J. Korean Math. Soc. **55** (2018), No. 2, pp. 343–372
https://doi.org/10.4134/JKMS.j170220
pISSN: 0304-9914 / eISSN: 2234-3008

GENERATION OF RAY CLASS FIELDS MODULO 2, 3, 4 OR 6 BY USING THE WEBER FUNCTION

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Abstract. Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Let E be an elliptic curve with complex multiplication by $\mathcal{O}_K$, and let h_E be the Weber function on E. Let $N \in \{2, 3, 4, 6\}$. We show that h_E alone when evaluated at a certain N-torsion point on E generates the ray class field of K modulo $N\mathcal{O}_K$. This would be a partial answer to the question raised by Hasse and Ramachandra.

1. Introduction

Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Corresponding to the elliptic curve E in $\mathbb{P}^2(\mathbb{C})$ with parametrization

$$\begin{array}{rcl} \varphi_E \ : \ \mathbb{C}/\mathcal{O}_K & \overset{\sim}{\to} & E \ : \ y^2 = 4x^3 - g_2(\mathcal{O}_K)x - g_3(\mathcal{O}_K) \\ z & \mapsto & [\wp(z;\, \mathcal{O}_K) : \wp'(z;\, \mathcal{O}_K) : 1] \end{array}$$

(see §2), the *Weber function* $h_E : E \to \mathbb{P}^1(\mathbb{C})$ is defined by

$$h_E(x,y) = \begin{cases} (g_2 g_3/\Delta)x & \text{if } K \neq \mathbb{Q}(\sqrt{-1}),\, \mathbb{Q}(\sqrt{-3}), \\ (g_2^2/\Delta)x^2 & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ (g_3/\Delta)x^3 & \text{if } K = \mathbb{Q}(\sqrt{-3}), \end{cases} \tag{1}$$

where $g_2 = g_2(\mathcal{O}_K)$, $g_3 = g_3(\mathcal{O}_K)$ and $\Delta = g_2^3 - 27g_3^2$. This function gives rise to an isomorphism of the quotient variety $E/\mathrm{Aut}(E)$ onto $\mathbb{P}^1(\mathbb{C})$ ([15, Theorem 7 in Chapter 1]).

For a proper nontrivial ideal $\mathfrak{m}$ of $\mathcal{O}_K$, let $K_\mathfrak{m}$ denote the ray class field of K modulo $\mathfrak{m}$. As a consequence of the theory of complex multiplication, Hasse proved in [5] that

$$K_\mathfrak{m} = K\left(j(E),\, h_E(P)\right) \ \text{for some } \mathfrak{m}\text{-torsion point } P \text{ on } E,$$

Received March 30, 2017; Revised August 16, 2017; Accepted September 5, 2017.
2010 *Mathematics Subject Classification.* Primary 11R37; Secondary 11G15, 11G16.
Key words and phrases. class field theory, complex multiplication, Weber function.
The first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (MISP) (2016R1A5A1008055 and 2017R1C1B2010652). The third author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea Government (MISP) (2017R1A2B1006578) and by Hankuk University of Foreign Studies Research Fund of 2017.

where $j(E) = 1728g_2^3/\Delta$ is the j-invariant of E (see also [3, §26]). In his letter to Hecke, Hasse further asked whether every finite abelian extension of K can be generated by a singular value of the Weber function without the j-invariant ([4, p. 91] or [17, p. 105]). He called this a "problem that lies rather deep", and Artin found this question "very interesting". On the other hand, in 1933 and 1936, Sugawara ([23] and [24]) first gave a partial answer to this question.

Recently, Koo et al. ([12] and [13]) showed by applying Kronecker's second limit formula that if $\mathfrak{m} = N\mathcal{O}_K$ for a positive integer N (> 1), then $K_{\mathfrak{m}}$ is generated soly by $h_E(\varphi_E(1/N))$ or $h_E(\varphi_E(2/N))$ except for the four cases where $N = 2, 3, 4, 6$. In this paper, we shall completely resolve these exceptional cases (Remark 4.6 (iv), Theorems 5.1, 5.3, 6.1, 6.5, Corollaries 7.1, 8.1, 8.2, 8.3 and Theorem 8.5) without using L-function arguments. To this end we shall express Fricke functions in terms of Siegel functions, and make use of Shimura's reciprocity law and inequality arguments.

2. Meromorphic modular functions

In this section, we shall introduce basic properties of Fricke functions and Siegel functions.

For a lattice Λ in $\mathbb{C}$, let E_Λ be the elliptic curve in $\mathbb{P}^2(\mathbb{C})$ given by the affine Weierstrass form

$$E_\Lambda \;:\; y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda),$$

where

$$g_2(\Lambda) = 60 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^4} \quad \text{and} \quad g_3(\Lambda) = 140 \sum_{\lambda\in\Lambda\setminus\{0\}} \frac{1}{\lambda^6}.$$

Then, there is a complex analytic isomorphism $\varphi_{E_\Lambda} : \mathbb{C}/\Lambda \to E_\Lambda$ of complex Lie groups defined by

$$\varphi_{E_\Lambda}(z + \Lambda) = [\wp(z;\,\Lambda) : \wp'(z;\,\Lambda) : 1],$$

where

$$\wp(z;\,\Lambda) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left\{ \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right\} \quad (z \in \mathbb{C}) \tag{2}$$

is the Weierstrass $\wp$-function relative to Λ and $\wp' = d\wp/dz$ ([19, Proposition 3.6(b) in Chapter VI]). Note that for $z_1, z_2 \in \mathbb{C}\setminus\Lambda$

$$\wp(z_1;\,\Lambda) = \wp(z_2;\,\Lambda) \iff z_1 \equiv \pm z_2 \pmod{\Lambda}$$

([2, Lemma 10.4]). Furthermore, the j-invariant of the elliptic curve E_Λ is given by

$$j(E_\Lambda) = 1728\frac{g_2(\Lambda)^3}{\Delta(\Lambda)}, \tag{3}$$

where $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$.

Proposition 2.1. *Given a lattice* $\Lambda = [\lambda_1,\, \lambda_2]$ *in* $\mathbb{C}$, *the polynomial*

$$4x^3 - g_2(\Lambda)x - g_3(\Lambda)$$

has three distinct roots $\wp(\lambda_1/2;\, \Lambda)$, $\wp(\lambda_2/2;\, \Lambda)$ *and* $\wp((\lambda_1+\lambda_2)/2;\, \Lambda)$.

