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ON GENERAL (α,β)-METRICS WITH ISOTROPIC E-CURVATURE

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Abstract. General (α,β)-metrics form a rich and important class of Finsler metrics. In this paper, we obtain a differential equation which characterizes a general (α,β)-metric with isotropic E-curvature, under a certain condition. We also solve the equation in a particular case.

1. Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics [3]. For a Finsler metric $F = F(x, y)$, its geodesics curves are given by the system of differential equations $\ddot{c}^i + 2G^i(c, \dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$. Taking a trace of Berwald curvature yields E-curvature (mean Berwald curvature). The E-curvature is one of the most important non-Riemannian quantities in Finsler geometry [6]. In [1], Chen and Shen studied the relationship between isotropic E-curvature and relatively isotropic Landsberg curvature on a Douglas manifold. Tayebi, Nankali and Peyghan proved that every m-root Cartan space of E-curvature reduces to weakly Berwald spaces [7].

The special Finsler metrics we are going to investigate are called general (α,β)-metrics which first introduced by C. Yu and H. Zhu in [10]. By definition, a general (α,β)-metric F can be expressed in the following form:

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha}),$$

where $\alpha := \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta := b_iy^i$ is a 1-form, $b := ||\beta_x||_\alpha$ and $\phi(b^2, s)$ is a positive smooth function. It is easy to see that (α,β)-metrics compose a special class in general (α,β)-metrics. Another special class is defined by α being an Euclidean metric $|y|$ and β being an inner product $\langle x, y\rangle$. In this case, the general (α,β)-metric F becomes a spherically symmetric

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Finsler metric in the following form

$$F = |y|\phi(|x|^2, \frac{\langle x, y\rangle}{|y|}),$$

which is first introduced by S. F. Rutz who studied the spherically symmetric Finsler metrics in 4-dimensional time-space and generalized the classic Birkhoff theorem in general relativity to the Finsler case [5]. Moreover, general (α, β)-metrics include part of Bryants metrics [10] and part of fourth root metrics [4]. Randers metrics can be expressed in the following form

$$F = \frac{\sqrt{(1-\bar{b}^2)\bar{\alpha}^2 + \bar{\beta}^2}}{1-\bar{b}^2} + \frac{\bar{\beta}}{1-\bar{b}^2},$$

where $\bar{\alpha}$ is also a Riemannian metric, $\bar{\beta}$ is a 1-form and $\bar{b} := ||\bar{\beta}||_{\bar{\alpha}}$. $(\bar{\alpha}, \bar{\beta})$ is called the navigation data of the Randers metric F. Tayebi and Rafie-rad showed that if a Randers metric $F = \alpha + \beta$ is an non trivial isotropic Berwald metric, then $\bar{\beta}$ is a conformal 1-form with respect to $\bar{\alpha}$ [8].

For general (α, β)-metrics, spray coefficients and related geometrical objects have been studied by C. Yu and H. Zhu [10]. C. Yu gave a local characterization of locally dually flat general (α, β)-metrics and construct some useful examples of dually flat general (α, β)-metrics in [9]. Yu and Zhu completely determined classification of general (α, β)-metrics with constant flag curvature under some suitable conditions and construct many new projectively flat Finsler metrics with flag curvature 1, 0 and -1 in [11]. Then Zhu characterized general (α, β)-metrics with isotropic Berwald-curvature in [12]. Recently, M. Zohrehvand and H. Maleki, have proved that every Landsberg general (α, β)-metric is a Berwald metric, under a certain condition [13].

