J. Korean Math. Soc. **55** (2018), No. 2, pp. 391–413
https://doi.org/10.4134/JKMS.j170252
pISSN: 0304-9914 / eISSN: 2234-3008

ON PSEUDO SEMI-PROJECTIVE SYMMETRIC MANIFOLDS

Uday Chand De and Pradip Majhi

Abstract. In this paper we introduce a new tensor named semi-projective curvature tensor which generalizes the projective curvature tensor. First we deduce some basic geometric properties of semi-projective curvature tensor. Then we study pseudo semi-projective symmetric manifolds $(PSPS)_n$ which recover some known results of Chaki [5]. We provide several interesting results. Among others we prove that in a $(PSPS)_n$ if the associated vector field ρ is a unit parallel vector field, then either the manifold reduces to a pseudosymmetric manifold or pseudo projective symmetric manifold. Moreover we deal with semi-projectively flat perfect fluid and dust fluid spacetimes respectively. As a consequence we obtain some important theorems. Next we consider the decomposability of $(PSPS)_n$. Finally, we construct a non-trivial Lorentzian metric of $(PSPS)_4$.

1. Introduction

It is well known that symmetric spaces play an important role in differential geometry. The study of locally symmetric Riemannian spaces was initiated in the late twenties by Cartan [4], who, in particular, obtained a classification of those spaces. Let (M^n, g) be a Riemannian manifold, i.e., a manifold M with the Riemannian metric g, and let ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric [4] if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). This condition of local symmetry is equivalent to the fact that at every point $P \in M$, the local geodesic symmetry $F(P)$ is an isometry [25]. The class of locally symmetric Riemannian manifolds is very natural generalization of the class of manifolds of constant curvature. During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways.

Received April 17, 2017; Revised December 5, 2017; Accepted December 29, 2017.
2010 *Mathematics Subject Classification.* 53C25, 53C35, 53C50, 53B30.
Key words and phrases. semi-projective curvature tensor, $(PSPS)_n$, decomposable of $(PSPS)_n$, dust fluid, energy-momentum tensor, perfect fluid, Einstein's field equation, pseudo semi-projective symmetric manifolds.

A non-flat pseudo-Riemannian manifold $(M^n, g), (n > 2)$ is said to be a pseudosymmetric [5] if its curvature tensor R of type $(0, 4)$ satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) = 2A(X)R(Y, Z, U, V) &+ A(Y)R(X, Z, U, V)\\ &+ A(Z)R(Y, X, U, V) + A(U)R(Y, Z, X, V)\\ &+ A(V)R(Y, Z, U, X),\end{aligned}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X, \rho) = A(X)$$

for all X, ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and $R(Y, Z, U, V) = g(\mathcal{R}(Y, Z)U, V)$, where $\mathcal{R}$ is the curvature tensor of type $(1, 3)$. The 1-form A is called the associated 1-form of the manifold. If $A = 0$, then the manifold reduces to a locally symmetric manifold in the sense of Cartan. An n-dimensional pseudosymmetric manifold is denoted by $(PS)_n$.

Gray [12] introduced the notion of cyclic parallel Ricci tensor and Codazzi type of Ricci tensor. A pseudo-Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{1.1}$$

Again a pseudo-Riemannian manifold is said to satisfy Codazzi type of Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfy the following condition

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z). \tag{1.2}$$

General relativity flows from the Einstein equation which implies that the energy-momentum tensor is of vanishing divergence. This requirement of the energy-momentum tensor is satisfied if this tensor is covariant constant, that is, $\nabla T = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. In the general theory of relativity, energy-momentum tensor plays an important role and the condition on energy-momentum tensor for a perfect fluid spacetime changes the nature of spacetime [28]. In a recent paper [6] Chaki and Roy studied general relavistic spacetime with covariant constant energy-momentum tensor. Recently, De and Velimirović [9] studied spacetimes with semisymmetric energy-momentum tensor.

The spacetime of general relativity and cosmology is regarded as a connected 4-dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of Lorentz manifold begins with the study of causal character of vectors of the manifold. It is due to this causality that Lorentz manifold becomes a convenient choice for the study of general relativity. Indeed by basing its study on Lorentzian manifold the general theory of relativity opens the way to the study of global questions about it ([2],

[8], [11], [13], [14]) and many others. Also several authors studied spacetimes in different way such as ([9], [15], [19], [29]) and many others.

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y). \tag{1.3}$$

The equation (1.3) of Einstein imply that matter determines the geometry of spacetime and conversely that the motion of matter is determined by the metric tensor of the space which is not flat.

In general relativity the matter content of the spacetime is described by the energy-momentum tensor. The matter content is assumed to be fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion.

In a perfect fluid spacetime, the energy-momentum tensor T of type $(0,2)$ is of the form ([25]):

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{1.4}$$

where σ and p are the energy density and the isotropic pressure respectively. The velocity vector field ρ metrically equivalent to the non-zero 1-form A is a time-like vector, that is, $g(\rho,\rho) = -1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity [13]. In addition, p and σ are related by an equation of state governing the particular sort of perfect fluid under consideration. In general, this is an equation of the form $p = p(\sigma, T_0)$, where T_0 is the absolute temperature. However, we shall only be concerned with situations in which T_0 is effectively constant so that the equation of state reduces to $p = p(\sigma)$. In this case, the perfect fluid is called isentropic [13]. Moreover, if $p = \sigma$, then the perfect fluid is termed as stiff matter (see [28], page 66).

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let M be an n-dimensional pseudo-Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in Euclidean space such that any geodesic of the pseudo-Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known Projective curvature tensor W vanishes.

Projective curvature tensor W in a pseudo-Riemannian manifold (M^n, g) $(n \geq 2)$ is defined by [25]

$$W(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.5}$$

where $\mathcal{R}$ is the Riemannian curvature tensor of type $(1,3)$ and S is the Ricci tensor of type $(0,2)$.

In 2012, Mantica and Molinari [16] defined a generalized $(0,2)$ symmetric $\mathcal{Z}$ tensor given by

$$\mathcal{Z}(X,Y) = S(X,Y) + \phi g(X,Y), \tag{1.6}$$

where ϕ is an arbitrary scalar function. In Refs. ([19], [20] and [21]) various properties of the $\mathcal{Z}$ tensor were pointed out.

Subsequently in 2013 Mantica and Suh [22] introduced a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold $(M^n, g)(n > 2)$ denoted by $\mathcal{Q}$ and defined by

$$\mathcal{Q}(X,Y)W = \mathcal{R}(X,Y)W - \frac{\psi}{(n-1)}[g(Y,W)X - g(X,W)Y], \tag{1.7}$$

where ψ is an arbitrary scalar function. Such a tensor $\mathcal{Q}$ is known as Q-curvature tensor. The notion of Q-tensor is also suitable to reinterpret some differential structures on a Riemannian manifold.