Proof. See [19, Proposition 3.6(a) in Chapter VI]. □

Remark 2.2. Thus, $\Delta(\Lambda) = g_2(\Lambda)^3 - 27g_3(\Lambda)^2$, as the discriminant of the polynomial $4x^3 - g_2(\Lambda)x - g_3(\Lambda)$, is always nonzero.

The modular group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm I_2\}$ acts on the complex upper-half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ by fractional linear transformations, namely, for each $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbb{Z})$

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} \quad (\tau \in \mathbb{H}).$$

For a positive integer N, let $\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv I_2 \pmod{N \cdot M_2(\mathbb{Z})}\}$ be the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of level N, which gives rise to the modular curve

$$X(N) = \overline{\Gamma}(N)\backslash\mathbb{H}^*,$$

where $\overline{\Gamma}(N) = \pm\Gamma(N)/\{\pm I_2\}$ and $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\mathrm{i}\infty\}$ ([18, Chapter 1]). If $\mathbb{C}(X(N))$ denotes the field of meromorphic functions on $X(N)$, then it is a Galois extension of $\mathbb{C}(X(1))$ with

$$\mathrm{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1))) \simeq \mathrm{PSL}_2(\mathbb{Z})/\overline{\Gamma}(N) \simeq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$$

([18, Proposition 6.1]). Observe that every function in $\mathbb{C}(X(N))$ has Fourier expansion with respect to $q^{1/N}$, where $q = e^{2\pi \mathrm{i}\tau}$ ([18, §2.1]). Let $\mathcal{F}_N$ be the subfield of $\mathbb{C}(X(N))$ consisting of functions whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$, where $\zeta_N = e^{2\pi \mathrm{i}/N}$. Then, $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1 = \mathbb{Q}(j(\tau))$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\},$$

where $j(\tau) = j([\tau,\, 1])$ is the elliptic modular function ([18, Theorem 6.6 and Proposition 6.9(1)]). Furthermore, $\mathrm{Gal}(\mathbb{C}(X(N))/\mathbb{C}(X(1)))$ can be considered as a subgroup of $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

For $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, the Fricke function $f_{\mathbf{v}}(\tau)$ is defined by

$$f_{\mathbf{v}}(\tau) = \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp(v_1\tau + v_2;\, [\tau,\, 1]) \quad (\tau \in \mathbb{H}), \tag{4}$$

where $g_2(\tau) = g_2([\tau,\, 1])$, $g_3(\tau) = g_3([\tau,\, 1])$ and $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$. And, the Siegel function $g_{\mathbf{v}}(\tau)$ is given by the infinite product expansion

$$\begin{aligned} g_{\mathbf{v}}(\tau) = & -e^{\pi \mathrm{i} v_2(v_1-1)} q^{(1/2)\mathbf{B}_2(v_1)}(1 - q^{v_1}e^{2\pi \mathrm{i} v_2}) \\ & \prod_{n=1}^{\infty}(1 - q^{n+v_1}e^{2\pi \mathrm{i} v_2})(1 - q^{n-v_1}e^{-2\pi \mathrm{i} v_2}) \quad (\tau \in \mathbb{H}), \end{aligned} \tag{5}$$

where $\mathbf{B}_2(x) = x^2 - x + 1/6$ is the Bernoulli second polynomial. We then see that $g_{\mathbf{v}}(\tau)$ has neither zeros nor poles on $\mathbb{H}$.

Proposition 2.3. *Let N be an integer such that $N \geq 2$, and let $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in (1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$.*

(i) *If $\mathbf{u} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ satisfies $\mathbf{u} \equiv \mathbf{v}$ or $-\mathbf{v} \pmod{\mathbb{Z}^2}$, then*

$$f_{\mathbf{u}}(\tau) = f_{\mathbf{v}}(\tau) \quad \textit{and} \quad g_{\mathbf{u}}(\tau)^{12N/\gcd(6,\,N)} = g_{\mathbf{v}}(\tau)^{12N/\gcd(6,\,N)}.$$

(ii) *$f_{\mathbf{v}}(\tau)$ and $g_{\mathbf{v}}(\tau)^{12N/\gcd(6,\,N)}$ belong to $\mathcal{F}_N$.*

(iii) *If $\gamma \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$, then*

$$f_{\mathbf{v}}(\tau)^{\gamma} = f_{\gamma^T\mathbf{v}}(\tau) \quad \textit{and} \quad \left(g_{\mathbf{v}}(\tau)^{12N/\gcd(6,\,N)}\right)^{\gamma} = g_{\gamma^T\mathbf{v}}(\tau)^{12N/\gcd(6,\,N)},$$

where γ^T means the transpose of γ.

(iv) *If $\mathbf{v}$ satisfies $\mathbf{v} \equiv \left[\begin{smallmatrix} v_1 \\ dv_2 \end{smallmatrix}\right]$ or $-\left[\begin{smallmatrix} v_1 \\ dv_2 \end{smallmatrix}\right] \pmod{\mathbb{Z}^2}$ for all $d \in (\mathbb{Z}/N\mathbb{Z})^*$, then $f_{\mathbf{v}}(\tau)$ and $g_{\mathbf{v}}(\tau)^{12N/\gcd(6,\,N)}$ have rational Fourier coefficients.*

Proof. See [18, (6.1.4), (6.1.6), Proposition 6.1 and Theorem 6.6] and [14, Proposition 1.3 in Chapter 2 and Theorems 5.2 and 5.3 in Chapter 3]. □

For $r \in \mathbb{R}$, let $\langle r \rangle$ be the fractional part of r in the interval $[0, 1)$.

Proposition 2.4. *Let $\mathbf{v} = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right] \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$.*

(i) *We have the q-order formula*

$$\mathrm{ord}_q\; g_{\mathbf{v}}(\tau) = \frac{1}{2}\mathbf{B}_2(\langle v_1 \rangle).$$

(ii) *If $\mathbf{u} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ with $\mathbf{u} \not\equiv \pm\mathbf{v} \pmod{\mathbb{Z}^2}$, then*

$$(f_{\mathbf{u}}(\tau) - f_{\mathbf{v}}(\tau))^6 = \frac{j(\tau)^2(j(\tau)-1728)^3}{2^{30}\cdot 3^{24}} \frac{g_{\mathbf{u}+\mathbf{v}}(\tau)^6 g_{\mathbf{u}-\mathbf{v}}(\tau)^6}{g_{\mathbf{u}}(\tau)^{12} g_{\mathbf{v}}(\tau)^{12}}.$$

Proof. (i) See [14, p. 31].

(ii) See [14, p. 51]. □

3. Fricke and Siegel functions of level 2, 3, 4 or 6

We shall develop several lemmas on Fricke and Siegel functions of low level for later use.