The goal of this paper is to study the isotropic E-curvature of general (α, β)-metrics, where β is a closed and conformal 1-form, i.e.,

$$b_{i|j} = ca_{ij}, \tag{1}$$

where $c = c(x) \neq 0$ is a scalar function on M and $b_{i|j}$ is the covariant derivation of β with respect to α. In fact we prove the following:

Theorem 1.1. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies* (1). *Then F is of isotropic E-curvature if and only if*

$$(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22}) = \rho(x)(n+1)(\phi - s\phi_2), \tag{2}$$

where $\rho(x) = \frac{k(x)}{c(x)}$, E and H are defined in (12) *and* (13), *respectively.*

In [2], Y. Chen and W. Song investigated projectively flat spherically symmetric Finsler metrics of isotropic$E$-curvature, which is correct for general (α, β)-metric as follows:

Corollary 1.2. *Let $F=\alpha\phi(b^2,\frac{\beta}{\alpha})$ be a projectively flat general (α,β)-metric with isotropic E-curvature. Suppose that β satisfies* (1)*. Then F is a Randers metric.*

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M. Every Finsler metric F induces a spray $G=y^i\frac{\partial}{\partial x^i}-2G^i\frac{\partial}{\partial y^i}$. The spray coefficients G^i are defined by

$$G^i:=\frac{1}{4}g^{il}\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\},$$

where $g_{ij}(x,y)=\left[\frac{1}{2}F^2\right]_{y^iy^j}$ and $(g^{ij})=(g_{ij})^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$.

For a Finsler metric F with spray coefficients G^i, the Berwald curvature $B=B_j{}^i{}_{kl}dx^j\otimes dx^k\otimes dx^l\otimes\frac{\partial}{\partial x^i}$ of F is defined by

$$B_j{}^i{}_{kl}:=\frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}. \tag{3}$$

F is called a Berwald metric if $B=0$. A Finsler metric F on a manifold M is said to be of isotropic Berwald curvature if its Berwald curvature $B_j{}^i{}_{kl}$ satisfies

$$B_j{}^i{}_{kl}=\tau(x)(F_{y^jy^k}\delta^i{}_l+F_{y^jy^l}\delta^i{}_k+F_{y^ly^k}\delta^i{}_l+F_{y^jy^ky^l}y^i), \tag{4}$$

where $\tau(x)$ is a scalar function on M. The E-curvature $E=E_{ij}dx^i\otimes dx^j$ of F is defined by

$$E_{ij}:=\frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}(\frac{\partial G^m}{\partial y^m}). \tag{5}$$

A Finsler metric F is said to have isotropic E-curvature if there is a scalar function $\kappa=\kappa(x)$ on M such that

$$E=\frac{1}{2}(n+1)\kappa F^{-1}h, \tag{6}$$

where h is a family of bilinear forms $h_y=h_{ij}dx^i\otimes dx^j$, which are defined by $h_{ij}:=FF_{y^iy^j}$.

In this paper, we use the indices 1 and 2 as the derivation with respect to b^2 and s, respectively.

Lemma 2.1 ([10])**.** *Let $F=\alpha\phi(b^2,\frac{\beta}{\alpha})$ be a general (α,β)-metric on an n-dimensional manifold M. Then the function F is a regular Finsler metric for any Riemannian metric α and any 1-form β if and only if $\phi(b^2,s)$ is a positive smooth function defined on the domain $|s|\le b<b_0$ for some positive number* (*maybe infinity*) *b_0 satisfying*

$$\phi-s\phi_2>0,\quad \phi-s\phi_2+(b^2-s^2)\phi_{22}>0, \tag{7}$$

when $n \geq 3$ *or*

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{8}$$

when $n = 2$.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Denote the coefficients of the covariant derivative of β with respect to α by $b_{i|j}$, and let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$r_{00} = r_{ij}y^i y^j, \quad s^i{}_0 = a^{ij}s_{jk}y^k, r_i = b^j r_{ji}, \quad s_i = b^j s_{ji},$$
$$r_0 = r_i y^i, \quad s_0 = s_i y^i, \quad r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^i r_i,$$

where $(a^{ij}) = (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$.

Clearly, β is a closed one-form if and only if $s_{ij} = 0$, and it is a conformal 1-form with respect to α, if and only if $b_{i|j} + b_{j|i} = ca_{ij}$, where $c = c(x)$ is a nonzero scalar function on M. Thus, we say that β is closed and conformal with respect to α, if $b_{i|j} = ca_{ij}$, where $c = c(x)$ is a nonzero scalar function on M.

