{22}$.
- If $c = c_2^+$, then $k \le 440$ and $m \le 4.07 \cdot 10^{22}$.
- If $c = c_3^-$, then $k \le 217$ and $m \le 2.76 \cdot 10^{22}$.
- If $c = c_3^+$, then $k \le 217$ and $m \le 2.76 \cdot 10^{22}$.

Next, we consider the case of $v_{2m} = w_{2n+1}$ with c_2^+ with $|z_0| = cr - st$ and $|z_1| = s$. Then we have the inequality from (15).
$$\sqrt{2x+1} - 2 < 4.778 \cdot 10^{14} \log(1.85x)^6 \cdot \log(1.62x)^7 \cdot \log(x+1).$$
Hence, $x < 2.34 \cdot 10^{44}$ and $c_2^- < 3.29 \cdot 10^{267}$. This means $k \le 213$ and $m \le 1.09 \cdot 10^{22}$.

Lastly, consider the case of $v_{2m+1} = w_{2n+1}$. There are three cases c_2^+, c_3^- and c_3^+. We have the following inequalities from (16).

- In the case of $c = c_2^+$, then
  $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.04x)^6 \cdot \log(1.76x)^7 \cdot \log(x+1).$
  Hence, $x < 4.91 \cdot 10^{44}$ and $c_2^+ < 5.05 \cdot 10^{269}$.
- In the case of $c = c_3^-$, then
  $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.39x)^{10} \cdot \log(2.14x)^{11} \cdot \log(x+1).$
  Hence, $x < 3.81 \cdot 10^{45}$ and $c_3^- < 1.95 \cdot 10^{459}$.
- In the case of $c = c_3^+$, then
  $\sqrt{x+1} - 1 < 4.778 \cdot 10^{14} \log(2.56x)^{10} \cdot \log(2.28x)^{11} \cdot \log(x+1).$
  Hence, $x < 3.82 \cdot 10^{45}$ and $c_3^+ < 3.86 \cdot 10^{459}$.

Also, using the inequality (14) and upper bounds of c, we have the following upper bound of k and m for each cases.

- In the case of $c = c_2^+$, then $k \le 215$ and $m \le 1.11 \cdot 10^{22}$.
- In the case of $c = c_3^-$, then $k \le 219$ and $m \le 3.09 \cdot 10^{22}$.
- In the case of $c = c_3^+$, then $k \le 219$ and $m \le 3.09 \cdot 10^{22}$.

5.2. The reduction method

We again apply Lemma 2.13 about diving logarithmic inequalities from Lemma 2.11 and Lemma 5.4 by $\log \alpha_2$, that is,

$$\begin{aligned}
0 &< m_1\kappa - n_1 + \mu_1 < A_1 B^{m_1},\\
0 &< m_1\kappa - n_1 + \mu_1 < A_2 B^{m_1},\\
0 &< m_1\kappa - n_2 + \mu_2 < A_1 B^{m_1},\\
0 &< m_2\kappa - n_2 + \mu_3 < A_1 B^{m_2},
\end{aligned}$$

where $m_1 := 2m,\ m_2 := 2m+1,\ n_1 := 2n,\ n_2 := 2n+1$ and

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2},\ \mu_1 = \frac{\log \alpha_3}{\log \alpha_2},\ \mu_2 = \frac{\log \alpha_4}{\log \alpha_2},\ \mu_3 = \frac{\log \alpha_5}{\log \alpha_2},$$

$$A_1 = \frac{(8/3)ac}{\log \alpha_2},\ A_2 = \frac{2.09}{\log \alpha_2},\ B = \alpha_1^2.$$

Let $M_1 := 2m \le 5.52 \cdot 10^{22}$ and $M_2 := 2m+1 \le 6.18 \cdot 10^{22}$. We have to examine $2{\cdot}203+2{\cdot}440+2{\cdot}440+2{\cdot}217+2{\cdot}217+2{\cdot}213+2{\cdot}215+2{\cdot}219+2{\cdot}219 = 4766$ cases. Similar to Section 4.2.2, we have the result as the following Table 2 by using the program **PARI/GP** running with 400 digit. We have the following upper bounds of m in Table 2 for each cases. Use these upper bounds and continue the program again, then we have $m \le 1$. Hence, we get the following theorem.

Theorem 5.5. *Let $k \ge 3$ be the integer and the set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c, d\}$ be a Diophantine quadruple with $c < d$. Then $d = d_+$.*

Corollary 5.6. *The set $\{F_{k-2}F_{k-1}, F_{k+1}F_{k+2}, c\}$ can be extended only to regular.*

TABLE 2. Results from **PARI/GP** running

Case of c	Initial values	Use the next convergent	Upper bound of m
c_1^+	$z_0 = z_1 = 1$	0 case	15
	$z_0 = z_1 = -1$	0 case	14
c_2^-	$z_0 = z_1 = 1$	353 cases ($k = 88, \ldots, 440$)	10
	$z_0 = z_1 = -1$	353 cases ($k = 88, \ldots, 440$)	10
c_2^-	$z_0 = cr - st, z_1 = -s$	169 cases ($k = 45, \ldots, 213$)	10
	$z_0 = st - cr, z_1 = s$	0 case	10
c_2^+	$z_0 = z_1 = 1$	354 cases ($k = 86, 88, \ldots, 440$)	9
	$z_0 = z_1 = -1$	0 case	8
c_2^+	$z_0 = t, z_1 = s$	200 cases ($k = 16, \ldots, 215$)	8
	$z_0 = -t, z_1 = -s$	171 cases ($k = 45, \ldots, 215$)	8
c_3^-	$z_0 = z_1 = 1$	0 case	6
	$z_0 = z_1 = -1$	0 case	6
c_3^-	$z_0 = t, z_1 = s$	204 cases ($k = 12, \ldots, 215$)	6
	$z_0 = -t, z_1 = -s$	0 case	7
c_3^+	$z_0 = z_1 = 1$	174 cases ($k = 44, \ldots, 217$)	6
	$z_0 = z_1 = -1$	0 case	6
c_3^+	$z_0 = t, z_1 = s$	0 case	6
	$z_0 = -t, z_1 = -s$	197 cases ($k = 23, \ldots, 219$)	6

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