For a positive integer N, let

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \pmod{N \cdot M_2(\mathbb{Z})} \right\},$$

and let $X_1(N) = \overline{\Gamma}_1(N)\backslash\mathbb{H}^*$ be the corresponding modular curve with $\overline{\Gamma}_1(N) = \pm\Gamma_1(N)/\{\pm I_2\}$. Let $\mathbb{C}(X_1(N))$ be the field of meromorphic functions on $X_1(N)$, and let $\mathbb{Q}(X_1(N))$ be its subfield consisting of functions with rational Fourier coefficients. Note that $X_1(N)$ has genus zero if and only if $N \in \{1, 2, \ldots, 10, 12\}$ ([9, Theorem 1]).

Proposition 3.1. *If $N \in \{2, 3\}$, then we have*

$$\mathbb{Q}(X_1(N)) = \mathbb{Q}\left(g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}\right).$$

Proof. For any $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1(N)$, we derive that

$$\begin{aligned}
\left(g^{12}_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]} \circ \gamma\right)(\tau) &= \left(g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}\right)^{\gamma} \quad \text{by Proposition 2.3(ii)} \\
&= g_{\gamma^T\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12} \quad \text{by Proposition 2.3(iii)} \\
&= g_{\left[\begin{smallmatrix} c/N \\ d/N \end{smallmatrix}\right]}(\tau)^{12} \\
&= g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12} \text{ by Proposition 2.3(i) and the fact } \gamma \in \Gamma_1(N).
\end{aligned}$$

This shows that $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ belongs to $\mathbb{C}(X_1(N))$. And, $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ has rational Fourier coefficients by Proposition 2.3(i) and (iv).

The modular curve $X_1(N)$ has two inequivalent cusps $\mathrm{i}\infty$ and 0 of width 1 and N, respectively ([10, Lemma 3]). We then see by Proposition 2.4(i) that the order of $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ at the cusp $\mathrm{i}\infty$ is

$$\text{width of } \mathrm{i}\infty \text{ on } X_1(N) \cdot \mathrm{ord}_q\, g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12} = 1 \cdot \left(12 \cdot \frac{1}{2}\mathbf{B}_2(0)\right) = 1.$$

Since $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ has neither zeros nor pole on $\mathbb{H}$, its order at the cusp 0 should be -1. This implies that $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ gives rise to an isomorphism

$$\begin{aligned}
X_1(N) &\to \mathbb{P}^1(\mathbb{C}) \\
\tau &\mapsto \left[g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12} : 1\right],
\end{aligned}$$

from which it follows that $\mathbb{C}(X_1(N)) = \mathbb{C}\left(g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}\right)$. Since $g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}$ has rational Fourier coefficients, we conclude by [11, Lemma 4.1] that

$$\mathbb{Q}(X_1(N)) = \mathbb{Q}\left(g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau)^{12}\right).$$

□

Lemma 3.2. *Let $\tau \in \mathbb{H}$ such that both $g_2(\tau)$ and $g_3(\tau)$ are nonzero. Then, the polynomial*

$$2916 \cdot 1728^3 x^3 - 27 \cdot 1728 j(\tau)(j(\tau) - 1728)x - j(\tau)(j(\tau) - 1728)^2$$

has three distinct roots $f_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)$, $f_{\left[\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right]}(\tau)$ and $f_{\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]}(\tau)$.

Proof. By Proposition 2.1 and the definition (4), the polynomial

$$4\left(\frac{\Delta(\tau)}{g_2(\tau)g_3(\tau)}x\right)^3 - g_2(\tau)\left(\frac{\Delta(\tau)}{g_2(\tau)g_3(\tau)}x\right) - g_3(\tau)$$

has three distinct roots $f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)$, $f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)$ and $f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau)$. By using the definition (3), one can express the above polynomial as

$$\begin{aligned}
&\left(\tfrac{\Delta(\tau)}{g_2(\tau)g_3(\tau)}\right)^3\left(4x^3 - \tfrac{g_2(\tau)^3g_3(\tau)^2}{\Delta(\tau)^2}x - \tfrac{g_2(\tau)^3g_3(\tau)^4}{\Delta(\tau)^3}\right)\\
=\ &\left(\tfrac{\Delta(\tau)}{g_2(\tau)g_3(\tau)}\right)^3\left\{4x^3 - \tfrac{j(\tau)}{1728}\left(\tfrac{j(\tau)}{27\cdot 1728} - \tfrac{1}{27}\right)x - \tfrac{j(\tau)}{1728}\left(\tfrac{j(\tau)}{27\cdot 1728} - \tfrac{1}{27}\right)^2\right\}\\
=\ &\left(\tfrac{\Delta(\tau)}{g_2(\tau)g_3(\tau)}\right)^3 \tfrac{1}{27^2\cdot 1728^3}\\
&\times\left\{2916\cdot 1728^3x^3 - 27\cdot 1728 j(\tau)(j(\tau)-1728)x - j(\tau)(j(\tau)-1728)^2\right\}.
\end{aligned}$$

This proves the lemma. □

Remark 3.3. Let $\tau \in \mathbb{H}$. Then we get that

$$\begin{aligned}
g_2(\tau) = 0 &\iff j(\tau) = 0 \iff \tau = \gamma(\zeta_3) \text{ for some } \gamma \in \mathrm{PSL}_2(\mathbb{Z}),\\
g_3(\tau) = 0 &\iff j(\tau) = 1728 \iff \tau = \gamma(\mathrm{i}) \text{ for some } \gamma \in \mathrm{PSL}_2(\mathbb{Z})
\end{aligned}$$

([2, Theorem 10.9 and p. 219]).