Motivated by the above studies in the present paper we define semi-projective curvature tensor $\mathcal{P}$ of type $(1,3)$ as follows:

$$\mathcal{P}(X,Y)U = \mathcal{R}(X,Y)U - \frac{\phi}{n-1}[S(Y,U)X - S(X,U)Y], \tag{1.8}$$

where ϕ is an arbitrary scalar function. We prefer the name 'semi-projective curvature tensor', since it is clear that for $\phi = 1$, semi-projective curvature tensor reduces to projective curvature tensor. If $\phi = 0$, then semi-projective curvature tensor and curvature tensor are equivalent. We can express (1.8) as follows:

$$\begin{aligned} P(X,Y,U,V) = R(X,Y,U,V) - \frac{\phi}{(n-1)}[S(Y,U)g(X,V) \\ - S(X,U)g(Y,V)], \end{aligned} \tag{1.9}$$

where $P(X,Y,U,V) = g(\mathcal{P}(X,Y)U,V)$ and $R(X,Y,U,V) = g(\mathcal{R}(X,Y)U,V)$. A non-flat pseudo-Riemannian manifold (M^n, g), $(n > 2)$ is said to be a pseudo semi-projective symmetric manifold if the semi-projective curvature tensor P of type $(0,4)$ satisfies the condition

$$\begin{aligned} (\nabla_X P)(Y,Z,U,V) = 2A(X)P(Y,Z,U,V) + A(Y)P(X,Z,U,V) \\ + A(Z)P(Y,X,U,V) + A(U)P(Y,Z,X,V) \\ + A(V)P(Y,Z,U,X), \end{aligned} \tag{1.10}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X).$$

An n-dimensional pseudo semi-projective symmetric manifold is denoted by $(PSPS)_n$, where P stands for pseudo, SP stands for semi-projective and S stands for symmetric.

If $\phi = 0$, then pseudo semi-projective symmetric manifold reduces to pseudosymmetric manifolds introduced by Chaki [5]. Moreover if $\phi = 1$, then pseudo semi-projective symmetric manifold includes pseudo projective symmetric manifolds $(PWS)_n$ introduced by Chaki et al. [7]. The present paper is organized as follows:

After introduction in Section 2, we study some basic geometric properties of semi-projective curvature. Section 3 is devoted to study of curvature property of $(PSPS)_n$. In Section 4, we study $(PSPS)_n$ admitting Codazzi type Ricci tensor. Sections 5 and 6 deal with Einstein $(PSPS)_n$ and $(PSPS)_n$ with $div\mathcal{P} = 0$ respectively. Section 7 is devoted to study of $(PSPS)_n$ admitting parallel vector field ρ. Among others we prove that in a $(PSPS)_n$ if the associated vector field ρ is a unit parallel vector field, then either the manifold reduces to a pseudosymmetric manifold or pseudoprojective symmetric manifold. Next in Section 8 we consider the decomposability of $(PSPS)_n$. Section 9 deals with semi-projectively flat spacetimes. Moreover in Sections 10 and 11 we consider semi-projectively flat perfect fluid and dust fluid spacetimes respectively. As a consequence we obtain some important theorems. Finally, we construct a non-trivial Lorentzian metric of $(PSPS)_4$.

2. Preliminaries

Let S and r denote the Ricci tensor of type (0,2) and the scalar curvature respectively and L denote the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is,

$$g(LX, Y) = S(X, Y). \tag{2.1}$$

In this section, some formulas are derived, which will be useful to the study of $(PSPS)_n$. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold where $1 \leq i \leq n$. In a pseudo Riemannian manifold the Ricci tensor S is defined by $S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,e_i)e_i, Y)$, where $\epsilon_i = g(e_i, e_i) = \pm 1$.

From (1.8) we can easily verify that the tensor $\mathcal{P}$ satisfies the following properties:

$$\begin{aligned} &\text{i)} \quad \mathcal{P}(X,Y)U = -\mathcal{P}(Y,X)U, \\ &\text{ii)} \quad \mathcal{P}(X,Y)U + \mathcal{P}(Y,U)X + \mathcal{P}(U,X)Y = 0. \end{aligned} \tag{2.2}$$

Also from (1.8) we have

$$\sum_{i=1}^{n} \epsilon_i P(X, Y, e_i, e_i) = 0 = \sum_{i=1}^{n} \epsilon_i P(e_i, e_i, U, V), \tag{2.3}$$

$$\begin{aligned} &\sum_{i=1}^{n} \epsilon_i P(e_i, Y, U, e_i) = (1-\phi)S(Y,U) = P_1(Y,U) \text{ (say)}, \\ &\sum_{i=1}^{n} \epsilon_i P(X, e_i, e_i, V) = (1 + \frac{\phi}{n-1})S(X,V) - \frac{r\phi}{n-1} g(X,V), \end{aligned} \tag{2.4}$$

$$= P_2(X,V), \text{ (say)} \tag{2.5}$$

where $r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i)$ is the scalar curvature and $\epsilon_i = g(e_i, e_i) = \pm 1$.

From (1.8) and (2.2) it follows that

$$\begin{aligned} &\text{(i)} \quad P(X,Y,U,V) = -P(Y,X,U,V),\\ &\text{(ii)} \quad P(X,Y,U,V) \neq P(X,Y,V,U),\\ &\text{(iii)} \quad P(X,Y,U,V) \neq P(U,V,X,Y),\\ &\text{(iv)} \quad P(X,Y,U,V) + P(Y,U,X,V) + P(U,X,Y,V) = 0, \end{aligned} \tag{2.6}$$

where $P(X,Y,U,V) = g(\mathcal{P}(X,Y)U,V)$.

Proposition 2.1. *A pseudo-Riemannian manifold is semi-projectively flat if and only if it is of constant curvature provided the scalar curvature is non-zero.*

Proof. The semi-projective curvature tensor is given by

$$\begin{aligned} P(X,Y,U,V) = R(X,Y,U,V) - \frac{\phi}{(n-1)}[&S(Y,U)g(X,V)\\ &-S(X,U)g(Y,V)], \end{aligned} \tag{2.7}$$

where ϕ is an arbitrary scalar function. If semi-projective curvature tensor vanishes, then

$$R(X,Y,U,V) = \frac{\phi}{(n-1)}[S(Y,U)g(X,V) - S(X,U)g(Y,V)]. \tag{2.8}$$

Taking a frame field and contracting Y and U in (2.8), we have

$$S(X,V) = \frac{r\phi}{(n-1+\phi)} g(X,V). \tag{2.9}$$

Again contracting X and V in (2.9), we get

$$r(n-1)(\phi - 1) = 0. \tag{2.10}$$

Therefore either $r = 0$ or $\phi = 1$. For $r = 0$, the semi-projective curvature tensor $\mathcal{P}$ is equivalent to the curvature tensor $\mathcal{R}$. Also for $\phi = 1$, the semi-projective curvature tensor $\mathcal{P}$ is equivalent to the projective curvature tensor W. Consequently semi-projectively flatness and projectively flatness are equivalent.

It is known that [25] a pseudo-Riemannian manifold is projectively flat if and only if it is space of constant curvature. Therefore a pseudo-Riemannian manifold is semi-projectively flat if and only if it is a manifold of constant curvature provided the scalar curvature is non-zero. This completes the proof. □

Proposition 2.2. *If the semi-projective curvature tensor is symmetric in the sense of Cartan, then the manifold reduces to a Ricci recurrent manifold.*

Proof. The semi-projective curvature tensor is given by

$$\mathcal{P}(Y,Z)U = \mathcal{R}(Y,Z)U - \frac{\phi}{(n-1)}[S(Z,U)Y - S(Y,U)Z], \tag{2.11}$$

where ϕ is an arbitrary scalar function. Differentiating (2.11) covariantly we get

$$(\nabla_X \mathcal{P})(Y,Z)U = (\nabla_X \mathcal{R})(Y,Z)U - \frac{d\phi(X)}{(n-1)}[S(Z,U)Y - S(Y,U)Z] - \frac{\phi}{(n-1)}[(\nabla_X S)(Z,U)Y - (\nabla_X S)(Y,U)Z]. \tag{2.12}$$

