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SMALL DATA SCATTERING OF HARTREE TYPE FRACTIONAL SCHRÖDINGER EQUATIONS IN DIMENSION 2 AND 3

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Abstract. In this paper we study the small-data scattering of the d dimensional fractional Schrödinger equations with $d = 2, 3$, Lévy index $1 < \alpha < 2$ and Hartree type nonlinearity $F(u) = \mu(|x|^{-\gamma} * |u|^2)u$ with $\max(\alpha, \frac{2d}{2d-1}) < \gamma \le 2$, $\gamma < d$. This equation is scaling-critical in $\dot{H}^{s_c}$, $s_c = \frac{\gamma-\alpha}{2}$. We show that the solution scatters in $H^{s,1}$ for any $s > s_c$, where $H^{s,1}$ is a space of Sobolev type taking in angular regularity with norm defined by $\|\varphi\|_{H^{s,1}} = \|\varphi\|_{H^s} + \|\nabla_{\mathbb{S}}\varphi\|_{H^s}$. For this purpose we use the recently developed Strichartz estimate which is L^2-averaged on the unit sphere $\mathbb{S}^{d-1}$ and utilize U^p-V^p space argument.

1. Introduction

In this paper we consider the following Cauchy problem for Hartree type fractional Schrödinger equations of the form:

$$\begin{cases} i\partial_t u = D^\alpha u + F(u) \ \text{ in } \ \mathbb{R}^{1+d}, \\ u(x,0) = \varphi(x) \ \ x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $D^\alpha = (-\Delta)^{\frac{\alpha}{2}}$, $1 < \alpha < 2$, $d = 2, 3$ and $F(u) = [V * |u|^2]u$ with $V = \mu|x|^{-\gamma}$, $0 < \gamma < d$, $\mu \in \mathbb{R} \setminus \{0\}$. The equation (1.1) has the scaling invariance structure in $\dot{H}^{s_c}$, $s_c = \frac{\gamma-\alpha}{2}$. Some basic notations are listed at the end of this section.

By Duhamel's formula, (1.1) is written as an integral equation

$$u = e^{-itD^\alpha}\varphi - i\int_0^t e^{-i(t-t')D^\alpha}(F(u(t')))\,dt'. \tag{1.2}$$

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Here we define the linear propagator e^{-itD^α} given by the solution to the linear problem $i\partial_t v = D^\alpha v$ with initial datum $v(0) = f$. It is formally given by

$$e^{-itD^\alpha} f = \mathcal{F}^{-1}(e^{-it|\xi|^\alpha}\mathcal{F}(f)) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^\alpha)}\widehat{f}(\xi)\,d\xi, \tag{1.3}$$

where $\widehat{f} = \mathcal{F}(f)$ denotes the Fourier transform of f and $\mathcal{F}^{-1}$ the inverse Fourier transform such that

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}^{-1}(g)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

The fractional Schrödinger equations have been derived to describe natural phenomena for the system of long-range lattice interaction ($1 < \alpha < 2$) [16], water waves ($\alpha = \frac{1}{2}, \frac{3}{2}$) [19], and fractional quantum mechanics ($1 < \alpha < 2$) [18]. We are concerned with small-data scattering theory of (1.1). In this paper the notion of scattering is defined as follows.

Definition 1.1. We say that a solution u to (1.1) scatters (to $u_\pm$) in a Hilbert space $\mathcal{H}$ if there exist $\varphi_\pm \in \mathcal{H}$ (with $u_\pm(t) = e^{-itD^\alpha}\varphi_\pm$) such that $\lim_{t\to\pm\infty}\|u(t) - u_\pm\|_{\mathcal{H}} = 0$.

There have been several scattering results for the Hartree type nonlinearity only for $d \geq 3$. The small data scattering can be obtained by Strichartz estimates [2, 3] as $\gamma > 2$ and by weighted space estimates [1, 9] as $1 < \gamma \leq 2$. In the weighted space regime one can take advantage of the smoothing effect of Hartree potential in the high frequency analysis. However, we have to pay the cost $d \geq 3$ and high regularity for using the smoothing of ∇V. To resolve the 2D or low regularity problem a different approach is required. The authors of this paper treated the cubic problem $F(u) = \mu|u|^2 u$ for $d \geq 3$ in [3, 4] by using $U^p - V^p$ space argument based on the endpoint Strichartz estimates which is angularly averaged. In this paper we apply the $U^p - V^p$ argument to the 2, 3-d Hartree problem.

To state the main theorem let us introduce angularly regular Sobolev space $H^{s,1}$. It is the set of all H^s functions whose angular derivative is also in H^s. The norm is defined by $\|f\|_{H^{s,1}} := \|f\|_{H^s} + \|\nabla_{\mathbb{S}} f\|_{H^s}$. Here $\nabla_{\mathbb{S}}$ is the gradient on the unit sphere and it can also be represented as $x \times \nabla$.

Theorem 1.2. *Let $d = 2, 3$, $1 < \alpha < 2$, $\max(\alpha, \frac{2d}{2d-1}) < \gamma \leq 2$, $\gamma < d$ and let $s > s_c$. Then there exists $\delta > 0$ such that for any $\varphi \in H^{s,1}$ with $\|\varphi\|_{H^{s,1}} \leq \delta$, (1.1) has a unique solution $u \in (C \cap L^\infty)(\mathbb{R}; H^{s,1})$ which scatters in $H^{s,1}$.*

We show Theorem 1.2 essentially on the basis of the nonlinear estimates, namely Proposition 4.1 below. The crucial part of the nonlinear estimate occurs in the case where high-high frequency interactions make low frequency outputs. Such case does not occur in the cubic problem. Thanks to the radial lemmas (Lemma 2.8 and Lemma 2.9 below) and endpoint Strichartz estimates ((2.1), (2.2), (2.3) below) we can circumvent the singularity in the low frequency part in case that $\frac{2d}{2d-1} < \alpha < 2$, $\alpha < \gamma < 2$ for $d = 2$ and $\alpha < \gamma \leq 2$ for $d = 3$.

The subcritical condition $s > s_c$ is necessary for the summability of square sum (3.2). If we consider the initial data in the Besov space $B_{2,1}^{s_c,1}$ with norm defined by $\|\varphi\|_{B_{2,1}^{s_c,1}} = \|\varphi\|_{B_{2,1}^{s_c}} + \|\nabla_{\mathbb{S}}\varphi\|_{B_{2,1}^{s_c}}$, one can easily get the small data GWP in $B_{2,1}^{s_c,1}$ and scattering in $H^{s_c,1}$. See Remark 1 below.

The scattering in the sense of Definition 1.1 does not occur in the long-range case $0 < \gamma < 1$ (see [8]). For the critical case $\gamma = 1$ it is highly expected that a modified scattering will occur. In fact, it turns out to be true for the 3-d case when α is close enough to 2 (see [5]). The short-range case $1 < \gamma \le \alpha$ still remains open. These two issues deserves to be taken into account within a regime of weighted spaces or U^p-V^p spaces.

Due to the lack of analysis technique for high-high-low case, the mass critical case $\gamma = \alpha$ is not attained by the current argument. But one can think of a modification of potential near zero frequency as in [1, 3]. Here let us consider a potential V in the sense:

(H) V is radial and smooth functions on $\mathbb{R}^d \setminus \{0\}$ such that for any nonnegative integers k

$$|\nabla_\xi^k(\widehat{V}(\xi))| \lesssim |\xi|^{-(d-\alpha_+)-k} \quad \text{for} \quad |\xi| \le 1,$$
$$|\nabla_\xi^k(\widehat{V}(\xi))| \lesssim |\xi|^{-(d-\alpha)-k} \quad \text{for} \quad |\xi| \ge 1,$$

where $\alpha_+ = \alpha + \varepsilon$ for an arbitrarily small $\varepsilon > 0$. The Yukawa potential $V = \mu\frac{e^{-m|x|}}{|x|}$ $(m > 0)$ is of type **(H)** since $\widehat{V}(\xi) = 4\pi\mu(m^2 + |\xi|^2)^{-1}$.