Lemma 2.2 ([10])**.** *The spray coefficients* G^i *of a general* (α, β)*-metric* $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ *are related to the spray coefficients* ${}^\alpha G^i$ *of* α *and given by*

$$\begin{aligned} G^i &= G^i_\alpha + \alpha Q s^i{}_0 + \{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0 + s_0)\}\frac{y^i}{\alpha} \\ &\quad + \{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Pi(r_0 + s_0)\}b^i - \alpha^2 R(r^i + s^i), \end{aligned} \tag{9}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \qquad R = \frac{\phi_1}{\phi - s\phi_2},$$
$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_2}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \qquad \Psi = \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})},$$
$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

By (1), we have

$$r_{00} = c\alpha^2, \ r_0 = c\beta, \ r = cb^2, \ r^i = cb^i, \ s^i{}_0 = 0, \ s_0 = 0, \ s^i = 0. \tag{10}$$

Substituting (10) into (9) yields

$$\begin{aligned} G^i &= G^i_\alpha + c\alpha\{\Theta(1 + 2Rb^2) + s\Omega\}y^i + c\alpha^2\{\Psi(1 + 2Rb^2) + s\Pi - R\}b^i, \\ &= G^i_\alpha + c\alpha E y^i + c\alpha^2 H b^i, \end{aligned} \tag{11}$$

where

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - H\frac{s\phi + (b^2 - s^2)\phi_2}{\phi}, \tag{12}$$

$$H := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}]}. \tag{13}$$

3. E-curvature of general (α,β)-metrics

In this section, we will compute the E-curvature of a general (α,β)-metric.

Proposition 3.1. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β satisfies (1). Then the E-curvature of F is given by*

$$\begin{aligned} E_{ij} = \frac{c}{2}\{ & \frac{1}{\alpha}[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}]b_ib_j \\ & - \frac{s}{\alpha^2}[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}](b_iy_j + b_jy_i) \\ & + \frac{1}{\alpha^3}[(n+1)s^2E_{22} - (n+1)(E - sE_2) + s^2(b^2 - s^2)H_{222} \\ & + (3s^2 - b^2)(H_2 - sH_{22})]y_iy_j \\ & + \frac{1}{\alpha}[(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22})]a_{ij}\}, \end{aligned} \tag{14}$$

where $c = c(x) \neq 0$ is a scalar function on M.

Proof. By (11), we can rewrite the spray coefficients of a general (α,β)-metric as

$$G^i = G^i_\alpha + cW^i, \tag{15}$$

where

$$W^i := \alpha E y^i + \alpha^2 H b^i. \tag{16}$$

Then, from (16), we have

$$\frac{\partial W^i}{\partial y^j} = \alpha_{y^j} E y^i + \alpha E_2 s_{y^j} y^i + \alpha E \delta^i{}_j + [\alpha^2]_{y^j} H b^i + \alpha^2 H_2 s_{y^j} b^i. \tag{17}$$

By taking $i = j$ in (17), we have

$$\begin{aligned} \frac{\partial W^m}{\partial y^m} &= \alpha_{y^m} E y^m + \alpha E_2 s_{y^m} y^m + \alpha E \delta^m_m + [\alpha^2]_{y^m} H b^m + \alpha^2 H_2 s_{y^m} b^m, \\ &= \alpha[(n+1)E + 2sH + (b^2 - s^2)H_2], \end{aligned} \tag{18}$$

where we have used

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \quad s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2}, \quad [\alpha^2]_{y^i} = 2y_i. \tag{19}$$

By simple calculations, we have

$$\alpha_{y^iy^j} = \frac{1}{\alpha}(a_{ij} - \frac{y_i}{\alpha}\frac{y_j}{\alpha}), \tag{20}$$

$$s_{y^iy^j} = -\frac{1}{\alpha^2}[s a_{ij} + \frac{1}{\alpha}(b_iy_j + b_jy_i) - \frac{3s}{\alpha^2} y_iy_j]. \tag{21}$$