Lemma 3.4. *We have the relation*

$$j(\tau) = \frac{\left(g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12} + 16\right)^3}{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}}.$$

Proof. See [2, Theorem 12.17]. □

Remark 3.5. It follows that

$$j(\tau)^2(j(\tau)-1728)^3 = \frac{\left(g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}+16\right)^6\left(g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}+64\right)^3\left(g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}-8\right)^6}{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{60}}.$$

Lemma 3.6. *We establish the expression*

$$f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau) = -\frac{1}{31104}\frac{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{36} + 72g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{24} + 384g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12} - 8192}{g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}}.$$

Proof. For convenience, set

$$f = f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau), \quad g = g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12} \quad \text{and} \quad j = j(\tau).$$

In a similar way to the first part of the proof of Proposition 3.1, one can show that f belongs to $\mathbb{Q}(X_1(2))$. Furthermore, f satisfies

$$2916 \cdot 1728^3 f^3 - 27 \cdot 1728 j(j-1728) f - j(j-1728)^2 = 0$$

by Lemma 3.2. It then follows from Lemma 3.4 that

$$\begin{aligned}&2916 \cdot 1728^3 g^3 f^3 - 27 \cdot 1728 g(g+16)^3\{(g+16)^3 - 1728g\} f \\ &- (g+16)^3\{(g+16)^3 - 1728g\}^2 = 0,\end{aligned}$$

which can be factored into

$$\begin{aligned}&(31104gf + g^3 + 72g^2 + 384g - 8192)\\ &\times\{483729408g^2f^2 + (-15552g^4 - 1119744g^3 - 5971968g^2 + 127401984g)f\\ &-g^6 - 72g^5 - 192g^4 + 20992g^3 - 1966080g + 8388608\} = 0.\end{aligned} \tag{6}$$

Since $\mathbb{Q}(X_1(2)) = \mathbb{Q}(g)$ by Proposition 3.1 and f lies in $\mathbb{Q}(X_1(2))$, we get $\deg(f, \mathbb{Q}(g)) = 1$. Note that the second factor in the left side of (6) is irreducible as a quadratic polynomial in f over $\mathbb{Q}(g)$ because its discriminant

$$\begin{aligned}&(-15552g^4 - 1119744g^3 - 5971968g^2 + 127401984g)^2\\ &- 4(483729408g^2)(-g^6 - 72g^5 - 192g^4 + 20992g^3 - 1966080g + 8388608)\\ =\ & 2176782336g^8 + 174142586880g^7 + \cdots + 2282521714753536g^3\end{aligned}$$

is not a square in $\mathbb{Q}(g)$. Thus we must have

$$31104gf + g^3 + 72g^2 + 384g - 8192 = 0.$$

Therefore, we achieve

$$f = -\frac{1}{31104}\frac{g^3 + 72g^2 + 384g - 8192}{g}. \qquad \square$$

Lemma 3.7. *We have*

$$f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau) = 0.$$

Proof. Let $T = \{(a, b) \in \mathbb{Z}^2 \mid 0 \le a, b \le 2\}$. For each $(a, b) \in T \setminus \{(0, 0)\}$, we attain by the definition (2) that

$$\begin{aligned}&\wp\left(\frac{a}{3}\tau + \frac{b}{3}; [\tau, 1]\right)\\ &= \sum_{(m,n)\in\mathbb{Z}^2} \frac{9}{\{(a-3m)\tau + (b-3n)\}^2} - \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^2}\\ &= \sum_{\substack{(k,\ell)\in\mathbb{Z}^2\setminus\{(0,0)\} \text{ such that}\\ k\equiv a \pmod 3,\, \ell\equiv b \pmod 3}} \frac{9}{(k\tau+\ell)^2} - \sum_{\substack{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\} \text{ such that}\\ m\equiv n\equiv 0 \pmod 3}} \frac{9}{(m\tau+n)^2}.\end{aligned}$$

We then derive that

$$\begin{aligned}
&\sum_{(a,\,b)\in T\setminus\{(0,\,0)\}} \wp\left(\frac{a}{3}\tau+\frac{b}{3};\,[\tau,\,1]\right)\\
&= \sum_{(a,\,b)\in T\setminus\{(0,\,0)\}} \left(\sum_{\substack{(k,\,\ell)\in\mathbb{Z}^2\setminus\{(0,\,0)\} \text{ such that}\\ k\equiv a \pmod 3,\, \ell\equiv b \pmod 3}} \frac{9}{(k\tau+\ell)^2} \right)\\
&\quad - 8 \sum_{\substack{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,\,0)\} \text{ such that}\\ m\equiv n\equiv 0 \pmod 3}} \frac{9}{(m\tau+n)^2}\\
&= \sum_{(a,\,b)\in T} \left(\sum_{\substack{(k,\,\ell)\in\mathbb{Z}^2\setminus\{(0,\,0)\} \text{ such that}\\ k\equiv a \pmod 3,\, \ell\equiv b \pmod 3}} \frac{9}{(k\tau+\ell)^2} \right)\\
&\quad - 9 \sum_{\substack{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,\,0)\} \text{ such that}\\ m\equiv n\equiv 0 \pmod 3}} \frac{9}{(m\tau+n)^2}\\
&= \sum_{(k,\,\ell)\in\mathbb{Z}^2\setminus\{(0,\,0)\}} \frac{9}{(k\tau+\ell)^2} - 9 \sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,\,0)\}} \frac{1}{(m\tau+n)^2}\\
&= 0.
\end{aligned}$$

Thus we obtain by Proposition 2.3(i) and the definition (4) that

$$\begin{aligned}
&f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau)\\
&= \frac{1}{2}\sum_{(a,\,b)\in T\setminus\{(0,\,0)\}} f_{\left[\begin{smallmatrix}a/3\\b/3\end{smallmatrix}\right]}(\tau)\\
&= \frac{g_2(\tau)g_3(\tau)}{2\Delta(\tau)} \sum_{(a,\,b)\in T\setminus\{(0,\,0)\}} \wp\left(\frac{a}{3}\tau+\frac{b}{3};\,[\tau,\,1]\right)\\
&= 0.
\end{aligned}$$

□

Lemma 3.8. *We have*

$$f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)=\frac{1}{2}\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}2/4\\1/4\end{smallmatrix}\right]}(\tau)\right).$$

Proof. Let $T=\{(0,\,1),\,(0,\,3),\,(2,\,1),\,(2,\,3)\}$. We attain by Proposition 2.3(i) and the definitions (2) and (4) that

$$\frac{1}{2}\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}2/4\\1/4\end{smallmatrix}\right]}(\tau)\right)$$

$$
\begin{aligned}
&= \frac{1}{4}\sum_{(a,\,b)\in T} f_{\left[\begin{smallmatrix} a/4 \\ b/4 \end{smallmatrix}\right]}(\tau) \\
&= \frac{g_2(\tau)g_3(\tau)}{4\Delta(\tau)}\sum_{(a,\,b)\in T} \wp\left(\frac{a}{4}\tau+\frac{b}{4};\,[\tau,\,1]\right) \\
&= \frac{g_2(\tau)g_3(\tau)}{4\Delta(\tau)}\sum_{(a,\,b)\in T}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{16}{\{(a-4m)\tau+(b-4n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right) \\
&= \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{4}{\{-2m\tau+(1-2n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right) \\
&= \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)}\wp\left(\frac{1}{2};\,[\tau,\,1]\right) \\
&= f_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau).
\end{aligned}
$$