By hypothesis, semi-projective curvature tensor is symmetric. Hence from the above equation

$$(\nabla_X \mathcal{R})(Y,Z)U = \frac{d\phi(X)}{(n-1)}[S(Z,U)Y - S(Y,U)Z] + \frac{\phi}{(n-1)}[(\nabla_X S)(Z,U)Y - (\nabla_X S)(Y,U)Z]. \tag{2.13}$$

Contracting Y in (2.13) we get

$$(\nabla_X S)(Z,U) = \frac{d\phi(X)}{(n-1)}(n-1)S(Z,U) + \frac{\phi}{(n-1)}(n-1)(\nabla_X S)(Z,U). \tag{2.14}$$

This implies

$$(1-\phi)(\nabla_X S)(Z,U) = d\phi(X)S(Z,U). \tag{2.15}$$

Again contracting Z and U in the above equation we get

$$(1-\phi)dr(X) = d\phi(X)r. \tag{2.16}$$

It follows that

$$(X \log r) = \frac{d\phi(X)}{1-\phi}. \tag{2.17}$$

From (2.15) and (2.17) it follows that

$$(\nabla_X S)(Z,U) = (X \log r)S(Z,U). \tag{2.18}$$

This completes the proof. □

Proposition 2.3. *For a semi-projective curvature tensor* $div\mathcal{P} = 0$ *and* $divW = 0$ *are equivalent provided* ϕ *is constant.*

Proof. The semi-projective curvature tensor is given by

$$\mathcal{P}(Y,Z)U = \mathcal{R}(Y,Z)U - \frac{\phi}{(n-1)}[S(Z,U)Y - S(Y,U)Z], \tag{2.19}$$

where ϕ is an arbitrary scalar function. Differentiating (2.11) covariantly we get

$$(\nabla_X \mathcal{P})(Y,Z)U = (\nabla_X \mathcal{R})(Y,Z)U - \frac{d\phi(X)}{(n-1)}[S(Z,U)Y - S(Y,U)Z]$$

$$-\frac{\phi}{(n-1)}[(\nabla_X S)(Z,U)Y - (\nabla_X S)(Y,U)Z]. \tag{2.20}$$

It follows that

$$(div\mathcal{P})(Y,Z)U = (div\mathcal{R})(Y,Z)U - \frac{1}{(n-1)}[S(Z,U)(Y\phi) - S(Y,U)(Z\phi)]$$
$$-\frac{\phi}{(n-1)}[(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)]. \tag{2.21}$$

Making use of $(div\mathcal{R})(Y,Z)U = (\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)$, the above equation implies

$$(div\mathcal{P})(Y,Z)U = (1 - \frac{\phi}{(n-1)})[(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)]$$
$$-\frac{1}{(n-1)}[S(Z,U)(Y\phi) - S(Y,U)(Z\phi)]. \tag{2.22}$$

Now if ϕ is constant, then the above equation reduces to

$$(div\mathcal{P})(Y,Z)U = (1 - \frac{\phi}{(n-1)})[(\nabla_Y S)(Z,U) - (\nabla_Z S)(Y,U)]. \tag{2.23}$$

This implies

$$(div\mathcal{P})(Y,Z)U = \frac{(n-1-\phi)}{(n-2)}(divW)(Y,Z)U. \tag{2.24}$$

This completes the proof. □

3. Some curvature properties of $(PSPS)_n (n > 2)$

In this section we prove that in a $(PSPS)_n(n > 2)$, the semi-projective curvature tensor $P(Y,Z,U,V)$ satisfies Bianchi's 2nd identity, that is,

$$(\nabla_X P)(Y,Z,U,V) + (\nabla_Y P)(Z,X,U,V) + (\nabla_Z P)(X,Y,U,V) = 0. \tag{3.1}$$

In view of (1.9), (1.10) and (3.1) we get

$$(\nabla_X P)(Y,Z,U,V) + (\nabla_Y P)(Z,X,U,V) + (\nabla_Z P)(X,Y,U,V)$$
$$= A(U)[P(Y,Z,X,V) + P(Z,X,Y,V) + P(X,Y,Z,V)]$$
$$+ A(V)[P(Y,Z,U,X) + P(Z,X,U,Y) + P(X,Y,U,Z)]. \tag{3.2}$$

Using Bianchi's 1st identity (2.6) in (3.2) we have

$$(\nabla_X P)(Y,Z,U,V) + (\nabla_Y P)(Z,X,U,V) + (\nabla_Z P)(X,Y,U,V)$$
$$= A(V)[S(Z,X)g(Y,V) - S(Y,X)g(Z,V) + S(X,Y)g(Z,V)$$
$$- S(Z,Y)g(X,V) + S(Y,Z)g(X,V) - S(X,Z)g(Y,V)](-\frac{\phi}{n-1})$$
$$= 0. \tag{3.3}$$

Thus we can state the following:

Theorem 3.1. *The semi-projective curvature tensor in* $(PSPS)_n(n>2)$ *satisfies Bianchi's 2nd identity.*

4. A $(PSPS)_n(n>2)$ admitting Codazzi type of Ricci tensor

In view of (1.9) we have

$$
\begin{aligned}
&(\nabla_X P)(Y,Z,U,V)+(\nabla_Y P)(Z,X,U,V)+(\nabla_Z P)(X,Y,U,V)\\
=&-\frac{\phi}{n-1}[(\nabla_X S)(Z,U)g(Y,V)-(\nabla_X S)(Y,U)g(Z,V)\\
&+(\nabla_Y S)(X,U)g(Z,V)-(\nabla_Y S)(Z,U)g(X,V)\\
&+(\nabla_Z S)(Y,U)g(X,V)-(\nabla_Z S)(X,U)g(Y,V)\\
&-\frac{(X\phi)}{n-1}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)]\\
&-\frac{(Y\phi)}{n-1}[S(X,U)g(Z,V)-S(Z,U)g(X,V)]\\
&-\frac{(Z\phi)}{n-1}[S(Y,U)g(X,V)-S(X,U)g(Y,V)]. \qquad (4.1)
\end{aligned}
$$

Assume that $(PSPS)_n$ admits Codazzi type of Ricci tensor, then from (4.1) we have

$$
\begin{aligned}
&(\nabla_X P)(Y,Z,U,V)+(\nabla_Y P)(Z,X,U,V)+(\nabla_Z P)(X,Y,U,V)\\
=&-\frac{(X\phi)}{n-1}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)]\\
&-\frac{(Y\phi)}{n-1}[S(X,U)g(Z,V)-S(Z,U)g(X,V)]\\
&-\frac{(Z\phi)}{n-1}[S(Y,U)g(X,V)-S(X,U)g(Y,V)]. \qquad (4.2)
\end{aligned}
$$

Using (3.3) in (4.2) we have

$$
\begin{aligned}
&-\frac{(X\phi)}{n-1}[S(Z,U)g(Y,V)-S(Y,U)g(Z,V)]\\
&-\frac{(Y\phi)}{n-1}[S(X,U)g(Z,V)-S(Z,U)g(X,V)]\\
&-\frac{(Z\phi)}{n-1}[S(Y,U)g(X,V)-S(X,U)g(Y,V)]=0. \qquad (4.3)
\end{aligned}
$$

Contracting Y and V in (4.3) yields

$$(X\phi)S(Z,U)=(Z\phi)S(X,U). \qquad (4.4)$$

Again contracting Z and U in (4.4) yields

$$(X\phi)r=g(LX,grad\phi). \qquad (4.5)$$

It follows that

$$S(grad\phi,X)=rg(grad\phi,X) \qquad (4.6)$$

for all X. This implies that r is an eigenvalue of S corresponding to the eigenvector $grad\phi$. Thus we conclude the following:

Theorem 4.1. *For a $(PSPS)_n$ admitting Codazzi type of Ricci tensor, r is an eigenvalue of the Ricci tensor S corresponding to the eigenvector $grad\phi$.*

If ϕ is constant, then from (4.1) and Bianchi's 2nd identity yields

$$\begin{aligned}&(\nabla_X S)(Z,U)g(Y,V) - (\nabla_X S)(Y,U)g(Z,V)\\&+(\nabla_Y S)(X,U)g(Z,V) - (\nabla_Y S)(Z,U)g(X,V)\\&+(\nabla_Z S)(Y,U)g(X,V) - (\nabla_Z S)(X,U)g(Y,V) = 0.\end{aligned} \tag{4.7}$$

Contracting Y and V in (4.7) yields

$$(\nabla_X S)(Z,U) = (\nabla_Z S)(X,U). \tag{4.8}$$

Therefore we are in a position to state the following:

Corollary 4.1. *In a $(PSPS)_n$ the Ricci tensor is of Codazzi type provided ϕ is constant.*

Again

$$P_1(Z,U) = (1-\phi)S(Z,U). \tag{4.9}$$

Contracting Z and U we have

$$p_1 = (1-\phi)r. \tag{4.10}$$

In $(PSPS)_n$, the semi-projective curvature tensor satisfies the following:

$$\begin{aligned}(\nabla_X P)(Y,Z,U,V) = {}& 2A(X)P(Y,Z,U,V) + A(Y)P(X,Z,U,V)\\&+ A(Z)P(Y,X,U,V) + A(U)P(Y,Z,X,V)\\&+ A(V)P(Y,Z,U,X),\end{aligned} \tag{4.11}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X).$$

Contracting Y and V in (4.11) we have

$$\begin{aligned}(\nabla_X P_1)(Z,U) = {}& 2A(X)P_1(Z,U) + P(X,Z,U,\rho)\\&+ A(Z)P_1(X,U) + A(U)P_1(Z,X) + P(\rho,Z,U,X).\end{aligned} \tag{4.12}$$

Again contracting Z and U in (4.12) we have

$$\nabla_X p_1 = 2A(X)p_1 + 2P_1(X,\rho) + 2P_2(X,\rho). \tag{4.13}$$

Therefore (4.10) and (4.13) yields

$$(1-\phi)dr(X) - d\phi(X)r = 2A(X)(1-\phi)r + 2P_1(X,\rho) + 2P_2(X,\rho). \tag{4.14}$$

Again using (4.9) and (2.5) in (4.14) we have

$$(1-\phi)dr(X) - d\phi(X)r = [2(1-\phi)r - \frac{r\phi}{n-1}]A(X) + 2[2-\phi+\frac{\phi}{n-1}]S(X,\rho). \tag{4.15}$$

It follows that
(4.16)

$$(1-\phi)dr(X) - d\phi(X)r = [2(1-\phi)r - \frac{r\phi}{n-1}]A(X) + 2[2-\phi+\frac{\phi}{n-1}]A(LX).$$

Thus we can state the following:

Theorem 4.2. *In a* $(PSPS)_n(n>2)$ *the following identity holds*:

$$(1-\phi)dr(X) - d\phi(X)r = [2(1-\phi)r - \frac{r\phi}{n-1}]A(X) + 2[2-\phi+\frac{\phi}{n-1}]A(LX).$$

In particular, let us assume that $\phi = 0$, then from Theorem 4.2 we have

$$dr(X) = 2A(X)r + 4A(LX). \tag{4.17}$$

Thus we recover the Chaki's result [5] as follows:

Corollary 4.2. *In a* $(PS)_n$ *the following identity holds*:

$$dr(X) = 2A(X)r + 4A(LX).$$

The above result was proved by Chaki [5].

5. Einstein $(PSPS)_n(n>2)$

In this section we consider Einstein $(PSPS)_n(n>2)$. Since for every Einstein manifold the scalar curvature r is constant, hence for Einstein $(PSPS)_n$ $(n>2)$ we have $dr(X)=0$. Therefore from Theorem 4.2 we have

$$-d\phi(X)r = 2A(X)(1-\phi)r + 4(1-\phi)A(LX). \tag{5.1}$$

Since in an Einstein manifold (M^n, g), we have $S(X,Y) = \frac{r}{n}g(X,Y)$, (5.1) can be written as

$$-d\phi(X)r = 2(1+\frac{1}{n})(1-\phi)A(X)r. \tag{5.2}$$

If ϕ is a constant, then $r=0$, as $A(X)\neq 0$. Therefore we can state the following:

Theorem 5.1. *An Einstein* $(PSPS)_n(n>2)$ *is of zero scalar curvature provided* ϕ *is constant.*

If possible, let $(PSPS)_n(n>2)$ be a space of constant curvature. Then we have

$$\mathcal{R}(X.Y)Z = k[g(Y,Z)X - g(X,Z)Y], \tag{5.3}$$

where k is a constant. Contracting X in (5.3), we have

$$S(Y,Z) = k(n-1)g(Y,Z). \tag{5.4}$$

Again contracting Y, Z in (5.4), we have

$$r = k(n-1)n. \tag{5.5}$$

Using (5.5) in (5.3) yields

$$\mathcal{R}(X,Y)Z = \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{5.6}$$

Since every space of constant curvature is an Einstein manifold, then from Theorem 5.1 we have $r = 0$. Hence from (5.6) it follows that $\mathcal{R}(X,Y)Z = 0$, which is in admissible by definitions. This leads to the following corollary of the above theorem under the assumption ϕ is constant:

Corollary 5.1. *A* $(PSPS)_n(n > 2)$ *can not be of constant curvature.*

6. $(PSPS)_n(n > 2)$ with $div\mathcal{P} = 0$

For $(PSPS)_n(n > 2)$, we have

$$\begin{aligned}(\nabla_X\mathcal{P})(Y,Z)U = {} & 2A(X)\mathcal{P}(Y,Z)U + A(Y)\mathcal{P}(X,Z)U \\ & + A(Z)\mathcal{P}(Y,X)U + A(U)\mathcal{P}(Y,Z)X \\ & + g(\mathcal{P}(Y,Z)U,X)\rho,\end{aligned} \tag{6.1}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X,\rho) = A(X).$$

Therefore

$$\begin{aligned}(div\mathcal{P})(Y,Z)U = {} & \sum_{i=1}^{n} g((\nabla_{e_i}\mathcal{P})(Y,Z)U,e_i), \\ = {} & \sum_{i=1}^{n}\{2A(e_i)g(\mathcal{P}(Y,Z)U,e_i) + A(Y)g(\mathcal{P}(e_i,Z)U,e_i) \\ & + A(Z)g(\mathcal{P}(Y,e_i)U,e_i) + A(U)g(\mathcal{P}(Y,Z)e_i,e_i) \\ & + g(\mathcal{P}(Y,Z)U,e_i)\}g(\rho,e_i)\end{aligned}$$

$$= 3A(\mathcal{P}(Y,Z)U) + A(Y)P_1(Z,U) - A(Z)P_1(Y,V). \tag{6.2}$$

Therefore $(div\mathcal{P})(Y,Z)U = 0$ implies

$$3A(\mathcal{P}(Y,Z)U) + A(Y)P_1(Z,U) - A(Z)P_1(Y,V) = 0. \tag{6.3}$$

Contracting Z and U we get

$$3P_2(Y,\rho) + (1-\phi)rA(Y) - P_1(Y,\rho) = 0. \tag{6.4}$$

Therefore using (2.4), (2.5) in (6.4) we have

$$S(Y,\rho) = \frac{[3\phi r - (n-1)(1-\phi)r]}{[3(n-1+\phi) - (n-1)(1-\phi)]}g(Y,\rho). \tag{6.5}$$