By using (18), we obtain

$$\frac{\partial}{\partial y^i}(\frac{\partial W^m}{\partial y^m}) = (n+1)\alpha_{y^i}E + (n+1)\alpha E_2 s_{y^i} + 2\alpha_{y^i}sH + 2\alpha s_{y^i}H + 2\alpha s H_2 s_{y^i} + \alpha_{y^i}(b^2-s^2)H_2 - 2\alpha s s_{y^i}H_2 + \alpha(b^2-s^2)H_{22}s_{y^i}. \tag{22}$$

It follows from (22) that

$$\begin{aligned}\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}(\frac{\partial W^m}{\partial y^m}) &= \alpha[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2-s^2)H_{222}]s_{y^i}s_{y^j}\\ &+ [(n+1)E_2 + 2H + (b^2-s^2)H_{22}](\alpha_{y^i}s_{y^j} + \alpha_{y^j}s_{y^i})\\ &+ \alpha[(n+1)E_2 + 2H + (b^2-s^2)H_{22}]s_{y^iy^j}\\ &+ [(n+1)E + 2sH + (b^2-s^2)H_2]\alpha_{y^iy^j}.\end{aligned} \tag{23}$$

Plugging (19), (20) and (21) into (23) and using Maple program, we obtain

$$\begin{aligned}\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}(\frac{\partial W^m}{\partial y^m}) &= \frac{1}{\alpha}[(n+1)E_{22} + 2(H_2 - sH_{22}) + (b^2-s^2)H_{222}]b_ib_j\\ &- \frac{s}{\alpha^2}[(n+1)E_{22} + 2(H_2 - sH_{22})\\ &+ (b^2-s^2)H_{222}](b_iy_j + b_jy_i)\\ &+ \frac{1}{\alpha^3}[(n+1)s^2E_{22} - (n+1)(E - sE_2) + s^2(b^2-s^2)H_{222}\\ &+ (3s^2-b^2)(H_2 - sH_{22})]y_iy_j\\ &+ \frac{1}{\alpha}[(n+1)(E - sE_2) + (b^2-s^2)(H_2 - sH_{22})]a_{ij}.\end{aligned} \tag{24}$$

It follows from $G^i_\alpha(x,y) = \frac{1}{2}\Gamma^i_{jk}y^jy^k$ that

$$\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}(\frac{\partial G^m_\alpha}{\partial y^m}) = 0. \tag{25}$$

By (5), (15), (16), (24) and (25), we obtain (14). □

3.1. Proof of Theorem 1.1

For a general (α,β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, where β is a closed and conformal 1-form, a direct computation yields

$$F_{y^i} = \alpha_{y^i}\phi + \alpha\phi_2 s_{y^i}, \tag{26}$$

$$F_{y^iy^j} = \alpha_{y^iy^j}\phi + (\alpha_{y^i}s_{y^j} + \alpha_{y^j}s_{y^i})\phi_2 + \alpha\phi_{22}s_{y^j}s_{y^i} + \alpha\phi_2 s_{y^iy^j}. \tag{27}$$

Plugging (19), (20) and (21) into (27), we obtain

$$\begin{aligned}F_{y^iy^j} &= \frac{1}{\alpha}(\phi - s\phi_2)a_{ij} - \frac{s\phi_{22}}{\alpha^2}(b_jy_i + b_iy_j) + \frac{\phi_{22}}{\alpha}b_ib_j\\ &- \frac{1}{\alpha^3}(\phi - s\phi_2 - s^2\phi_{22})y_iy_j.\end{aligned} \tag{28}$$

From (6), we have

$$\frac{\partial}{\partial y^j}\frac{\partial}{\partial y^i}\Big(\frac{\partial G^m}{\partial y^m}\Big) = (n+1)kF_{y^iy^j}. \tag{29}$$