□

Lemma 3.9. *We have*

$$
f_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)=\frac{1}{4}\left(f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 1/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 3/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 5/6 \end{smallmatrix}\right]}(\tau)\right).
$$

Proof. Let $T=\{(0,\,1),\,(0,\,3),\,(0,\,5),\,(2,\,1),\,(2,\,3),\,(2,\,5),\,(4,\,1),\,(4,\,3),\,(4,\,5)\}$. We derive by Proposition 2.3(i) and the definitions (2) and (4) that

$$
\begin{aligned}
&\frac{1}{4}\left(f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 1/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 3/6 \end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix} 2/6 \\ 5/6 \end{smallmatrix}\right]}(\tau)\right) \\
&= \frac{1}{8}\sum_{(a,\,b)\in T} f_{\left[\begin{smallmatrix} a/6 \\ b/6 \end{smallmatrix}\right]}(\tau)-\frac{1}{8}f_{\left[\begin{smallmatrix} 0 \\ 3/6 \end{smallmatrix}\right]}(\tau) \\
&= \frac{g_2(\tau)g_3(\tau)}{8\Delta(\tau)}\sum_{(a,\,b)\in T} \wp\left(\frac{a}{6}\tau+\frac{b}{6};\,[\tau,\,1]\right)-\frac{1}{8}f_{\left[\begin{smallmatrix} 0 \\ 3/6 \end{smallmatrix}\right]}(\tau) \\
&= \frac{g_2(\tau)g_3(\tau)}{8\Delta(\tau)}\sum_{(a,\,b)\in T}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{36}{\{(a-6m)\tau+(b-6n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right) \\
&\quad -\frac{1}{8}f_{\left[\begin{smallmatrix} 0 \\ 3/6 \end{smallmatrix}\right]}(\tau) \\
&= \frac{9g_2(\tau)g_3(\tau)}{8\Delta(\tau)}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{4}{\{-2m\tau+(1-2n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right) \\
&\quad -\frac{1}{8}f_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau) \\
&= \frac{9g_2(\tau)g_3(\tau)}{8\Delta(\tau)}\wp\left(\frac{1}{2};\,[\tau,\,1]\right)-\frac{1}{8}f_{\left[\begin{smallmatrix} 0 \\ 1/2 \end{smallmatrix}\right]}(\tau)
\end{aligned}
$$

$= f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau).$ □

Lemma 3.10. *We have*

$$f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)=\frac{1}{3}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}3/6\\1/6\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}3/6\\4/6\end{smallmatrix}\right]}(\tau)\right).$$

Proof. Let $T = \{(0,\,1),\,(0,\,4),\,(3,\,1),\,(3,\,4)\}$. We achieve by Proposition 2.3(i) and the definitions (2) and (4) that

$$\begin{aligned}
&\frac{1}{3}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}3/6\\1/6\end{smallmatrix}\right]}(\tau)+f_{\left[\begin{smallmatrix}3/6\\4/6\end{smallmatrix}\right]}(\tau)\right)\\
&=\frac{1}{3}\sum_{(a,\,b)\in T} f_{\left[\begin{smallmatrix}a/6\\b/6\end{smallmatrix}\right]}(\tau)-\frac{1}{3}f_{\left[\begin{smallmatrix}0\\4/6\end{smallmatrix}\right]}(\tau)\\
&=\frac{g_2(\tau)g_3(\tau)}{3\Delta(\tau)}\sum_{(a,\,b)\in T}\wp\left(\frac{a}{6}\tau+\frac{b}{6};\,[\tau,\,1]\right)-\frac{1}{3}f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)\\
&=\frac{g_2(\tau)g_3(\tau)}{3\Delta(\tau)}\sum_{(a,\,b)\in T}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{36}{\{(a-6m)\tau+(b-6n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right)\\
&\quad-\frac{1}{3}f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)\\
&=\frac{4g_2(\tau)g_3(\tau)}{3\Delta(\tau)}\left(\sum_{(m,\,n)\in\mathbb{Z}^2}\frac{9}{\{-3m\tau+(1-3n)\}^2}-\sum_{(m,\,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(m\tau+n)^2}\right)\\
&\quad-\frac{1}{3}f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)\\
&=\frac{4g_2(\tau)g_3(\tau)}{3\Delta(\tau)}\wp\left(\frac{1}{3};\,[\tau,\,1]\right)-\frac{1}{3}f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)\\
&=f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau).
\end{aligned}$$

□

Lemma 3.11. *We have the following identities*:

(i) $g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau)^{12}=-2^{12}$.

(ii) $g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau)^{12}=3^{6}$.

Proof. For convenience, let

$$\begin{aligned}
G_2(\tau)&=g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau)^{12},\\
G_3(\tau)&=g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau)^{12}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau)^{12}.
\end{aligned}$$

Let $N \in \{2, 3\}$. By the definition (5), the leading coefficient of $G_N(\tau)$ is given by

$$\begin{cases} -2^{12} & \text{if } N = 2, \\ 3^6 & \text{if } N = 3. \end{cases}$$

One can readily show by Proposition 2.3(i)–(iii) that $G_N(\tau)$ is invariant under the action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$, which implies that $G_N(\tau)$ belongs to $\mathcal{F}_1$. Since the width of $\mathrm{i}\infty$ on the modular curve $X(1)$ is 1, we see by Proposition 2.4(i) that

$$\begin{aligned} \mathrm{ord}_{\mathrm{i}\infty}\, G_N(\tau) &= \mathrm{ord}_q\, G_N(\tau) \\ &= \begin{cases} \left\{12 \cdot \dfrac{1}{2}(\mathbf{B}_2(0) + 2\mathbf{B}_2(1/2))\right\} & \text{if } N = 2, \\ \left\{12 \cdot \dfrac{1}{2}(\mathbf{B}_2(0) + 3\mathbf{B}_2(1/3))\right\} & \text{if } N = 3 \end{cases} \\ &= 0. \end{aligned}$$

Thus $G_N(\tau)$ is holomorphic functions on $X(1)$; and hence it is a constant, namely,

$$G_2(\tau) = -2^{12} \quad \text{and} \quad G_3(\tau) = 3^6. \qquad \square$$

4. Class fields over imaginary quadratic fields

In this section, we shall briefly review an explicit version of Shimura's reciprocity law.

Let K be an imaginary quadratic field of discriminant d_K, and let $\mathcal{O}_K$ be its ring of integers. For a nontrivial ideal $\mathfrak{m}$ of $\mathcal{O}_K$, let $K_\mathfrak{m}$ denote the ray class field of K modulo $\mathfrak{m}$. In particular, $K_{\mathcal{O}_K}$ becomes the maximal unramified abelian extension of K, called the Hilbert class field of K and denoted by H_K. For more details, we refer to [2, §5 and §8] or [7, Chapter V].