It follows that

$$S(Y,\rho) = \lambda g(Y,\rho), \tag{6.6}$$

where $\lambda = \frac{[3\phi r-(n-1)(1-\phi)r]}{[3(n-1+\phi)-(n-1)(1-\phi)]}$ a scalar. In view of the above results we are a position to state the following:

Theorem 6.1. *For a $(PSPS)_n (n > 2)$ with $div\mathcal{P} = 0$, λ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ.*

7. $(PSPS)_n (n > 2)$ admitting a parallel vector field

In this section we obtain a sufficient condition for a $(PSPS)_n$ to be a $(PS)_n$ or $(PWS)_n$. For this we require a notion of parallel vector field defined as follows:

A vector field V is said to parallel [10] if

$$\nabla_X V = 0. \tag{7.1}$$

We now suppose that a $(PSPS)_n$ admitting a unit parallel vector field ρ, that is,

$$\nabla_X \rho = 0. \tag{7.2}$$

Applying Ricci identity to (7.2) we have

$$\mathcal{R}(X, Y)\rho = 0. \tag{7.3}$$

Contracting Y in (7.3) we have

$$S(Y, \rho) = 0. \tag{7.4}$$

Therefore (2.4) and (7.4) yield

$$P_1(X, \rho) = (1 - \phi)S(X, \rho) = 0. \tag{7.5}$$

Again from definition of $(PSPS)_n$, we have

$$\begin{aligned}(\nabla_X P)(Y, Z, U, V) = 2A(X)P(Y, Z, U, V) &+ A(Y)P(X, Z, U, V)\\ &+ A(Z)P(Y, X, U, V) + A(U)P(Y, Z, X, V)\\ &+ A(V)P(Y, Z, U, X),\end{aligned} \tag{7.6}$$

where A is a non-zero 1-form, ρ is a vector field defined by

$$g(X, \rho) = A(X).$$

Therefore

$$\begin{aligned}(\nabla_X P_1)(Z, U) = &\sum_{i=1}^{n} (\nabla_X P)(e_i, Z, U, e_i)\\ = &\sum_{i=1}^{n} \{2A(X)P(e_i, Z, U, e_i) + A(e_i)P(X, Z, U, e_i)\\ &+ A(Z)P(e_i, X, U, e_i) + A(U)P(e_i, Z, X, e_i)\\ &+ A(e_i)P(e_i, Z, U, X)\},\\ = &\,2A(X)P_1(Z, U) + A(Z)P_1(X, U) + A(U)P_1(Z, X)\\ &+ P(X, Z, U, \rho) + P(\rho, Z, U, X).\end{aligned} \tag{7.7}$$

Therefore substituting $U = \rho$ in (7.7) we get

$$(\nabla_X P_1)(Z,\rho) = 2A(X)P_1(Z,\rho) + A(Z)P_1(X,\rho) + A(\rho)P_(Z,X) + P(X,Z,\rho,\rho) + P(\rho,Z,\rho,X). \tag{7.8}$$

Using $P(X,Z,\rho,\rho) = 0 = P(\rho,Z,\rho,X)$ in (7.8) we have

$$(\nabla_X P_1)(Z,\rho) = A(\rho)P_1(Z,X). \tag{7.9}$$

Using (7.5) in the above equation we get

$$P_1(Z,X) = 0. \tag{7.10}$$

It follows that

$$(1-\phi)S(Z,X) = 0. \tag{7.11}$$

Therefore either $\phi = 1$ or $S(X,Z) = 0$. For $\phi = 1$, $(PSPS)_n$ reduces to pseudo projective symmetric manifolds, that is, $(PWS)_n$. Also for $S = 0$, $(PSPS)_n$ reduces to pseudosymmetric manifolds, that is, $(PS)_n$. Therefore we can state the following:

Theorem 7.1. *In a $(PSPS)_n$ if the associated vector field ρ is a unit parallel vector field, then either the manifold reduces to a pseudosymmetric manifold or pseudo projective symmetric manifold.*

8. Decomposable $(PSPS)_n$

A pseudo-Riemannian manifold (M^n, g) is said to be decomposable or a product manifold ([10]) if it can be expressed as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighbourhood of the pseudo-Riemannian manifold (M^n, g), the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{8.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \ldots, x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, \ldots, x^n$ denoted by x^*; $a, b, c, \ldots$ run from 1 to p and $\alpha, \beta, \gamma, \ldots$ run from $p+1$ to n.

The two parts of (8.1) are the metrics of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

Let (M^n, g) be a pseudo-Riemannian manifold such that $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ are two components of this manifold. Here throughout this section each object denoted by a 'bar' is assumed to be from M_1 and each object denoted by 'star' is assumed to be from M_2.

Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$ and $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, the following relations hold [10]:

$R(X^*, \bar{Y}, \bar{Z}, \bar{U}) = 0 = R(\bar{X}, Y^*, \bar{Z}, U^*) = R(\bar{X}, Y^*, Z^*, U^*)$,
$(\nabla_{X^*} R)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} R)(\bar{Y}, Z^*, \bar{U}, V^*) = (\nabla_{X^*} R)(\bar{Y}, Z^*, \bar{U}, V^*)$,
$R(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{R}(\bar{X}, \bar{Y}, \bar{Z}, \bar{U})$; $R(X^*, Y^*, Z^*, U^*) = R^*(X^*, Y^*, Z^*, U^*)$,

$S(\bar{X},\bar{Y})=\bar{S}(\bar{X},\bar{Y}); S(X^*,Y^*)=S^*(X^*,Y^*)$,
$(\nabla_{\bar{X}}S)(\bar{Y},\bar{Z})=(\bar{\nabla}_{\bar{X}}S)(\bar{Y},\bar{Z}); (\nabla_{X^*}S)(Y^*,Z^*)=(\nabla^*_{X^*}S)(Y^*,Z^*)$,
and $r=\bar{r}+r^*$, where $r,\bar{r}$ and r^* are scalar curvatures of M, M_1 and M_2 respectively.

Let us consider a pseudo-Riemannian manifold (M^n,g), which is a decomposable $(PSPS)_n$.

Then $M^n=M_1^p\times M_2^{n-p}(2\le p\le n-2)$.