Now we introduce the second scattering result concerning the hybrid type potential.

Theorem 1.3. *Let $d = 2, 3$ and $\frac{2d}{2d-1} < \alpha < 2$. Suppose that V satisfies* **(H)**. *If $\varphi \in H^{s,1}$ for some $s > \frac{\alpha_+ - \alpha}{2}$ and $\|\varphi\|_{H^{s,1}}$ is sufficiently small, then there exists a unique solution $u \in (C \cap L^\infty)(\mathbb{R}; H^{s,1})$ which scatters in $H^{s,1}$.*

This paper is organized as follows: In Section 2 we give several preliminary lemmas on 2-d endpoint Strichartz estimates, $U^p - V^p$ space, and radial functions. We prove Theorem 1.2 in Section 3 via nonlinear estimates. In Section 4 we provide a proof for the crucial nonlinear estimate. In the last section we prove Theorem 1.3.

Basic notations.

• Littlewood-Paley operators: (1) Homogeneous type. $\dot\beta \in C_{0,rad}^\infty$ with $\dot\beta\ddot\beta = \dot\beta$ and $\ddot\beta(\xi) = \dot\beta(\xi/2) + \dot\beta(\xi) + \dot\beta(2\xi)$. $\mathcal{F}(\dot P_N f)(\xi) = \dot\beta(\xi/N)\widehat{f}$ for any dyadic number N. Let $\ddot P_N = \dot P_{N/2} + \dot P_N + \dot P_{2N}$. Then $\dot P_N \ddot P_N = \dot P_N$. (2) Inhomogeneous type. $P_1 = 1 - \sum_{N>1}\dot P_N$ and $P_N = \dot P_N$ for $N > 1$.

• Fractional derivatives: $D^s = (-\Delta)^{\frac{s}{2}} = \mathcal{F}^{-1}|\xi|^s\mathcal{F}$, $\Lambda^s = (1-\Delta)^{\frac{s}{2}} = \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}$ for $s > 0$.

• Function spaces: $\dot H^s_r = D^{-s}L^r$, $\dot H^s = \dot H^s_2$, $H^s_r = \Lambda^{-s/2}L^r$, $H^s = H^s_2$, $L^r = L^r_x(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $1 \le r \le \infty$. The Besov space $B^s_{p,q}$ is defined by the norm $\|f\|_{B^s_{p,q}} = (\sum_{N\ge1} N^{qs}\|P_N f\|^q_{L^p_x})^{\frac1q}$ for $s \in \mathbb{R}, 1 \le p, q \le \infty$.
• Mixed-normed spaces: For a Banach space X, $u \in L^q_I X$ if and only if $u(t) \in X$ for a.e. $t \in I$ and $\|u\|_{L^q_I X} := \|\|u(t)\|_X\|_{L^q_I} < \infty$. Especially, we denote $L^q_I L^r_x = L^q_t(I; L^r_x(\mathbb{R}^d))$, $L^q_{I,x} = L^q_I L^q_x$ and $L^q_t L^r_x = L^q_{\mathbb{R}} L^r_x$.
• As usual different positive constants depending only on d, α are denoted by the same letter C, if not specified. $A \lesssim B$ and $A \gtrsim B$ means that $A \le CB$ and $A \ge C^{-1}B$, respectively for some $C > 0$. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

2. Preliminaries

2.1. Strichartz estimates

Let the pair (q, r) satisfy that $2 \le q, r \le \infty$, $\frac2q + \frac dr = \frac d2$ and $(q, r, d) \ne (2, \infty, 2)$. Then we have for any $0 < \alpha \ne 1 < 2$ that

$$N^{-\frac{2-\alpha}{q}}\|e^{-itD^\alpha}\dot P_N\varphi\|_{L^q_t L^r_x} \lesssim \|\dot P_N\varphi\|_{L^2_x}. \tag{2.1}$$

For this see [10].

The endpoint estimate can be extended to a wider range provided an angular average is taken into account. More precisely, let $d = 2, 3, 1 < \alpha < 2$, and let $6 < r \le \infty$, $r_* = 2$ if $d = 2$ and $\frac{10}{3} < r < 6$, $2 < r_* < \frac{4r}{10-r}$ if $d = 3$. Then there holds

$$N^{-\frac{d-\alpha}{2}+\frac dr}\|e^{-itD^\alpha}\dot P_N\varphi\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} \lesssim \|\dot P_N\varphi\|_{L^2_x}. \tag{2.2}$$

In particular, if $\frac{2d}{d-\alpha} < r < \frac{2d}{d-2}$, then we have from (2.2) that

$$\begin{aligned}\|e^{-itD^\alpha}P_1\varphi\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} &\lesssim \sum_{0<N\le1}\|e^{-itD^\alpha}\dot P_N\varphi\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta}\\ &\lesssim \sum_{0<N\le1} N^{\frac{d-\alpha}{2}-\frac dr}\|\dot P_N\varphi\|_{L^2_x} \lesssim \|P_1\varphi\|_{L^2_x}.\end{aligned} \tag{2.3}$$

Here the norm $L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta$ with $r < \infty$ is defined by

$$\|u\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} = \left(\int_{\mathbb{R}}\left\|\Big(\int_{S^{d-1}}|u(t,\rho\theta)|^{r_*}\,d\theta\Big)^{\frac{1}{r_*}}\right\|^2_{L^r_\rho(\rho^{d-1}d\rho)}\right)^{\frac12}.$$

If $r = \infty$, then we define $\mathcal{L}^\infty_\rho = L^\infty_\rho$. For (2.2) see Theorem 1.1 and Corollary 2.9 of [12]. See also [6,13] for some general Strichartz estimates associated with angular regularity.

2.2. U^p and V^p spaces

We list some useful properties of U^p and V^p spaces. For complete a description see [14,17].

Let $\mathcal{Z}$ be the set of all finite partitions $-\infty < t_0 < \cdots < t_K \le \infty$ of $\mathbb{R}$. If $t_K = \infty$, then we use the convention $u(t_K) = 0$ for all functions $u : \mathbb{R} \to L^2_x$.

Definition 2.1. Let $1 \le p < \infty$. A U^p-atom is defined by a step function $a : \mathbb{R} \to L^2_x$ of the form

$$a(t) = \sum_{k=1}^{K} \chi_{[t_{k-1},t_k)}(t)\phi_{k-1},$$

where χ is the characteristic function,

$$\{t_k\} \in \mathcal{Z},\ \ \{\phi_k\}_{k=0}^{K-1} \subset L^2_x \quad \text{with} \quad \sum_{k=0}^{K-1} \|\phi_k\|^p_{L^2_x} = 1.$$

The atomic space $U^p(\mathbb{R}; L^2_x)$ is defined as the set of functions $u : \mathbb{R} \to L^2_x$ of the form

$$U^p(\mathbb{R}; L^2_x) = \left\{ u = \sum_{j=1}^{\infty} \lambda_j\, a_j \,\middle|\, a_j \ \text{ are } U^p\text{-atoms and } \ \{\lambda_j\} \in \ell^1 \right\},$$

with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \,\middle|\, u = \sum \lambda_j a_j \right\}.$$

Definition 2.2. Let $1 \le p < \infty$. We define $V^p(\mathbb{R}; L^2_x)$ as the normed space of all functions $v : \mathbb{R} \to L^2_x$ such that $\lim_{t\to\pm\infty} v(t)$ exist and for which the norm

$$\|v\|_{V^p} := \sup_{\{t_k\}\in\mathcal{Z}} \left(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|^p_{L^2_x} \right)^{\frac{1}{p}}$$

is finite. $V^p_-(\mathbb{R}; L^2_x)$ denotes the normed space of all function $v \in V^p(\mathbb{R}; L^2_x)$ with $v(-\infty) = 0$. $V^p_{-,rc}(\mathbb{R}; L^2_x)$ is the closed subspace of all right continuous $V^p_-(\mathbb{R}; L^2_x)$ functions.