Lemma 4.1. *We have the degree formula*

$$[K_\mathfrak{m} : H_K] = \frac{\phi(\mathfrak{m})\omega(\mathfrak{m})}{\omega_K},$$

where ϕ is the multiplicative Euler function for ideals given by

$$\phi(\mathfrak{p}^n) = (\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p}) - 1)\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{p})^{n-1}$$

for each prime ideal power $\mathfrak{p}^n$ $(n \geq 1)$, $\omega(\mathfrak{m})$ is the number of units in $\mathcal{O}_K$ which are congruent to 1 *modulo $\mathfrak{m}$, and $\omega_K = \omega(\mathcal{O}_K)$.*

Proof. See [16, Theorem 1 in Chapter VI]. $\square$

Throughout this section, we let N be a positive integer and $\mathfrak{m} = N\mathcal{O}_K$. Further, we fix an element τ_K of $\mathbb{H}$ as

$$\tau_K = \begin{cases} \sqrt{d_K}/2 & \text{if } d_K \equiv 0 \pmod 4, \\ (-1 + \sqrt{d_K})/2 & \text{if } d_K \equiv 1 \pmod 4; \end{cases} \tag{7}$$

and hence $\mathcal{O}_K = \mathbb{Z}[\tau_K]$ ([2, p. 103]).

Proposition 4.2. *Let $E = E_{\mathcal{O}_K}$ be the elliptic curve with parametrization $\varphi_E : \mathbb{C}/\mathcal{O}_K \to E$ given in §2, and let h_E be the Weber function on E defined in* (1).

(i) $H_K = K\left(j(E)\right) = K\left(j(\tau_K)\right)$.
(ii) $K_{\mathfrak{m}} = K\left(f(\tau_K) \mid f \in \mathcal{F}_N \text{ is finite at } \tau_K\right)$.
(iii) *If* $N \geq 2$, *then* $K_{\mathfrak{m}} = H_K\left(h_E(\varphi_E(1/N))\right)$.
(iv) *If* $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ *and* $N \geq 2$, *then*

$$K_{\mathfrak{m}} = K\left(g_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K)^{12Nn/\gcd(6,\,N)}\right)$$

for any nonzero integer n.

Proof. (i) See [15, Theorem 1 in Chapter 10].
(ii) See [15, Corollary to Theorem 2 in Chapter 10].
(iii) See [15, Corollary to Theorem 7 in Chapter 10].
(iv) See [8, Theorem 4.2]. □

Remark 4.3. If $K \neq \mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3})$ and $N \geq 2$, then we get by the definitions (1) and (4) that

$$h_E(\varphi_E(1/N)) = f_{\left[\begin{smallmatrix} 0 \\ 1/N \end{smallmatrix}\right]}(\tau_K).$$

Let $\min(\tau_K, \mathbb{Q}) = x^2 + bx + c \in \mathbb{Z}[x]$, and let

$$W_{K,\,N} = \left\{ \begin{bmatrix} t - bs & -cs \\ s & t \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \mid t,\, s \in (\mathbb{Z}/N\mathbb{Z}) \right\}$$

which is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Proposition 4.4 (Shimura's reciprocity law)**.** *There is a surjection*

$$\begin{array}{rcl} W_{K,\,N} & \to & \mathrm{Gal}(K_{\mathfrak{m}}/H_K) \\ \gamma & \mapsto & \left(f(\tau_K) \mapsto f^{\gamma}(\tau_K) \mid f \in \mathcal{F}_N \text{ is finite at } \tau_K\right) \end{array}$$

whose kernel is

$$\begin{cases} \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \pm\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-1}), \\ \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \pm\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},\ \pm\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\} & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} & \text{if } K \neq \mathbb{Q}(\sqrt{-1}),\, \mathbb{Q}(\sqrt{-3}). \end{cases}$$

Proof. See [22, §3]. □

Let $\mathrm{C}(d_K)$ be the form class group of properly equivalence classes of primitive positive definite quadratic forms of discriminant d_K under Dirichlet composition. See [2, Theorem 3.9] for the details. Note that every element of $\mathrm{C}(d_K)$

is uniquely represented by a reduced form $Q = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]$ characterized by the condition

$$(-a < b \le a < c \text{ or } 0 \le b \le a = c) \quad \text{and} \quad b^2 - 4ac = d_K \tag{8}$$

([2, Theorem 2.8]). For each reduced form $Q = ax^2 + bxy + cy^2$, let

$$\tau_Q = \frac{-b + \sqrt{d_K}}{2a} \in \mathbb{H}. \tag{9}$$

If $d_K \equiv 0 \pmod 4$, then for each prime p we set

$$\gamma_{Q,\,p} = \begin{cases} \begin{bmatrix} a & b/2 \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} -b/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a - b/2 & -c - b/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases} \tag{10}$$

Otherwise, if $d_K \equiv 1 \pmod 4$, then we set

$$\gamma_{Q,\,p} = \begin{cases} \begin{bmatrix} a & (b-1)/2 \\ 0 & 1 \end{bmatrix} & \text{if } p \nmid a, \\ \begin{bmatrix} -(b+1)/2 & -c \\ 1 & 0 \end{bmatrix} & \text{if } p \mid a \text{ and } p \nmid c, \\ \begin{bmatrix} -a - (b+1)/2 & -c + (1-b)/2 \\ 1 & -1 \end{bmatrix} & \text{if } p \mid a \text{ and } p \mid c. \end{cases} \tag{11}$$

Then, there exists a unique matrix r_Q in $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying $r_Q \equiv r_{Q,\,p} \pmod{N \cdot \mathbb{Z}_p}$ for all primes p by the Chinese remainder theorem.

Proposition 4.5 (Shimura's reciprocity law)**.** *We have an injective map*

$$\begin{array}{ccl} \mathrm{C}(d_K) & \to & \mathrm{Gal}(K_{\mathfrak{m}}/K) \\ Q^{-1} & \mapsto & \left(f(\tau_K) \mapsto f^{\gamma_Q}(\tau_Q) \mid f \in \mathcal{F}_N \text{ is finte at } \tau_K\right). \end{array}$$

Here, Q^{-1} is a reduced form which is the inverse of Q. And, the restriction to H_K followed by the above map gives rise to an isomorphism of $\mathrm{C}(d_K)$ *onto* $\mathrm{Gal}(H_K/K)$*.*

Proof. See [22, §6]. □

Remark 4.6. (i) The identity of $\mathrm{C}(d_K)$ is represented by the reduced form

$$\begin{cases} x^2 - \dfrac{d_K}{4}y^2 & \text{if } d_K \equiv 0 \pmod 4, \\ x^2 + xy + \dfrac{1 - d_K}{4}y^2 & \text{if } d_K \equiv 1 \pmod 4 \end{cases}$$

([2, Theorem 3.9]). One can readily show by the definitions (7), (9), (10) and (11) that the identity of $\mathrm{C}(d_K)$ corresponds to the identity of $\mathrm{Gal}(K_{\mathfrak{m}}/K)$ via the map given in Proposition 4.5.