Now from (1.8), we get

$$\begin{aligned}P(Y^*,\bar{Z},\bar{U},\bar{V})=0&=P(\bar{Y},Z^*,U^*,V^*)\\&=P(\bar{Y},Z^*,\bar{U},\bar{V})=P(\bar{Y},\bar{Z},U^*,\bar{V}),\end{aligned}\tag{8.2}$$

$$P(Y^*,Z^*,\bar{U},\bar{V})=0=P(\bar{Y},\bar{Z},U^*,V^*),\tag{8.3}$$

and

$$P(Y^*,\bar{Z},U^*,\bar{V})=\frac{\phi}{(n-1)}[S(Y^*,U^*)g(\bar{Z},\bar{V})].\tag{8.4}$$

Again from (1.10), we get

$$\begin{aligned}(\nabla_{\bar{X}}\bar{P})(\bar{Y},\bar{Z},\bar{U},\bar{V})=2A(\bar{X})\bar{P}(\bar{Y},\bar{Z},\bar{U},\bar{V})&+A(\bar{Y})\bar{P}(\bar{X},\bar{Z},\bar{U},\bar{V})\\+A(\bar{Z})\bar{P}(\bar{Y},\bar{X},\bar{U},\bar{V})&+A(\bar{U})\bar{P}(\bar{Y},\bar{Z},\bar{X},\bar{V})\\+A(\bar{V})\bar{P}(\bar{Y},\bar{Z},\bar{U},\bar{X}).&\end{aligned}\tag{8.5}$$

Replacing $\bar{X}$ by X^* in (8.5) we get

$$2A(X^*)P(\bar{Y},\bar{Z},\bar{U},\bar{V})=0,\tag{8.6}$$

$$A(V^*)P(\bar{Y},\bar{Z},\bar{U},\bar{X})=0.\tag{8.7}$$

Similarly, we have

$$A(Y^*)P(\bar{X},\bar{Z},\bar{U},\bar{V})=0,\tag{8.8}$$

$$A(U^*)P(\bar{Y},\bar{Z},\bar{X},\bar{V})=0.\tag{8.9}$$

Now putting $\bar{X}=X^*$, $\bar{Y}=Y^*$ in (8.5) we get

$$\phi[A(\bar{U})S(Y^*,X^*)g(\bar{Z},\bar{V})-A(\bar{V})S(\bar{Z},\bar{U})g(Y^*,X^*)]=0.\tag{8.10}$$

Similarly, putting $\bar{X}=X^*$, $\bar{U}=U^*$ in (8.5) we obtain

$$\phi[A(\bar{Y})S(X^*,U^*)g(\bar{Z},\bar{V})-A(\bar{Z})S(X^*,U^*)g(\bar{Y},\bar{V})]=0.\tag{8.11}$$

Also from (1.10), we obtain

$$\begin{aligned}&(\nabla_{X^*}P)(Y^*,Z^*,U^*,V^*)\\&=2A(X^*)P(Y^*,Z^*,U^*,V^*)+A(Y^*)P(X^*,Z^*,U^*,V^*)\\&\quad+A(Z^*)P(Y^*,X^*,U^*,V^*)+A(U^*)P(Y^*,Z^*,X^*,V^*)\\&\quad+A(V^*)P(Y^*,Z^*,U^*,X^*).\end{aligned}\tag{8.12}$$

From (8.12), it follows that

$$A(\bar{X})P(Y^*, Z^*, U^*, V^*) = 0, \tag{8.13}$$

$$A(\bar{Y})P(X^*, Z^*, U^*, V^*) = 0, \tag{8.14}$$

$$A(\bar{Z})P(Y^*, X^*, U^*, V^*) = 0, \tag{8.15}$$

$$A(\bar{U})P(Y^*, Z^*, X^*, V^*) = 0, \tag{8.16}$$

$$A(\bar{V})P(Y^*, Z^*, U^*, X^*) = 0. \tag{8.17}$$

From equations (8.6)-(8.9) we have two cases, namely,

I) $A = 0$ on M_2,

II) M_1 is semi-projectively flat.

At first we consider the case (I). Then from (8.12), it follows that

$$(\nabla_{X^*}P)(Y^*, Z^*, U^*, V^*) = 0,$$

that is,

$$\begin{aligned}(\nabla_{X^*}R)&(Y^*, Z^*, U^*, V^*)\\ &- \frac{(X^*\phi)}{(n-1)}[S(Z^*, U^*)g(Y^*, V^*) - S(Y^*, U^*)g(Z^*, V^*)] = 0.\end{aligned} \tag{8.18}$$

Setting $Z^* = U^* = e^*_\alpha$ in (8.18) and taking summation over α, $p+1 \le \alpha \le n$, we obtain

$$(\nabla_{X^*}S)(Y^*, V^*) - \frac{(X^*\phi)}{n-1}[r^*g(Y^*, V^*) - S(Y^*, V^*)] = 0, \tag{8.19}$$

since $r = \bar{r} + r^*$ and if we take $(X^*\phi) = 0$ in M_2, then from (8.19) we have

$$(\nabla_{X^*}S)(Y^*, V^*) = 0.$$

This implies that M_2 is Ricci symmetric if ϕ is constant in M_2.

For case (II). Since M_1 is semi-projectively flat, therefore it is a manifold of constant curvature. Hence we can state the following:

Theorem 8.1. *Let (M^n, g) be a pseudo-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \le p \le n-2)$. If M is a $(PSPS)_n$ then the following statements hold*:

(I) *In the case of $A = 0$ on M_2, the manifold M_2 is Ricci symmetric, provided $\phi = constant$ in M_2.*

(II) *When M_1 is semi-projectively flat, then M_1 is a manifold of constant curvature.*

Similarly, from equations (8.13)-(8.17), we obtain:

Theorem 8.2. *Let (M^n, g) be a pseudo-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(PSPS)_n$, then the following statements hold*:

(I) *In the case of $A = 0$ on M_1, the manifold M_1 is Ricci symmetric, provided $\phi = constant$ in M_1.*

(II) *When M_2 is semi-projectively flat, then M_2 is a manifold of constant curvature.*

If possible, let M_2 be an Einstein manifold, then we have

$$S(Y^*, V^*) = \frac{r^*}{(n-p-1)} g(Y^*, V^*). \tag{8.20}$$

It follows that

$$(\nabla_X^* S)(Y^*, V^*) = \frac{(X^* r^*)}{(n-p-1)} g(Y^*, V^*). \tag{8.21}$$

Therefore from (8.19), (8.20) and (8.21) we have

$$\frac{1}{n-p-1}[(X^* r^*) - \frac{r^*(n-p-2)}{n-1}(X^* \phi)] g(Y^*, V^*) = 0. \tag{8.22}$$

Since $g(Y^*, V^*) \neq 0$ on M_2, then from (8.22) we have

$$(X^* r^*) = \frac{r^*(n-p-2)}{n-1}(X^* \phi). \tag{8.23}$$

Thus from (8.23) one can conclude that if M_2 is Einstein, then r^* is constant if and only if ϕ is constant. But for an n-dimensional $(n > 2)$ Einstein manifold the scalar curvature is constant. Hence if M_2 is Einstein, then ϕ is constant. Thus we are in a position to state the following:

Theorem 8.3. *Let (M^n, g) be a pseudo-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(PSPS)_n$ and M_2 is Einstein, then ϕ is constant in M_2.*

Similarly, equations (8.13)-(8.17) we obtain:

Theorem 8.4. *Let (M^n, g) be a pseudo-Riemannian manifold such that $M = M_1^p \times M_2^{n-p}, (2 \leq p \leq n-2)$. If M is a $(PSPS)_n$ and M_1 is Einstein, then ϕ is constant in M_1.*

9. Semi-projectively flat spacetimes

From Proposition 2.1, it follows that a spacetime (M^4, g) is semi-projectively flat if and only if it is of constant curvature provided the scalar curvature is non-zero. In this case in view of (2.9) the Ricci tensor is of the form

$$S(X, Y) = \frac{r}{4} g(X, Y), \tag{9.1}$$

and

$$R(X, Y)Z = \frac{r}{12}[g(Y, Z)X - g(X, Z)Y].$$

Thus we conclude the following:

Theorem 9.1. *A semi-projectively flat spacetime with non-zero scalar curvature is a space of constant curvature.*

Remark. The space of constant curvature play a significant role in cosmology. The simplest cosmological model is obtained by making the assumption that the universe is isotropic and homogeneous. This is known as cosmological principle. This principle, when translated into the language of Differential geometry, asserts that the three dimensional position space is a space of maximal symmetry [28], that is, a space of constant curvature whose curvature depends upon time. The cosmological solution of Einstein equations which contain a three dimensional spacelike surface of a constant curvature are the Robertson-Walker metrics, while four dimensional space of constant curvature is the de Sitter model of the universe ([24], [28]).