Lemma 2.3. (1) *$U^p(\mathbb{R}; L^2_x)$ and $V^p(\mathbb{R}; L^2_x)$ are Banach spaces.*

(2) *For $1 \le p < q < \infty$ the embedding $U^p(\mathbb{R}; L^2_x) \hookrightarrow U^q(\mathbb{R}; L^2_x) \hookrightarrow L^\infty_t L^2_x$ is continuous.*

(3) *Every $u \in U^p(\mathbb{R}; L^2_x)$ is right-continuous and* $\lim_{t\to-\infty} u(t) = 0$.

(4) *For $1 \le p < \infty$ the embedding $U^p(\mathbb{R}; L^2_x) \hookrightarrow V^p_{-,rc}(\mathbb{R}; L^2_x)$ is continuous.*

(5) *For $1 < p < \infty$ $(U^p(\mathbb{R}; L^2_x))^* = V^{p'}(\mathbb{R}; L^2_x)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Let $B(u, v)$ denote the duality form between $U^p(\mathbb{R}; L^2_x)$ and $V^{p'}(\mathbb{R}; L^2_x)$.

Lemma 2.4. *Let $1 < p < \infty$. Let $u \in V^1_-(\mathbb{R}; L^2_x)$ be absolutely continuous on compact intervals and $v \in V^{p'}(\mathbb{R}; L^2_x)$. Then*

$$B(u,v) = -\int_{-\infty}^{\infty} \big\langle u'(t), v(t)\big\rangle\, dt.$$

Now we introduce adapted space U^p_α and V^p_α to e^{-itD^α}.

Definition 2.5. We define $U^p_\alpha(\mathbb{R}; L^2_x)$ as the spaces of all functions $u : \mathbb{R} \to L^2_x$ such that $e^{itD^\alpha} u \in U^p(\mathbb{R}; L^2_x)$ with norm $\|u\|_{U^p_\alpha} := \|e^{itD^\alpha} u\|_{U^p}$. Likewise, we define $V^p_\alpha(\mathbb{R}; L^2_x)$ and its norm $\|v\|_{V^p_\alpha} := \|e^{itD^\alpha} v\|_{V^p}$.

Lemma 2.3 is extended to the spaces $U^p_\alpha(\mathbb{R}; L^2_x)$ and $V^p_\alpha(\mathbb{R}; L^2_x)$.

Lemma 2.6. *For any $v \in L^\infty_t L^2_x$ we have $\|v\|_{L^\infty_t L^2_x} \lesssim \|v\|_{V^2_\alpha}$.*

Lemma 2.7 (Transfer principle)**.** *Let $T : L^2_x \to L^1_{loc}(\mathbb{R}^d; \mathbb{C})$ be a linear operator satisfying that*

$$\|T(e^{-itD^\alpha}\phi)\|_{L^q_t X} \lesssim \|\phi\|_{L^2_x}$$

for some $1 \le q \le \infty$ and a Banach space $X \subset L^1_{loc}$. Then

$$\|T(u)\|_{L^q_t X} \lesssim \|u\|_{U^q_\alpha}.$$

Proof of Lemma 2.7. Without loss of generality we may assume that $e^{itD^\alpha}u(t)$ is a U^q-atom with the partition $t_0, \dots, t_{K-1}$ and $t_K = \infty$. That is to say,

$$e^{itD^\alpha}u(t) = \sum_{k=1}^{K} \chi_{[t_{k-1}, t_k)}\phi_{k-1}$$

for $\phi_k \in L^2$ with $\sum_{k=0}^{K-1} \|\phi_k\|^q_{L^2} = 1$. Then we have only to show that $\|T(u)\|_{L^q_t X} \lesssim 1$. In fact, by assumption we have

$$\begin{aligned}
\|T(u)\|^q_{L^q_t X} &= \|T(e^{-itD^\alpha}\sum_{k=1}^{K} \chi_{[t_k-1, t_k)}\phi_{k-1})\|^q_{L^q_t X}\\
&= \sum_{k=1}^{K} \|T(e^{-itD^\alpha}\phi_{k-1})\|^q_{L^q_{[t_{k-1},t_k)} X}\\
&\lesssim \sum_{k=0}^{K-1} \|\phi_k\|^q_{L^2_x} \lesssim 1.
\end{aligned}$$

□

2.3. Some useful lemmata

Lemma 2.8. *Let ψ, f be smooth and let ψ be radially symmetric. Then*

$$\nabla_{\mathbb{S}}(\psi * f) = \psi * \nabla_{\mathbb{S}} f.$$

Lemma 2.9 (Lemma 7.1 of [7]). *If $\psi(x)$ is radially symmetric, then*

$$\|\psi * f\|_{\mathcal{L}_\rho^p L_\theta^q} \le \|f\|_{\mathcal{L}_\rho^{p_1} L_\theta^{q_1}} \|\psi\|_{L_x^{p_2}}$$

for all $p_1, p_2, p, q, q_1 \in [1, \infty]$ satisfying

$$\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p}, \quad \frac{1}{q_1} + \frac{1}{p_2} - 1 \le \frac{1}{q}.$$

The next is on the convolution inequality for sequence.

Lemma 2.10. *For any $f \in \ell^1(\mathbb{Z})$ and $g \in \ell^2(\mathbb{Z})$ we have*

$$\|f * g\|_{\ell^2(\mathbb{Z})} \le \|f\|_{\ell^1(\mathbb{Z})} \|g\|_{\ell^2(\mathbb{Z})}.$$

The final one is on the Sobolev inequality on the unit sphere.

Lemma 2.11. *For any $d - 1 < \widetilde{r} < \infty$*

$$\|f\|_{L_\theta^\infty} \lesssim \|f\|_{L_\theta^{\widetilde{r}}} + \|\nabla_{\mathbb{S}} f\|_{L_\theta^{\widetilde{r}}}, \quad \|f\|_{L_\theta^{\widetilde{r}}} \lesssim \|f\|_{L_\theta^2} + \|\nabla_{\mathbb{S}} f\|_{L_\theta^2}.$$

For this see [11, 20].

3. Proof of Theorem 1.2

Let us define the Banach space $X^{s,1}$ for $s \in \mathbb{R}$ by

$$X^{s,1} := \left\{ u : \mathbb{R} \to L^2 \;\middle|\; P_N u, \nabla_{S^2} P_N u \in U_\alpha^2(\mathbb{R}; L_x^2) \ \ \forall N \ge 1 \right\}$$

with the norm

$$\|u\|_{X^{s,1}} = \left(\sum_{N \ge 1} N^{2s} (\|P_N u\|_{U_\alpha^2} + \|P_N(\nabla_{S^2} u)\|_{U_\alpha^2})^2 \right)^{\frac{1}{2}}.$$

Let $X_+^{s,1}$ be the restricted space defined by

$$X_+^{s,1} = \left\{ u \in C([0,\infty); H^s) \;\middle|\; \chi_{[0,\infty)}(t) u(t) \in X^{s,1} \right\}$$

with norm $\|u\|_{X_+^{s,1}} := \|\chi_{[0,\infty)} u\|_{X^{s,1}}$.