Suppose F is of isotropic E-curvature. By (14) and (28), (29), we obtain

$$\frac{1}{\alpha^3}(A_{ij}\alpha^2 + B_{ij}\alpha + C_{ij}) = 0, \tag{30}$$

where

$$\begin{aligned}
A_{ij} := {} & \{(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\}b_ib_j \\
& + \{(n+1)[(E - sE_2) - \rho(x)(\phi - s\phi_2)] \\
& + (b^2 - s^2)(H_2 - sH_{22})\}a_{ij}, \\
B_{ij} := {} & -s\{(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) \\
& + (b^2 - s^2)H_{222}\}(b_iy_j + b_jy_i), \\
C_{ij} := {} & \{(n+1)[(s^2E_{22} - E + sE_2) + \rho(x)(\phi - s\phi_2 - s^2\phi_{22})] \\
& + s^2(b^2 - s^2)H_{222} + (3s^2 - b^2)(H_2 - sH_{22})\}y_iy_j.
\end{aligned}$$

From (30), we conclude that

$$\begin{aligned}
& A_{ij}\alpha^2 + C_{ij} = 0, \\
& B_{ij} = 0.
\end{aligned}$$

For $s \neq 0$, from $(A_{ij}\alpha^2 + C_{ij})y^iy^j = 0$, we have

$$\begin{aligned}
& \{(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\}\alpha^4s^2 \\
& + \{(n+1)[(E - sE_2) - \rho(x)(\phi - s\phi_2)] + (b^2 - s^2)(H_2 - sH_{22})\}\alpha^4 \\
& + \{(n+1)[(s^2E_{22} - E + sE_2) + \rho(x)(\phi - s\phi_2 - s^2\phi_{22})] \\
& + s^2(b^2 - s^2)H_{222} + (3s^2 - b^2)(H_2 - sH_{22})\}\alpha^4 = 0.
\end{aligned}$$

Simplifying this, yields

$$2[(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}]\alpha^4s^2 = 0. \tag{31}$$

Thus

$$(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222} = 0. \tag{32}$$

On the other hand, from $(A_{ij}\alpha^2 + C_{ij})b^ib^j = 0$, we have

$$\begin{aligned}
& \{(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\}\alpha^2b^4 \\
& + \{(n+1)[(E - sE_2) - \rho(x)(\phi - s\phi_2)] + (b^2 - s^2)(H_2 - sH_{22})\}\alpha^2b^2 \\
& + \{(n+1)[(s^2E_{22} - E + sE_2) + \rho(x)(\phi - s\phi_2 - s^2\phi_{22})] \\
& + s^2(b^2 - s^2)H_{222} + (3s^2 - b^2)(H_2 - sH_{22})\}\beta^2 = 0.
\end{aligned}$$

By considering (32), one can see that

$$[(n+1)(E - sE_2) - \rho(x)(\phi - s\phi_2) + (b^2 - s^2)(H_2 - sH_{22})](b^2\alpha^2 - \beta^2) = 0. \tag{33}$$

Thus

$$(n+1)(E - sE_2) - \rho(x)(\phi - s\phi_2) + (b^2 - s^2)(H_2 - sH_{22}) = 0. \tag{34}$$

From $B_{ij}y^iy^j = 0$, we have

$$2s\{(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}\}\alpha^2\beta = 0. \tag{35}$$

Hence, it is easy to see from (35) that

$$(n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222} = 0. \tag{36}$$

Note that

$$\begin{aligned} &[(n+1)(E - sE_2) - \rho(x)(\phi - s\phi_2) + (b^2 - s^2)(H_2 - sH_{22})]_2 \\ &= (n+1)(E_{22} - \rho(x)\phi_{22}) + 2(H_2 - sH_{22}) + (b^2 - s^2)H_{222}. \end{aligned}$$

Therefore, (34) implies that (32) and (36) hold. Thus, if a general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is of isotropic E-curvature, then (34) holds. Conversely, if F satisfies (34), then (29) holds, namely F is of isotropic E-curvature.