Let us consider a spacetime satisfying the Einstein's field equation with cosmological constant

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{9.2}$$

where S and r denote the Ricci tensor and scalar curvature respectively. λ is the cosmological constant, κ is the gravitational constant and $T(X,Y)$ is the energy-momentum tensor.

Using (9.1) and (9.2) we obtain

$$T(X,Y) = \frac{1}{\kappa}[\lambda - \frac{r}{4}]g(X,Y). \tag{9.3}$$

Taking covariant derivative of (9.3) we get

$$(\nabla_Z T)(X,Y) = -\frac{1}{4\kappa}dr(Z)g(X,Y). \tag{9.4}$$

Since semi-projectively flat spacetime is Einstein, therefore the scalar curvature r is constant. Hence

$$dr(X) = 0 \tag{9.5}$$

for all X.

Equations (9.4) and (9.5) together yield

$$(\nabla_Z T)(X,Y) = 0.$$

Thus we can state the following:

Theorem 9.2. *In a semi-projectively flat spacetime with non-zero scalar curvature satisfying Einstein's field equation with cosmological constant, the energy-momentum tensor is covariant constant.*

In [6] Chaki et al. proved that in a general relativistic space time $\nabla T = 0$ implies $\nabla S = 0$. Conversely if $\nabla S = 0$, it is clear that in our case $\nabla T = 0$. Thus we can state the following:

Corollary 9.1. *In a semi-projectively flat spacetime with non-zero scalar curvature* $\nabla T = 0$ *and* $\nabla S = 0$ *are equivalent.*

10. Semi-projectively flat perfect fluid spacetimes

Now we consider the matter distribution is perfect fluid whose velocity vector field is the vector field ρ corresponding to the 1-form A of the spacetime. Therefore the energy-momentum tensor T of type $(0,2)$ is of the form ([25]):

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{10.1}$$

where σ and p are the energy density and the isotropic pressure respectively.

Hence from the Einstein's field equation we get

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa[pg(X,Y) + (\sigma + p)A(X)A(Y)]. \tag{10.2}$$

Contracting X and Y in the above equation we have

$$r = \kappa(\sigma - 3p). \tag{10.3}$$

Using (2.9) in (10.2) we have

$$\frac{r\phi}{3+\phi}g(X,Y) - \frac{r}{2}g(X,Y) = \kappa[pg(X,Y) + (\sigma + p)A(X)A(Y)]. \tag{10.4}$$

Putting $Y = \rho$ in (10.4) and making use of $A(X) \neq 0$, we have

$$r + \kappa\sigma = \frac{9+\phi}{2(3+\phi)}r. \tag{10.5}$$

Also equations (10.3) and (10.5) yield

$$(\phi + 1)\sigma = (\phi - 3)p. \tag{10.6}$$

Thus in view of the above we can state the following:

Theorem 10.1. *In a semi-projectively flat perfect fluid spacetime with non-zero scalar curvature obeying Einstein's field equation without cosmological constant the energy density and the isotropic pressure are related by* (10.6).

Remark 1. In a semi-projectively flat spacetime with non-zero scalar curvature, ϕ is equal to 1. Therefore the equation (10.6) reduces to $\sigma = -p$. It follows that $p = -\sigma$, that is, of the form $p = p(\sigma)$. Hence we conclude that the fluid is isentropic [13].

Remark 2. In perfect fluid spacetime the equation of state parameter $\omega = \frac{p}{\sigma} = -1$. The dark energy is usually described by an equation of stateparameter $\omega = \frac{p}{\sigma}$, the ratio of the spatially homogeneous dark energy pressure p to its energy density ρ. A value $\omega < -\frac{1}{3}$ is required for cosmic acceleration. The simplest explanation for dark energy is a cosmological constant, for which $\omega = -1$ [3].

Now we consider spacetimes with $div\mathcal{P} = 0$ with the additional condition ϕ is constant. Then from equation (2.22) it follows that the Ricci tensor is of Codazzi type. In [26] Chaki and Ray proved the following:

Theorem 10.2. *If the Ricci tensor of the perfect fluid spacetime is a Codazzi tensor, then the velocity vector field U of the fluid is hypersurface orthogonal and energy density is constant over a hypersurface orthogonal to U. Further the fluid has vanishing vorticity and vanishing shear.*

If the Ricci tensor is Codazzi then the divergence of the conformal curvature tensor vanishes. In [18] (see also [23]) the authors showed that a perfect fluid with closed fluid velocity and vanishing conformal curvature tensor is a generalized Robertson-walker spacetime, i.e., a space-time endowed with the metric

$$ds^2 = -dt^2 + f(t)^2 \tilde{g}_{\alpha\beta} dx^\alpha dx^\beta,$$

being $\tilde{g}_{\alpha\beta}$ is the metric tensor of a $n-1$ dimensional Riemannian manifold. For the most recent review on generalized Robertson-Walker space-times see the survey [17] and references therein. So we have

Theorem 10.3. *A perfect fluid space-time with $divP = 0$ and ϕ constant is a generalized Robertson-walker space-time.*

It has been proved by Barnes [1] if a perfect fluid spacetimes is vorticity free and shear-free and velocity vector field U of the fluid is hypersurface orthogonal and energy density is constant over a hyper surface orthogonal to U, then the possible local cosmological structures of the spacetime are of Petrov type I, D or O (conformally flat). Thus we can state the following:

Theorem 10.4. *If the semi-projective curvature tensor is divergence free in a perfect fluid spacetime with ϕ=constant, then the possible local cosmological structures of the spacetime are of Petrov type I, D or O (conformally flat).*

11. Semi-projectively flat dust fluid spacetime

In a dust or pressureless fluid spacetime, the energy-momentum tensor is of the form [27]

$$T(X,Y) = \sigma A(X)A(Y), \tag{11.1}$$

where σ is the energy density of the dust-like matter and A is a non-zero 1-form such that $g(X,\rho) = A(X)$ for all X, A being the velocity vector field of the flow, that is, $g(\rho,\rho) = -1$.

Using (1.4) and (11.1) we obtain

$$\frac{r\phi}{3+\phi} g(X,Y) - \frac{r}{2} g(X,Y) = \kappa A(X)A(Y). \tag{11.2}$$

Taking a frame field and contracting over X and Y leads to

$$4\left(\frac{r\phi}{3+\phi} - \frac{r}{2}\right) = -\kappa\sigma. \tag{11.3}$$

Again, if we put $X = Y = \rho$ in (11.2), we get

$$-(\frac{r\phi}{3+\phi} - \frac{r}{2}) = \kappa\sigma. \tag{11.4}$$

Thus combining the equations (11.3) and (11.4), we finally obtain that

$$\sigma = 0. \tag{11.5}$$

Thus from (11.1) and (11.5) we conclude that

$$T(X, Y) = 0.$$

This means that the spacetime is devoid of matter. Thus we can state the following:

Theorem 11.1. *A semi-projectively flat dust fluid spacetime satisfying Einstein's field equation without cosmological constant is vacuum.*

12. Example of a $(PSPS)_4$

We consider a Lorentzian manifold (M^4, g) endowed with the Lorentzian metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2, \tag{12.1}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2 are non zero.