Let $\mathcal{D}_+(\delta)$ be a complete metric space $\{u \in X_+^{s,1} \mid \|u\|_{X_+^{s,1}} \le \delta\}$ equipped with the metric $d(u,v) := \|u - v\|_{X_+^{s,1}}$. Then we will show that the nonlinear functional $\Psi(u) = e^{-itD^\alpha} \varphi + \mathcal{N}_\alpha(u)$ is a contraction on $\mathcal{D}_+(\delta)$, where

$$\mathcal{N}_\alpha(u) = -i \int_0^t e^{-i(t-t')D^\alpha} [V * |u|^2] u \, dt'.$$

We will show that if $s > s_c = \frac{2-\alpha}{2}$,

$$\begin{aligned} &\|\mathcal{N}_\alpha(u)\|_{X_+^{s,1}} \lesssim \|u\|_{X_+^{s,1}}^3, \\ &\|\mathcal{N}_\alpha(u) - \mathcal{N}_\alpha(u)\|_{X_+^{s,1}} \lesssim (\|u\|_{X_+^{s,1}} + \|v\|_{X_+^{s,1}})^2 \|u - v\|_{X_+^{s,1}}. \end{aligned} \tag{3.1}$$

Clearly, $\|e^{-itD^\alpha}\varphi\|_{X_+^{s,1}} \lesssim \|\varphi\|_{H^{s,1}}$ and thus we can find δ small enough for Ψ to be a contraction mapping on $\mathcal{D}_+(\delta)$.

Since $e^{itD^\alpha}P_N\mathcal{N}_\alpha(u)$ and $e^{itD^\alpha}P_N\nabla_{\mathbb{S}}\mathcal{N}_\alpha(u)$ are in $V^2_{-,rc}(\mathbb{R};L^2)$ from (4) of Lemma 2.3, and

$$\sum_{N\ge 1} N^{2s}(\|e^{itD^\alpha}P_N\mathcal{N}_\alpha(u)\|_{V^2} + \|e^{itD^\alpha}P_N\nabla_{\mathbb{S}}\mathcal{N}_\alpha(u)\|_{V^2})^2 < \infty$$

from (3.1), $\lim_{t\to+\infty} e^{itD^\alpha}\mathcal{N}_\alpha(u)$ exists in $H^{s,1}$. Define a scattering state u^+ with

$$\varphi^+ := \varphi + \lim_{t\to+\infty} e^{itD^\alpha}\mathcal{N}_\alpha(u).$$

By time symmetry we can argue in a similar way for the negative time. Thus we get the desired result.

Now it remains to show (3.1). Since the second part of (3.1) follows from the argument of the first part, we omit it. We may assume that $u(t) = 0$ for $-\infty < t < 0$. Since clearly $\int_0^t e^{it'D^\alpha}P_N[(V*|u|^2)u]\,dt' \in V^1_-(\mathbb{R};L^2_x)$ and differentiable, from the duality form and Lemma 2.4 it follows that

$$\begin{aligned}\|P_N\mathcal{N}_\alpha(u)\|_{U^2_\alpha} &= \|e^{itD^\alpha}P_N\mathcal{N}_\alpha(u)\|_{U^2} = \sup_{\|v\|_{V^2}\le 1}\Big|B(e^{itD^\alpha}P_N\mathcal{N}_\alpha(u),v)\Big| \\ &= \sup_{\|v\|_{V^2}\le 1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[V*|u|^2]u(t)\overline{e^{-itD^\alpha}P_Nv(t)}\,dxdt\Big| \\ &= \sup_{\|v\|_{V^2_\alpha}\le 1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}[V*|u|^2]u(t)P_N\overline{v(t)}\,dxdt\Big|\end{aligned}$$

and similarly

$$\|P_N\nabla_{\mathbb{S}}\mathcal{N}_\alpha(u)\|_{U^2_\alpha} = \sup_{\|v\|_{V^2_\alpha}\le 1}\Big|\int_{\mathbb{R}}\int_{\mathbb{R}^d}\nabla_{\mathbb{S}}([V*|u|^2]u(t))P_N\overline{v(t)}\,dxdt\Big|.$$

By Littlewood-Paley decomposition we get

(3.2)
$$\begin{aligned}&\|\mathcal{N}_\alpha(u)\|^2_{X^{s,1}} \\ &\lesssim \sum_{N\ge1}N^{2s}\Bigg[\sup_{\|v\|_{V^2_\alpha}\le 1}\Bigg(\sum_{N_1,N_2,N_3\ge 1}\Big|\iint[V*P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t)P_N\overline{v(t)}\,dxdt\Big|\Bigg)^2 \\ &\qquad + \sup_{\|v\|_{V^2_\alpha}\le 1}\Bigg(\sum_{N_1,N_2,N_3\ge 1}\Big|\iint\nabla_{\mathbb{S}}([V*P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t))P_N\overline{v(t)}\,dxdt\Big|\Bigg)^2\Bigg].\end{aligned}$$

The nonlinear estimate (Proposition 4.1 below) yields that for any $\varepsilon > 0$

$$\begin{aligned}&\Big|\iint [V * P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t)P_N\overline{v(t)}\,dxdt\Big| \\ &+\Big|\iint \nabla_{\mathbb{S}}([V * P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t))P_N\overline{v(t)}\,dxdt\Big| \\ &\lesssim (N_{\min}N_{\mathrm{med}})^{\frac{\gamma-\alpha}{2}}\prod_{i=1}^{3}(\|P_{N_i}u\|_{U^2_\alpha}+\|P_{N_i}\nabla_{\mathbb{S}}u\|_{U^2_\alpha}),\end{aligned} \tag{3.3}$$

where $N_{\min} = \min(N_1, N_2, N_3)$, $N_{\max} = \max(N_1, N_2, N_3)$ and $N_{\min} \le N_{\mathrm{med}} \le N_{\max}$. Since $\nabla_{\mathbb{S}} = x \times \nabla$ and hence $\nabla_{\mathbb{S}}(vw) = (\nabla_{\mathbb{S}}v)w + v(\nabla_{\mathbb{S}}w)$, we can apply Lemma 2.8 and Proposition 4.1 in the next section with $u_i = u$.

The self-mapping property (3.1) can be shown by exactly the same way as in [15]. But we brief on it for the reader's convenience. Let us split the summation of RHS of (3.2) into three parts as follows:

$$\text{RHS of } (3.2) =: \Sigma_1 + \Sigma_2 + \Sigma_3,$$

where

$$\Sigma_1 = \sum_{N_3\sim N}, \quad \Sigma_2 = \sum_{N_3\ll N}, \quad \Sigma_3 = \sum_{N_3\gg N}.$$

For Σ_1 we use Lemma 2.10 w.r.t. N_3, N and Cauchy-Schwarz inequality w.r.t. N_1, N_2 together with (3.3) to get

$$\begin{aligned}\Sigma_1 &\le \sum_{N\ge1} N^{2s}\Bigg(\sum_{\substack{N_1,N_2,N_3\ge1\\N_3\sim N}}(N_1N_2)^{\frac{\gamma-\alpha}{2}-s}(N_1N_2)^s\prod_{i=1}^{3}(\|P_{N_i}u\|_{U^2_\alpha}+\|P_{N_i}\nabla_{\mathbb{S}}u\|_{U^2_\alpha})\Bigg)^2 \\ &\lesssim \|u\|^2_{X^{s,1}}\Bigg(\sum_{N_1,N_2\ge1}(N_1N_2)^{\frac{\gamma-\alpha}{2}-s}(N_1N_2)^s\prod_{j=1}^{2}(\|P_{N_i}u\|_{U^2_\alpha}+\|P_{N_i}(\nabla_{\mathbb{S}}u)\|_{U^2_\alpha}\Bigg)^2 \\ &\lesssim \|u\|^6_{X^{s,1}}.\end{aligned}$$

For Σ_2 and Σ_3 we use the fact that if $N_3 \ll N$ or $N_3 \gg N$, then $N \lesssim \max(N_1, N_2)$ or $N_3 \lesssim \max(N_1, N_2)$, respectively. Applying Lemma 2.10 w.r.t. $\max(N_1, N_2), N$ and then Cauchy-Schwarz inequality w.r.t. $\min(N_1, N_2), N_3$, we get

$$\begin{aligned}&\Sigma_2 + \Sigma_3 \\ &\lesssim \sum_{N\ge1}\Bigg((\sum_{N_3\ll N}+\sum_{N_3\gg N})\left(\frac{N}{\max(N_1,N_2)}\right)^s(\min(N_1,N_2)N_3)^{\frac{\gamma-\alpha}{2}-s} \\ &\quad (\min(N_1,N_2)N_3)^s(\max(N_1,N_2))^s\times\prod_{i=1}^{3}(\|P_{N_i}u\|_{U^2_\alpha}+\|P_{N_i}\nabla_{\mathbb{S}}u\|_{U^2_\alpha})\Bigg)^2 \\ &\lesssim \|u\|^6_{X^{s,1}}.\end{aligned}$$

This shows (3.1) and completes the proof of Theorem 1.2.