Corollary 3.2. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies (1). Then F is of vanishing E-curvature if and only if*

$$(n+1)(E - sE_2) + (b^2 - s^2)(H_2 - sH_{22}) = 0. \tag{37}$$

3.2. Proof of Corollary 1.2

Suppose that a projectively flat general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ has isotropic E-curvature and β satisfies (1), then $H = 0$, (2) can be written as:

$$E - sE_2 = \rho(x)(\phi - s\phi_2). \tag{38}$$

By solving (38), we get

$$E = \theta s + \rho(x)\phi, \tag{39}$$

where $\theta = \theta(b^2)$ is a scalar function on M. From (12), we know

$$E = \frac{1}{2\phi}(\phi_2 + 2s\phi_1). \tag{40}$$

Thus if the projectively flat general (α, β)-metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ has isotropic E-curvature, ϕ satisfies

$$\frac{1}{2\phi}(\phi_2 + 2s\phi_1) = \theta s + \rho(x)\phi, \tag{41}$$

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = 0. \tag{42}$$

Differentiating (41) with respect to s, we get

$$2\phi_1 + 2s\phi_{12} + \phi_{22} = 2\phi\theta + 2\phi_2\theta s + 4\phi\phi_2\rho(x). \tag{43}$$

Plugging (42) into (43), we know

$$2\phi_1 = \phi\theta + \phi_2\theta s + 2\phi\phi_2\rho(x). \tag{44}$$

Multiplying (44) by s and subtract with (41), we have

$$(\theta s^2 + 2\phi\rho(x)s + 1)\phi_2 = \phi\theta s + 2\phi^2\rho(x). \tag{45}$$

For a fixed b^2, (45) is equivalent to the following equation

$$Xd\phi + Yds = 0, \tag{46}$$

where $X = \theta s^2 + 2\phi\rho(x)s + 1$ and $Y = -\phi\theta s - 2\phi^2\rho(x)$. By a direct computation,

$$\frac{\partial X}{\partial s} = 2\theta s + 2\phi\rho(x), \quad \frac{\partial Y}{\partial \phi} = -\theta s - 4\phi\rho(x). \tag{47}$$

Thus

$$\frac{1}{Y}\left(\frac{\partial X}{\partial s} - \frac{\partial Y}{\partial \phi}\right) = -\frac{3}{\phi}. \tag{48}$$

By (48), the integrating factor $u(\phi)$ of (46) can be easily obtained,

$$u(\phi) = \frac{1}{\phi^3}. \tag{49}$$

Multiplying (46) by $u(\phi)$, yields

$$\frac{1}{\phi^3}(\theta s^2 + 2\phi\rho(x)s + 1)d\phi - \frac{1}{\phi^3}(\phi\theta s + 2\phi^2\rho(x))ds = 0. \tag{50}$$

So

$$d\left(\frac{1}{\phi^2}\theta s^2 + \frac{4}{\phi}\rho(x)s + \frac{1}{2\phi^2}\right) = 0,$$

suppose that $\mathcal{X}(b^2) = \frac{1}{\phi^2}\theta s^2 + \frac{4}{\phi}\rho(x)s + \frac{1}{2\phi^2}$, we obtain

$$\phi^2\mathcal{X}(b^2) - 4\phi\rho(x)s - (\theta s^2 + \frac{1}{2}) = 0. \tag{51}$$

Thus

$$\phi(b^2, s) = \frac{2\rho(x)s \pm \sqrt{(4\rho^2(x) + \mathcal{X}(b^2)\theta)s^2 + \frac{1}{2}\mathcal{X}(b^2)}}{\mathcal{X}(b^2)}. \tag{52}$$

Due to $F \geq 0$, we have

$$\phi(b^2, s) = \frac{2\rho(x)s + \sqrt{(4\rho^2(x) + \mathcal{X}(b^2)\theta)s^2 + \frac{1}{2}\mathcal{X}(b^2)}}{\mathcal{X}(b^2)}. \tag{53}$$

It follows that

$$F = \frac{2\rho(x)\beta + \sqrt{(4\rho^2(x) + \mathcal{X}(b^2)\theta)\beta^2 + \frac{1}{2}\mathcal{X}(b^2)\alpha^2}}{\mathcal{X}(b^2)}. \tag{54}$$

This means F is a Randers metric. Conversely, if F satisfies (54), then (29) holds, namely $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ has isotropic E-curvature. The proof of corollary is completed.

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