(ii) Let $Q = ax^2 + bxy + cy^2$ be a reduced form of discriminant d_K. Then, the condition (8) implies

$$a \leq \sqrt{\frac{|d_K|}{3}}$$

([2, p. 29]). Moreover, if Q is a nonidentity, then we get $a \geq 2$.

(iii) Let h_K be the class number of K. It is well known that

$$h_K = 1 \iff d_K = -3,\ -4,\ -7,\ -8,\ -11,\ -19,\ -43,\ -67,\ -163$$

due to Heegner ([6]), Baker ([1]) and Stark ([20] and [21]).

(iv) Let $N \geq 2$. If $K = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, then we attain by (iii) and Proposition 4.2(iii) that $H_K = K$ and

$$K_\mathfrak{m} = K(h_E(\varphi_E(1/N))).$$

5. Generation of ray class fields modulo 2

We are ready to prove our main theorems. Henceforth we assume that K is an imaginary quadratic field different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$. Furthermore, we let $E = E_{\mathcal{O}_K}$ be the elliptic curve with parametrization $\varphi_E : \mathbb{C}/\mathcal{O}_K \to E$ stated in §2, and let h_E be the Weber function on E.

Let $\mathfrak{m} = 2\mathcal{O}_K$. By Proposition 4.4 we have

$$\begin{aligned} \mathrm{Gal}(K_\mathfrak{m}/H_K) &\simeq W_{K,2}/\{\pm I_2\} \\ &= \begin{cases} \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}\right\} & \text{if } d_K \equiv 0 \pmod 8, \\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\} & \text{if } d_K \equiv 4 \pmod 8, \\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}\right\} & \text{if } d_K \equiv 5 \pmod 8, \\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} & \text{if } d_K \equiv 1 \pmod 8. \end{cases} \end{aligned} \tag{12}$$

Let $E = E_{\mathcal{O}_K}$ be the elliptic curve with parametrization $\varphi_E : \mathbb{C}/\mathcal{O}_K \to E$ stated in §2, and let h_E be the Weber function on E.

Theorem 5.1. *Suppose that* 2 *does not split in* K. *Then,* $h_E(\varphi_E(1/2))$ *generates* $K_\mathfrak{m}$ *over* K.

Proof. Since 2 is ramified or inert in K, we get $d_K \equiv 0 \pmod 4$ or $d_K \equiv 5 \pmod 8$. And, it follows from (12) and Proposition 2.3(iii) that the Galois conjugates of $f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)$ over H_K are given by

$$\begin{cases} f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 0 \pmod 8,\\ f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 4 \pmod 8,\\ f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 5 \pmod 8.\end{cases}$$

Let

$$F = K\left(h_E(\varphi_E(1/2))\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)\right)$$

which is a subfield of $K_{\mathfrak{m}}$ by Proposition 4.2(ii). Now that F is an abelian extension of K, it contains all Galois conjugates of $f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)$ over K. Furthermore, since

$$f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K) = 0$$

by Lemma 3.2, F contains all three values $f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)$, $f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K)$ and $f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K)$. Thus we achieve by Lemma 3.2 that

$$\begin{aligned} F \ni & \frac{f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K)f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K)}{f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}(\tau_K)f_{\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right]}(\tau_K)}\\ &= \frac{j(\tau_K)(j(\tau_K)-1728)^2}{-27\cdot 1728 j(\tau_K)(j(\tau_K)-1728)}\\ &= \frac{-j(\tau_K)+1728}{27\cdot 1728}. \end{aligned}$$

Therefore, we conclude by Proposition 4.2(i) and (iii) that $F = K_{\mathfrak{m}}$. This proves the theorem. □

Lemma 5.2. *Let* $u + v\mathrm{i} \in \mathbb{H}$, *and let* $t = e^{-2\pi v}$.

(i) $\left|g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(u+v\mathrm{i})\right| \leq 2t^{1/12}e^{-2+2/(1-t)}$.

(ii) $\left|g_{\left[\begin{smallmatrix}1/2\\s\end{smallmatrix}\right]}(u+v\mathrm{i})\right| \leq t^{-1/24}e^{t^{1/2}(1+2/(1-t))}$ *for any* $s \in \mathbb{Q}$.

Proof. (i) We derive that

$$\begin{aligned} \left|g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(u+v\mathrm{i})\right| &\leq 2t^{(1/2)\mathbf{B}_2(0)}\prod_{n=1}^{\infty}(1+t^n)^2 \quad \text{by the definition (5)}\\ &\leq 2t^{1/12}\prod_{n=1}^{\infty}e^{2t^n} \quad \text{because } 1+x<e^x \text{ for all } x>0 \end{aligned}$$

$$= 2t^{1/12}e^{-2+2/(1-t)}.$$

(ii) We see that

$$\begin{aligned}
&\left|g_{\left[\begin{smallmatrix}1/2\\ s\end{smallmatrix}\right]}(u+v\mathrm{i})\right|\\
&\leq t^{(1/2)\mathbf{B}_2(1/2)}(1+t^{1/2})\prod_{n=1}^{\infty}(1+t^{n+1/2})(1+t^{n-1/2}) \quad \text{by the definition (5)}\\
&\leq t^{-1/24}(1+t^{1/2})\prod_{n=1}^{\infty}(1+t^{n-1/2})^2\\
&\leq t^{-1/24}e^{t^{1/2}}\prod_{n=1}^{\infty}e^{2t^{n-1/2}} \quad \text{since } 1+x<e^x \text{ for all } x>0\\
&= t^{-1/24}e^{t^{1/2}(1+2/(1-t))}.
\end{aligned}$$

□

Theorem 5.3. *If 2 splits in K, then $h_E(\varphi_E(1/2))$ generates $K_{\mathfrak{m}}$ over K.*

Proof. Since 2 splits in K, we have $d_K \equiv 1 \pmod 8$ and $K_{\mathfrak{m}} = H_K$ by Lemma 4.1 or (12). If $h_K = 1$, then the theorem is obvious. So, we let $h_K \geq 2$, and hence $d_K \leq -15$ by Remark 4.6(iii).