Remark 1. Let us define the Banach space $Y^{s_c,1}$ by

$$Y^{s_c,1} := \Big\{u : [0,\infty) \to L^2 \,\Big|\, P_N u, \nabla_{\mathbb{S}} P_N u \in U^2_\alpha(\mathbb{R}; L^2_x) \ \ \forall N \ge 1\Big\}$$

with the norm

$$\|u\|_{Y^{s,1}} = \sum_{N\ge 1} N^{s_c}(\|P_N u\|_{U^2_\alpha} + \|P_N(\nabla_{\mathbb{S}} u)\|_{U^2_\alpha}).$$

Instead of (3.2), we need to estimate

(3.4)

$$\begin{aligned}
&\|\mathcal{N}_\alpha(u)\|_{Y^{s_c,1}}\\
&\lesssim \sum_{N\ge 1} N^s \Bigg[\sup_{\|v\|_{V^2_\alpha}\le 1}\Bigg(\sum_{N_1,N_2,N_3\ge 1}\Big|\iint [V * P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t)P_N\overline{v(t)}\,dxdt\Big|\Bigg)\\
&\qquad\qquad + \sup_{\|v\|_{V^2_\alpha}\le 1}\Bigg(\sum_{N_1,N_2,N_3\ge 1}\Big|\iint \nabla_{\mathbb{S}}([V * P_{N_1}uP_{N_2}\overline{u}]P_{N_3}u(t))P_N\overline{v(t)}\,dxdt\Big|\Bigg)\Bigg]\\
&\lesssim \|u\|^3_{Y^{s_c,1}_+}.
\end{aligned}$$

The main part Σ_0 of the summation is the case $N \gg N_3$. $\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3$, where $\Sigma_1 = \sum_{N_1\ll N_2}$, $\Sigma_2 = \sum_{N_1\gg N_2}$ and $\Sigma_3 = \sum_{N_1\sim N_2}$. If $N_1 \ll N_2 (N_1 \gg N_2)$, then $N_2 \sim N (N_1 \sim N)$, respectively, and hence by (3.3)

$$\begin{aligned}
&\Sigma_1 + \Sigma_2\\
&\lesssim \Bigg(\sum_{\substack{N_1\ll N_2\\ N_3\ll N\sim N_2}} + \sum_{\substack{N_1\gg N_2\\ N_3\ll N\sim N_1}}\Bigg) N^{s_c}(N_1N_3)^{s_c}\prod_{i=1}^3(\|P_{N_i}u\|_{U^2_\alpha} + \|P_{N_i}(\nabla_{\mathbb{S}}u)\|_{U^2_\alpha}\\
&\lesssim \|u\|^2_{Y^{s_c,1}}\Bigg(\sum_{N_2\ge 1}\sum_{N\sim N_2} N^{s_c}(\|P_{N_2}u\|_{U^2_\alpha} + \|P_{N_2}(\nabla_{\mathbb{S}}u)\|_{U^2_\alpha})\\
&\qquad\qquad + \sum_{N_1\ge 1}\sum_{N\sim N_1} N^{s_c}(\|P_{N_1}u\|_{U^2_\alpha} + \|P_{N_1}(\nabla_{\mathbb{S}}u)\|_{U^2_\alpha}\Bigg)\\
&\lesssim \|u\|^3_{Y^{s_c,1}}.
\end{aligned}$$

If $N_1 \sim N_2$, then $N \lesssim N_1 \sim N_2$. Therefore

$$\Sigma_3 \lesssim \sum_{\substack{N_1\sim N_2\\ N_3\ll N_2}}\sum_{N\lesssim N_2} N^{s_c}(N_1N_3)^{s_c}\prod_{i=1}^3(\|P_{N_i}u\|_{U^2_\alpha} + \|P_{N_i}(\nabla_{\mathbb{S}}u)\|_{U^2_\alpha}) \lesssim \|u\|^3_{Y^{s_c,1}}.$$

4. Nonlinear estimate

Proposition 4.1. *Assume that* $u_1, u_2, \nabla_S u_i (i = 1, 2, 3) \in U^2_\alpha$, $v \in V^2_\alpha$. *Let* $\mathbf{u}_i = P_{N_i} u_i$, $\mathbf{v} = P_N v$ *for* $N_i, N \geq 1$, $i = 1, 2, 3$ *and let* $\widetilde{\mathbf{u}_i} = \mathbf{u}_i$ *or* $\overline{\mathbf{u}_i}$. *Then for all* $N_i, N \geq 1$ *we have*

$$\begin{aligned}&\Big|\iint [V * (\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})](\nabla_S \widetilde{\mathbf{u}_3})\mathbf{v}\, dxdt\Big| \\ &\lesssim (N_{\min}N_{\rm med})^{\frac{\gamma-\alpha}{2}} \prod_{i=1}^{2} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_S \mathbf{u}_i\|_{U^2_\alpha})\|\nabla_S \mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&\Big|\iint [V * ((\nabla_S\widetilde{\mathbf{u}_1})\widetilde{\mathbf{u}_2})]\widetilde{\mathbf{u}_3}\mathbf{v}\, dxdt\Big| \\ &\lesssim (N_{\min}N_{\rm med})^{\frac{\gamma-\alpha}{2}} \|\nabla_S \mathbf{u}_1\|_{U^2_\alpha}\prod_{i=2}^{3} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_S \mathbf{u}_i\|_{U^2_\alpha})\|\mathbf{v}\|_{V^2_\alpha}.\end{aligned} \tag{4.2}$$

Proof of (4.1). We may assume that $N_1 \leq N_2$ by summation symmetry. By Littlewood-Paley decomposition we have

$$\begin{aligned}&\Big|\iint [V * (\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})](\nabla_{\mathbb{S}} \widetilde{\mathbf{u}_3})\mathbf{v}\, dxdt\Big| \\ &\leq \sum_{M>0} \Big|\iint \big[\dot{P}_M(V * (\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))\ddot{P}_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\, dxdt\Big| \\ &=: \sum_{M>0} \mathfrak{N}_M.\end{aligned}$$

We first consider the case (123): $N_1 \leq N_2 \leq N_3$. This case can be split into two parts (i) $N_1 \ll N_2 \sim M$ or $N_1 \sim N_2 \sim M$; (ii) $N_1 \sim N_2 \gg M$.