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{22} = -x^1,\ \ \Gamma^2_{33} = -\frac{x^2}{(x^1)^2},\ \ \Gamma^2_{12} = \frac{1}{x^1},\ \ \Gamma^3_{23} = \frac{1}{x^2},$$

$$R_{1332} = -\frac{x^2}{x^1},\ \ S_{12} = -\frac{1}{x^1 x^2}.$$

We shall now show that this M^4 is a pseudo semi-projective symmetric spacetime i.e., it satisfies the defining relation (1.10).

In this example we consider the scalar ϕ as follows:

$$\phi = \begin{cases} \frac{5}{3x^1}, & \text{for non-zero components of the curvature tensor} \\ 0, & \text{for vanishing components of the curvature tensor.} \end{cases}$$

Then only the non vanishing component for semi-projective curvature tensor and its covarient derivatives are given by

$$P_{1332} = -\frac{x^2}{x^1},\ P_{1332,1} = \frac{2x^2}{(x^1)^2},\ P_{1332,2} = \frac{1}{x^1}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} -\frac{2}{3x^1}, & \text{for } \ i = 1 \\ -\frac{1}{3x^2}, & \text{for } \ i = 2 \\ 0, & \text{for } \ i = 3, 4 \end{cases}$$

at any point $x \in M$. In our (M^4, g), (1.10) reduces with these 1-forms to the following equations:

$$P_{1332,1} = 2A_1P_{1332} + A_1P_{1332} + A_3P_{1132} + A_3P_{1312} + A_2P_{1331} \tag{12.2}$$

and

$$P_{1332,2} = 2A_2P_{1332} + A_1P_{2332} + A_3P_{1232} + A_3P_{1322} + A_2P_{1332} \tag{12.3}$$

It can be easily verified that the equations (12.2) and (12.3) are true.
So, the manifold under consideration is a pseudo semi-projective symmetric spacetime, that is, $(PSPS)_4$.

Thus we can state the following:

Theorem 12.1. *Let $(\mathbb{R}^4, g)$ be a 4-dimensional Lorentzian manifold with the Lorentzian metric g given by*

$$ds^2 = g_{ij}dx^idx^j = (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^2)^2(dx^3)^2 - (dx^4)^2,$$

where $i,\ j = 1, 2, 3, 4$ and x^1, x^2 are non zero. Then $(\mathbb{R}^4, g)$ is a pseudo semi-projective symmetric spacetime.

Acknowledgment: The authors are thankful to the referees for their valuable suggestions and comments towards the improvement of the paper.

References

[1] A. Barnes, *On shear free normal flows of a perfect fluid*, General Relativity and Gravitation **4** (1973), no. 2, 105–129.
[2] J. K. Beem and P. E. Ehrlich, *Global Lorentzian Geometry*, Monographs and Textbooks in Pure and Applied Math., **67**, Marcel Dekker, Inc., New York, 1981.
[3] R. R. Caldwell, M. Kaminonkowski, and N. N. Weinberg, *Phantom energy and cosmic doomsday*, arxiv:astro-ph/0302506v1.
[4] E. Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **54** (1926), 214–264.
[5] M. C. Chaki, *On pseudosymmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. **33** (1987), no. 1, 53–58.
[6] M. C. Chaki and S. Ray, *Space-times with covariant-constant energy-momentum tensor*, Internat. J. Theoret. Phys. **35** (1996), no. 5, 1027–1032.
[7] M. C. Chaki and S. K. Saha, *On pseudo-projective Ricci symmetric manifolds*, Bulgar. J. Phys. **21** (1994), no. 1-2, 1–7.
[8] C. J. S. Clarke, *Singularities: global and local aspects*, in Topological properties and global structure of space-time (Erice, 1985), 61–71, NATO Adv. Sci. Inst. Ser. B Phys., **138**, Plenum, New York, 1986.
[9] U. C. De and L. Velimirović, *Spacetimes with semisymmetric energy-momentum tensor*, Internat. J. Theoret. Phys. **54** (2015), no. 6, 1779–1783.
[10] F. A. Ficken, *The Riemannian and affine differential geometry of product-spaces*, Ann. of Math. (2) **40** (1939), 892–913.
[11] R. Geroch, *Space-time structure from a global viewpoint*, in General relativity and cosmology (Proc. Internat. School of Physics "Enrico Fermi", Italian Phys. Soc., Varenna, 1969), 71–103, Academic Press, New York, 1971.
[12] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata **7** (1978), no. 3, 259–280.

[13] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge University Press, London, 1973.
[14] P. S. Joshi, *Global Aspects in Gravitation and Cosmology*, International Series of Monographs on Physics, **87**, The Clarendon Press, Oxford University Press, New York, 1993.
[15] S. Mallick and U. C. De, *Spacetimes admitting W_2-curvature tensor*, Int. J. Geom. Methods Mod. Phys. **11** (2014), no. 4, 1450030, 8 pp.
[16] C. A. Mantica and L. G. Molinari, *Weakly Z-symmetric manifolds*, Acta Math. Hungar. **135** (2012), no. 1-2, 80–96.
[17] ______, *Generalized Robertson-Walker spacetimes—a survey*, Int. J. Geom. Methods Mod. Phys. **14** (2017), no. 3, 1730001, 27 pp.
[18] C. A. Mantica, L. G. Molinari, and U. C. De, *A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time*, J. Math. Phys. **57** (2016), no. 2, 022508, 6 pp.
[19] C. A. Mantica and Y. J. Suh, *Pseudo-Z symmetric space-times*, J. Math. Phys. **55** (2014), no. 4, 042502, 12 pp.
[20] ______, *Pseudo Z symmetric Riemannian manifolds with harmonic curvature tensors*, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 1, 1250004, 21 pp.
[21] ______, *Recurrent Z forms on Riemannian and Kaehler manifolds*, Int. J. Geom. Methods Mod. Phys. **9** (2012), no. 7, 1250059, 26 pp.
[22] ______, *Pseudo-Q-symmetric Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys. **10** (2013), no. 5, 1350013, 25 pp.
[23] C. A. Mantica, Y. J. Suh, and U. C. De, *A note on generalized Robertson-Walker space-times*, Int. J. Geom. Methods Mod. Phys. **13** (2016), no. 6, 1650079, 9 pp.
[24] J. V. Narlikar, *Lectures on general relativity and cosmology*, Macmillan Co. of India, Ltd., New Delhi, 1979.
[25] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.
[26] S. Ray, *Certain investigations in relativity ans cosmology by differential geometric method*, University of Kalyani, April,1999, 29 pp.
[27] S. K. Srivastava, *General Relativity and Cosmology*, Prentice-Hall of India Private Limited, New Delhi, 2008.
[28] H. Stephani, *General Relativity*, translated from the German by Martin Pollock and John Stewart, Cambridge University Press, Cambridge, 1982.
[29] F. Zengin, *M-projectively flat spacetimes*, Math. Rep. (Bucur.) **14(64)** (2012), no. 4, 363–370.

Uday Chand De
Department of Pure Mathematics
University of Calcutta
35, Ballygaunge Circular Road
Kolkata -700019, West Bengal, India
Email address: uc_de@yahoo.com

Pradip Majhi
Department of Pure Mathematics
University of Calcutta
35, Ballygaunge Circular Road
Kolkata -700019, West Bengal, India
Email address: mpradipmajhi@gmail.com