Case (i) *of* (123). If $d = 2$, then by using the embedding lemmas (Lemma 2.3 and Lemma 2.6) and the Sobolev embedding $H^1_\theta(\mathbb{S}^1) \hookrightarrow L^\infty_\theta(\mathbb{S}^1)$, we have

$$\begin{aligned}&\mathfrak{N}_M \\ &\lesssim M^{-(2-\gamma)}\|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^1_tL^\infty_x}\|\ddot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L^\infty_tL^1_x} \\ &\lesssim M^{-(2-\gamma)}(N_1N_2)^{\frac{2-\alpha}{2}}N_1^{\frac{-(2-\alpha)}{2}}\|\mathbf{u}_1\|_{L^2_tL^\infty_x}N_2^{\frac{-(2-\alpha)}{2}}\|\mathbf{u}_2\|_{L^2_tL^\infty_x}\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L^\infty_tL^2_x}\|\mathbf{v}\|_{L^\infty_tL^2_x} \\ &\lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}}\prod_{i=1}^{2}(N_i^{\frac{-(2-\alpha)}{2}}\|\mathbf{u}_i\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta} + N_i^{\frac{-(2-\alpha)}{2}}\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}.\end{aligned}$$

Then by using Lemma 2.7 with $X = \mathcal{L}^\infty_\rho L^2_\theta$ and combining with (2.2) and (2.3), we get

$$\mathfrak{N}_M \lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}}\prod_{i=1}^{2}(\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}.$$

Thus since for (i) of (123) $M \sim N_2$, we get

$$\sum_{\text{(i) of (123)}} \mathfrak{N}_M \lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}} \prod_{i=1}^{2} (\|\mathbf{u}_i\|_{U_\alpha^2} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U_\alpha^2}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U_\alpha^2} \|\mathbf{v}\|_{V_\alpha^2}.$$

If $d = 3$, then by Bernstein's inequality we have

$$\begin{aligned} \mathfrak{N}_M &\lesssim M^{-(3-\gamma)} \|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L_t^1 L_x^\infty} \|\ddot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L_t^\infty L_x^1} \\ &\lesssim M^{-(3-\gamma)} M \|\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L_t^1 L_x^3} \|\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v}\|_{L_t^\infty L_x^1}. \end{aligned}$$

Then the endpoint Strichartz estimate of (2.1) and Lemma 2.7 with $X = L_x^6$ give us

$$\begin{aligned} \mathfrak{N}_M &\lesssim M^{\gamma-2}(N_1N_2)^{\frac{2-\alpha}{2}} \prod_{i=1,2} (N_i^{\frac{-(2-\alpha)}{2}} \|\mathbf{u}_i\|_{L_t^2 L_x^6}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L_t^\infty L_x^2} \|\mathbf{v}\|_{L_t^\infty L_x^2} \\ &\lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}} \prod_{i=1,2} (N_i^{\frac{-(2-\alpha)}{2}} \|\mathbf{u}_i\|_{L_t^2 L_x^6}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L_t^\infty L_x^2} \|\mathbf{v}\|_{L_t^\infty L_x^2} \\ &\lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}} \prod_{i=1}^{2} \|\mathbf{u}_i\|_{U_\alpha^2} \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U_\alpha^2} \|\mathbf{v}\|_{V_\alpha^2}. \end{aligned}$$

Thus since for (i) of (123) $M \sim N_2$, we get

$$\sum_{\text{(i) of (123)}} \mathfrak{N}_M \lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}} \prod_{i=1}^{2} \|\mathbf{u}_i\|_{U_\alpha^2} \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U_\alpha^2} \|\mathbf{v}\|_{V_\alpha^2}.$$

Case (ii) *of* (123). At first we estimate

$$\mathfrak{N}_M \lesssim M^{-(d-\gamma)} \|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_3})\|_{L_t^1 L_x^\infty} \|\ddot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L_t^\infty L_x^1}.$$

Let us choose r such that $\frac{d-\gamma}{2} < \frac{d}{r} < \frac{d-\alpha}{2}$. Applying Lemma 2.9 with $p_1 = \frac{r}{2}, p_2 = \frac{r}{r-2}$ and $q_1 = q = \infty$ to the radial kernel of $\dot{P}_M$, we get

$$\begin{aligned} \mathfrak{N}_M &\lesssim M^{-(d-\gamma)} \|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L_t^1 L_x^\infty} \|\ddot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L_t^\infty L_x^1} \\ &\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L_t^1 \mathcal{L}_\rho^{\frac{r}{2}} L_\theta^\infty} \|\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3}\mathbf{v}\|_{L_t^\infty L_x^1}. \end{aligned}$$

From the Sobolev inequality (Lemma 2.11) and the embedding lemmas (Lemma 2.3 and Lemma 2.6) it follows that

$$\begin{aligned} \mathfrak{N}_M &\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\mathbf{u}_1\|_{L_t^2 \mathcal{L}_\rho^r L_\theta^\infty} \|\mathbf{u}_2\|_{L_t^2 \mathcal{L}_\rho^r L_\theta^\infty} \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L_t^\infty L_x^2} \|\mathbf{v}\|_{L_t^\infty L_x^2} \\ &\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_2)^{\frac{d-\alpha}{2}-\frac{d}{r}} \\ &\quad \times \prod_{j=1,2} N_i^{-\frac{d-\alpha}{2}+\frac{d}{r}} (\|\mathbf{u}_i\|_{L_t^2 \mathcal{L}_\rho^r L_\theta^d} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{L_t^2 \mathcal{L}_\rho^r L_\theta^d}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2} \\ &\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_2)^{\frac{d-\alpha}{2}-\frac{3}{r}} \end{aligned}$$

$$\times \prod_{j=1,2}(\|\mathbf{u}_i\|_{U^2_\alpha}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L^\infty_t L^2_x}\|v\|_{L^\infty_t L^2_x}.$$

Then we conclude that

$$\sum_{\text{(ii) of (123)}}\mathfrak{N}^{\text{(ii)}}_M(1,2,3)\lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}}\prod_{i=1}^{2}(\|\mathbf{u}_i\|_{U^2_\alpha}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}.$$

We have shown the proposition in the case (123). Now let us consider the case (132): $N_1\le N_3\le N_2$, which can be split into (i): $N_3\ll N_2\sim M$ or $N_3\sim N_2\sim M$ and (ii): $N_3\sim N_2\gg M$.

We first consider the case (i) of (132). If $d=2$, then we change the role of $\mathbf{u}_2$ and $\mathbf{u}_3$ in this case. Applying Lemma 2.9 with $p_2=1$ to the radial kernels of $\dot{P}_M$ and $\ddot{P}_M$, we get

$$\begin{aligned}\mathfrak{N}_M &\lesssim M^{-(2-\gamma)}\|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^2_t\mathcal{L}^2_\rho L^\infty_\theta}\|\ddot{P}_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\|_{L^2_t\mathcal{L}^2_\rho L^1_\theta}\\ &\lesssim M^{-(2-\gamma)}\|\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^2_t\mathcal{L}^2_\rho L^\infty_\theta}\|(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v}\|_{L^2_t\mathcal{L}^2_\rho L^1_\theta}\\ &\lesssim M^{-(2-\gamma)}\|\mathbf{u}_1\|_{L^2_tL^\infty_x}\|\mathbf{u}_2\|_{L^\infty_t\mathcal{L}^2_\rho L^\infty_\theta}\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta}\|\mathbf{v}\|_{L^\infty_tL^2_x}.\end{aligned}$$

The Sobolev embedding $H^1_\theta(\mathbb{S}^1)\hookrightarrow L^\infty_\theta(\mathbb{S}^1)$ applied to the norms for $\mathbf{u}_1,\mathbf{u}_2$ yields

$$\begin{aligned}\mathfrak{N}_M \lesssim M^{-(2-\gamma)}&(\|\mathbf{u}_1\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta}+\|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta})\\ &(\|\mathbf{u}_2\|_{L^\infty_tL^2_x}+\|\nabla_{\mathbb{S}}\mathbf{u}_2\|_{L^\infty_tL^2_x})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L^2_t\mathcal{L}^\infty_\rho L^2_\theta}\|\mathbf{v}\|_{L^\infty_tL^2_x}.\end{aligned}$$

By the transfer principle for $\mathbf{u}_1,\mathbf{u}_3$ and the embeddings $U^2_\alpha(\mathbb{R};L^2_x)$, $V^2_\alpha(\mathbb{R};L^2_x)$ $\hookrightarrow L^\infty_tL^2_x$, we have that

$$\begin{aligned}\mathfrak{N}_M \lesssim M^{-(2-\gamma)}&(N_1N_3)^{\frac{2-\alpha}{2}}(\|\mathbf{u}_1\|_{U^2_\alpha}+\|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U^2_\alpha})\\ &(\|\mathbf{u}_2\|_{U^2_\alpha}+\|\nabla_{\mathbb{S}}\mathbf{u}_2\|_{U^2_\alpha})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha},\end{aligned}$$

which gives us

$$\sum_{\text{(i) of (132)}}\mathfrak{N}^{\text{(i)}}_M(1,3,2)\lesssim (N_1N_3)^{\frac{\gamma-\alpha}{2}}\prod_{i=1,2}(\|\mathbf{u}_i\|_{U^2_\alpha}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha})\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}.$$

If $d=3$, then Bernstein's inequality yields

$$\begin{aligned}\mathfrak{N}_M &\lesssim M^{-(3-\gamma)}\|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^2_tL^2_x}\|\widetilde{P}_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\|_{L^2_tL^2_x}\\ &\lesssim M^{-(3-\gamma)+1}\|\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^2_tL^{\frac{3}{2}}_x}\|(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v}\|_{L^2_tL^{\frac{3}{2}}_x}\\ &\lesssim M^{\gamma-2}\|\mathbf{u}_1\|_{L^2_tL^6_x}\|\mathbf{u}_2\|_{L^\infty_tL^2_x}\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{L^2_tL^6_x}\|\mathbf{v}\|_{L^\infty_tL^2_x}.\end{aligned}$$

Using the transfer principle for $\mathbf{u}_1,\mathbf{u}_3$ associated with the endpoint Strichartz estimate of (2.1), we have that

$$\sum_{\text{(i) of (132)}}\mathfrak{N}^{\text{(i)}}_M(1,3,2)\lesssim (N_1N_3)^{\frac{\gamma-\alpha}{2}}\prod_{i=1,2}\|\mathbf{u}_i\|_{U^2_\alpha}\|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha}\|\mathbf{v}\|_{V^2_\alpha}.$$

On the other hand, from (ii) and the support condition it follows that $N_3 \sim N_2 \sim N_1 \gg M$. We perform a similar estimate to the case (ii) of (123) as follows:

$$
\begin{aligned}
&\sum_{\text{(ii) of (132)}} \mathfrak{N}_M \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}} \prod_{i=1,2} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha} \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha} \|\mathbf{v}\|_{V^2_\alpha} \\
&\lesssim (N_1N_3)^{\frac{\gamma-\alpha}{2}} \prod_{i=1,2} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha} \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha} \|\mathbf{v}\|_{V^2_\alpha}.
\end{aligned}
$$

Let us treat the final case (312): $N_3 \le N_1 \le N_2$. This is split into the four parts: ($N_3 \sim N_1 \le N_2$); ($N_3 \ll N_1 \ll N_2$); ($N_3 \ll N_1 \sim N_2$ and $N_3 \lesssim M$); ($M \ll N_3 \ll N_1 \sim N_2$). For simplicity we only consider the last case: $M \ll N_3 \ll N_1 \sim N_2$. The remaining cases can be handled similarly. Let us take r such that $\frac{d-\gamma}{2d} < \frac{1}{r} < \frac{d-\alpha}{2d}$. Applying Lemma 2.9 with $p = 2, p_1 = \frac{2r}{r+2}, p_2 = \frac{r}{r-1}, q_1 = r, q = \frac{2r_*}{r_*-2}$ to the norm of $\mathbf{u}_1\mathbf{u}_2$ and $p = 2, p_1 = \frac{2r}{r+2}, p_2 = \frac{r}{r-1}, q_1 = q = \frac{2r_*}{r_*+2}$ to the norm of $\mathbf{u}_3\mathbf{v}$, respectively, we get

$$
\begin{aligned}
\mathfrak{N}_M &\lesssim M^{-(d-\gamma)} \|\dot{P}_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^2_t\mathcal{L}^2_\rho L_\theta^{\frac{2r_*}{r_*-2}}} \|\widetilde{P}_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\|_{L^2_t\mathcal{L}^2_\rho L_\theta^{\frac{2r_*}{r_*+2}}} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^2_t\mathcal{L}_\rho^{\frac{2r}{r+2}} L^r_\theta} \|(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v}\|_{L^2_t\mathcal{L}_\rho^{\frac{2r}{r+2}} L_\theta^{\frac{2r_*}{r_*+2}}} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}} \prod_{j=1,2} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha} \|v\|_{V^2_\alpha}
\end{aligned}
$$

and hence

$$
\sum_{\text{(Last)}} \mathfrak{N}_M \lesssim (N_1N_3)^{\frac{\gamma-\alpha}{2}} \prod_{j=1,2} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha}) \|\nabla_{\mathbb{S}}\mathbf{u}_3\|_{U^2_\alpha} \|v\|_{V^2_\alpha}.
$$

This proves (4.1). □

Proof of (4.2). By Littlewood-Paley decomposition we have

$$
\begin{aligned}
&\Big| \iint [V * (\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})]\widetilde{\mathbf{u}_3}\mathbf{v}\, dxdt \Big| \\
&\le \sum_{M>0} \Big| \iint \big[\dot{P}_M(V * (\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))\ddot{P}_M(\widetilde{\mathbf{u}_3}\mathbf{v})\, dxdt \Big| \\
&=: \sum_{M>0} \mathfrak{N}_M.
\end{aligned}
$$

We first consider the cases (123): $N_1 \le N_2 \le N_3$ and (213): $N_2 \le N_1 \le N_3$. These cases can be split into two parts, $\max(N_1, N_2) \sim M$ and $N_1 \sim N_2 \gg M$. For both cases we use the embedding lemmas (Lemma 2.3 and Lemma 2.6), Lemma 2.9 with $p_2 = \frac{r}{r-2}$, $\frac{d-\gamma}{2d} < \frac{1}{r} < \frac{d-\alpha}{2d}$, the Sobolev inequality Lemma 2.11, and then Lemma 2.7 with $X = \mathcal{L}^r_\rho L^{r_*}_\theta$, to obtain that

$$\begin{aligned}
\mathfrak{N}_M &\lesssim M^{-(d-\gamma)} \|\dot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^1_t\mathcal{L}^\infty_\rho L^2_\theta} \|\ddot{P}_M(\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L^\infty_t\mathcal{L}^1_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^1_t\mathcal{L}^{\frac{r}{2}}_\rho L^2_\theta} \|\widetilde{\mathbf{u}_3}\mathbf{v}\|_{L^\infty_t\mathcal{L}^1_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_2)^{\frac{d-\alpha}{2}-\frac{d}{r}} N_1^{-\frac{d-\alpha}{2}+\frac{d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} \\
&\quad N_2^{-\frac{d-\alpha}{2}+\frac{d}{r}} \|\mathbf{u}_2\|_{L^2_t\mathcal{L}^r_\rho L^\infty_\theta} \|\mathbf{u}_3\|_{L^\infty_t\mathcal{L}^2_\rho L^\infty_\theta} \|\mathbf{v}\|_{L^\infty_t L^2_x} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_2)^{\frac{d-\alpha}{2}-\frac{d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U^2_\alpha} \prod_{i=2}^{3} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha}) \|\mathbf{v}\|_{V^2_\alpha},
\end{aligned}$$

and hence

$$\sum_{(123)\text{ or }(213)} \mathfrak{N}_M \lesssim (N_1N_2)^{\frac{\gamma-\alpha}{2}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U^2_\alpha} \prod_{i=2}^{3} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha}) \|\mathbf{v}\|_{V^2_\alpha}.$$

Now we consider the cases (132): $N_1 \le N_3 \le N_2$ and (231): $N_2 \le N_3 \le N_1$. These can be split into (i): $N_3 \ll \max(N_1, N_2) \sim M$ or $N_3 \sim \max(N_1, N_2) \sim M$ and (ii): $N_3 \sim N_2 \sim N_1 \gg M$. The first case (i) can be easily treated with the Strichartz estimate (2.1). We consider (ii) separately; $N_1 \le N_2$ and $N_2 \le N_1$.

If $N_1 \le N_2$, then with the same r as above we estimate

$$\begin{aligned}
\mathfrak{N}_M &\lesssim M^{-(d-\gamma)} \|\dot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^2_t\mathcal{L}^2_\rho L^2_\theta} \|\ddot{P}_M(\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L^2_t\mathcal{L}^2_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^2_t\mathcal{L}^{\frac{2r}{r+2}}_\rho L^{\frac{rr_*}{r+r_*}}_\theta} \|\widetilde{\mathbf{u}_3}\mathbf{v}\|_{L^2_t\mathcal{L}^{\frac{2r}{r+2}}_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} \|\mathbf{u}_2\|_{L^\infty_t\mathcal{L}^2_\rho L^r_\theta} \|\mathbf{u}_3\|_{L^2_t\mathcal{L}^r_\rho L^\infty_\theta} \|\mathbf{v}\|_{L^\infty_t L^2_x} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}} N_1^{-\frac{d-\alpha}{2}+\frac{d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^2_t\mathcal{L}^r_\rho L^{r_*}_\theta} \|\mathbf{u}_2\|_{L^\infty_t\mathcal{L}^2_\rho L^r_\theta} \\
&\quad N_3^{-\frac{d-\alpha}{2}+\frac{d}{r}} \|\mathbf{u}_3\|_{L^2_t\mathcal{L}^r_\rho L^\infty_\theta} \|\mathbf{v}\|_{L^\infty_t L^2_x} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_1N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U^2_\alpha} \prod_{i=2}^{3} (\|\mathbf{u}_i\|_{U^2_\alpha} + \|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U^2_\alpha}) \|\mathbf{v}\|_{V^2_\alpha}.
\end{aligned}$$

On the other hand, if $N_2 \le N_1$, then

$$\begin{aligned}
\mathfrak{N}_M &\lesssim M^{-(d-\gamma)} \|\dot{P}_M(\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{L^2_t\mathcal{L}^2_\rho L^2_\theta} \|\ddot{P}_M(\widetilde{\mathbf{u}_3}\mathbf{v})\|_{L^2_t\mathcal{L}^2_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}\|_{L^2_t\mathcal{L}^{\frac{2r}{r+2}}_\rho L^2_\theta} \|\widetilde{\mathbf{u}_3}\mathbf{v}\|_{L^2_t\mathcal{L}^{\frac{2r}{r+2}}_\rho L^2_\theta} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^\infty_t L^2_x} \|\mathbf{u}_2\|_{L^2_t\mathcal{L}^r_\rho L^\infty_\theta} \|\mathbf{u}_3\|_{L^2_t\mathcal{L}^r_\rho L^\infty_\theta} \|\mathbf{v}\|_{L^\infty_t L^2_x} \\
&\lesssim M^{-(d-\gamma)+\frac{2d}{r}} (N_2N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}} \|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{L^\infty_t L^2_x} N_2^{-\frac{d-\alpha}{2}+\frac{d}{r}} \|\mathbf{u}_2\|_{L^\infty_t\mathcal{L}^r_\rho L^\infty_\theta}
\end{aligned}$$

$$N_3^{-\frac{d-\alpha}{2}+\frac{d}{r}}\|\mathbf{u}_3\|_{L_t^2\mathcal{L}_\rho^r L_\theta^\infty}\|\mathbf{v}\|_{L_t^\infty L_x^2}$$
$$\lesssim M^{-(d-\gamma)+\frac{2d}{r}}(N_2N_3)^{\frac{d-\alpha}{2}-\frac{d}{r}}\|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U_\alpha^2}\prod_{i=2}^3(\|\mathbf{u}_i\|_{U_\alpha^2}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U_\alpha^2})\|\mathbf{v}\|_{V_\alpha^2}.$$

By taking $\frac{d-\gamma}{2d}<\frac{1}{r}<\frac{d-\alpha}{2d}$ we get

$$\sum_{(132)\text{ or }(231)}\mathfrak{N}_M\lesssim(\min(N_1,N_2)N_3)^{\frac{\gamma-\alpha}{2}}\|\nabla_{\mathbb{S}}\mathbf{u}_1\|_{U_\alpha^2}\prod_{i=2}^3(\|\mathbf{u}_i\|_{U_\alpha^2}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U_\alpha^2})\|\mathbf{v}\|_{V_\alpha^2}.$$

The final cases (312): $N_3\le N_1\le N_2$ and (321): $N_3\le N_2\le N_1$ are split into the four parts: $(N_3\sim\min(N_1,N_2))$; $(N_3\ll\min(N_1,N_2)\ll\max(N_1,N_2))$; $(N_3\ll N_1\sim N_2$ and $N_3\lesssim M)$; $(M\ll N_3\ll N_1\sim N_2)$. Since each case can be handled by a similar way to the cases (132) and (231), we leave the details to the readers. This completes the proof of (4.2). □

5. Proof of Theorem 1.3

In view of the summation argument in Section 3 and Remark 1, we have only to show the nonlinear estimate, Proposition 4.1 for V satisfying **(H)**. Let us first observe that $\psi_M:=M^{d-\gamma}\mathcal{F}^{-1}(\ddot\beta(\frac{\cdot}{M})\widehat{V})$ is integrable and its integral is independent of M. Here $\gamma=\alpha_+$ or α. In fact, by the hypothesis of V

$$\begin{aligned}\int|\psi_M|&=\int_{|x|\le M^{-1}}|\psi_M|+\int_{|x|>M^{-1}}|\psi_M|\lesssim M^{-1}\||x|^{d-1}\psi_M\|_{L^\infty}\\&\quad+M\||x|^{d+1}\psi_M\|_{L^\infty}\\&\lesssim M^{-1}M^{d-\gamma}\|\nabla_\xi^{d-1}(\ddot\beta(\frac{\xi}{M})\widehat{V}(\xi))\|_{L^1}+MM^{d-\gamma}\|\nabla_\xi^{d+1}(\ddot\beta(\frac{\xi}{M})\widehat{V}(\xi))\|_{L^1}\\&\lesssim 1.\end{aligned}$$

Since $\dot P_M(V*(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))=M^{-(d-\gamma)}\psi_M*(\dot P_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))$, $\|\dot P_M(V*(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))\|_{\mathcal{L}_\rho^rL_\theta^{r_*}}\lesssim M^{-(d-\gamma)}\|\dot P_M(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2})\|_{\mathcal{L}_\rho^rL_\theta^{r_*}}$ for any $1\le r,r_*\le\infty$. And hence,

$$\mathfrak{N}_M\lesssim M^{-(d-\alpha_+)}\|\dot P_M(V*(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))\|_{L_t^q\mathcal{L}_\rho^rL_\theta^{r_*}}\|\ddot P_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\|_{L_t^{q'}\mathcal{L}_\rho^{r'}L_\theta^{r_*'}}$$

for $M\le 1$ and

$$\mathfrak{N}_M\lesssim M^{-(d-\alpha)}\|\dot P_M(V*(\widetilde{\mathbf{u}_1}\widetilde{\mathbf{u}_2}))\|_{L_t^q\mathcal{L}_\rho^rL_\theta^{r_*}}\|\ddot P_M((\nabla_{\mathbb{S}}\widetilde{\mathbf{u}_3})\mathbf{v})\|_{L_t^{q'}\mathcal{L}_\rho^{r'}L_\theta^{r_*'}}$$

for $M>1$. Following the proof of Proposition 4.1, we actually get that

$$\sum_{M\le1}\mathfrak{N}_M\lesssim(N_{\min}N_{\mathrm{med}})^{\frac{\alpha_+-\alpha}{2}}\prod_{i=1}^3(\|\mathbf{u}_i\|_{U_\alpha^2}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U_\alpha^2})$$

and

$$\sum_{M>1}\mathfrak{N}_M\lesssim\ln(1+N_{\min}N_{\mathrm{med}})\prod_{i=1}^3(\|\mathbf{u}_i\|_{U_\alpha^2}+\|\nabla_{\mathbb{S}}\mathbf{u}_i\|_{U_\alpha^2}).$$

So by using the summation technique we get the desired result.

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