Suppose on the contrary that $h_E(\varphi_E(1/2)) = f_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K)$ does not generate H_K over K. Then, by Proposition 4.5, there is a nonidentity reduced form $Q = ax^2 + bxy + cy^2$ of discriminant d_K which leaves $f_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K)$ fixed. Let

$$f = f_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K), \quad g = g_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau_K)^{12} \quad \text{and} \quad h = g^{Q^{-1}}.$$

Then we get by Propositions 4.5 and 2.3(iii) that

$$h = \left(g_{\left[\begin{smallmatrix}0\\ 1/2\end{smallmatrix}\right]}(\tau)^{12}\right)^{\gamma_Q}(\tau_Q) = g_{\left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right]}\left(\frac{-b+\sqrt{d_K}}{2a}\right)^{12} \quad \text{for some } \begin{bmatrix}r\\ s\end{bmatrix} \in (1/2)\mathbb{Z}^2\setminus\mathbb{Z}^2.$$

Now, we establish that

$$\begin{aligned}
|g| &\leq 2^{12}e^{-\pi\sqrt{|d_K|}}e^{-24+24/(1-e^{-\pi\sqrt{|d_K|}})} \quad \text{by Lemma 5.2(i)}\\
&\leq 2^{12}e^{-\pi\sqrt{|d_K|}}e^{-24+24/(1-e^{-\pi\sqrt{15}})} \quad \text{by the fact } d_K \leq -15\\
&< 4097e^{-\pi\sqrt{|d_K|}}
\end{aligned}$$

and

$$\begin{aligned}
|h| &\leq \begin{cases} 2^{12}e^{-\pi\sqrt{|d_K|}/a}e^{-24+24/(1-e^{-\pi\sqrt{|d_K|}/a})} & \text{if } \langle r\rangle = 0,\\ e^{\pi\sqrt{|d_K|}/2a}e^{12e^{-\pi\sqrt{|d_K|}/2a}(1+2/(1-e^{-\pi\sqrt{|d_K|}/a}))} & \text{if } \langle r\rangle = 1/2\end{cases}\\
&\quad \text{by Lemma 5.2(ii)}\\
&\leq \begin{cases} 2^{12}e^{-\pi\sqrt{3}}e^{-24+24/(1-e^{-\pi\sqrt{3}})} & \text{if } \langle r\rangle = 0,\\ e^{\pi\sqrt{|d_K|}/4}e^{12e^{-\pi\sqrt{3}/2}(1+2/(1-e^{-\pi\sqrt{3}}))} & \text{if } \langle r\rangle = 1/2,\end{cases}
\end{aligned}$$

$$
\begin{aligned}
&\text{because } \sqrt{3} \leq \sqrt{|d_K|}/a \text{ by Remark 4.6(ii)} \\
&< \begin{cases} 20 & \text{if } \langle r \rangle = 0, \\ 11e^{\pi\sqrt{|d_K|}/4} & \text{if } \langle r \rangle = 1/2 \end{cases} \\
&\leq 11e^{\pi\sqrt{|d_K|}/4}.
\end{aligned}
$$

We obtain by the fact $f = f^{Q^{-1}}$ and Lemma 3.6 that

$$\frac{g^3 + 72g^2 + 384g - 8192}{g} = \frac{h^3 + 72h^2 + 384h - 8192}{h},$$

from which we attain

$$(g - h)(g^2h + gh^2 + 72gh + 8192) = 0.$$

Since g generates $K_{\mathfrak{m}}$ over K by Proposition 4.2(iv), we see that $g \neq h$; and hence

$$g^2h + gh^2 + 72gh + 8192 = 0.$$

We then derive that

$$
\begin{aligned}
8192 &= |g^2h + gh^2 + 72gh| \\
&\leq |g^2h| + |gh^2| + 72|gh| \\
&< \left(4097e^{-\pi\sqrt{|d_K|}}\right)^2 \left(11e^{\pi\sqrt{|d_K|}/4}\right) + \left(4097e^{-\pi\sqrt{|d_K|}}\right) \left(11e^{\pi\sqrt{|d_K|}/4}\right)^2 \\
&\quad + 72 \left(4097e^{-\pi\sqrt{|d_K|}}\right) \left(11e^{\pi\sqrt{|d_K|}/4}\right) \\
&= 4097^2 \cdot 11e^{-7\pi\sqrt{|d_K|}/4} + 4097 \cdot 11^2 e^{-\pi\sqrt{|d_K|}/2} \\
&\quad + 72 \cdot 4097 \cdot 11e^{-3\pi\sqrt{|d_K|}/4} \\
&< 1484 \quad \text{because } |d_K| \geq 15,
\end{aligned}
$$

which yields a contradiction. As for the last inequality in the above, we rely on evaluation using Maple 2016 software. Therefore, we conclude that $h_E(\varphi_E(1/2))$ generates $K_{\mathfrak{m}} = H_K$ over K. □

6. Generation of ray class fields modulo 3

Let $\mathfrak{m} = 3\mathcal{O}_K$. By Proposition 4.4 we know that

$$\mathrm{Gal}(K_{\mathfrak{m}}/H_K) \simeq W_{K,3}/\{\pm I_2\}$$

$$(13)\qquad = \begin{cases} \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}2&0\\1&2\end{bmatrix}\right\} & \text{if } d_K \equiv 0 \pmod{12},\\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2&2\\1&0\end{bmatrix}, \begin{bmatrix}0&2\\1&1\end{bmatrix}\right\} & \text{if } d_K \equiv 9 \pmod{12},\\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&2\\1&1\end{bmatrix}, \begin{bmatrix}2&2\\1&2\end{bmatrix}, \begin{bmatrix}0&2\\1&0\end{bmatrix}\right\} & \text{if } d_K \equiv 8 \pmod{12},\\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}2&1\\1&0\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&2\end{bmatrix}\right\} & \text{if } d_K \equiv 5 \pmod{12},\\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}\right\} & \text{if } d_K \equiv 4 \pmod{12},\\ \left\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&2\end{bmatrix}\right\} & \text{if } d_K \equiv 1 \pmod{12}. \end{cases}$$

Theorem 6.1. *Suppose that* 3 *does not split in* K. *Then,* $h_E(\varphi_E(1/3))$ *generates* $K_{\mathfrak{m}}$ *over* K.

Proof. Since 3 is ramified or inert in K, we have $d_K \equiv 0$ or $2 \pmod 3$. It follows from (13) and Proposition 2.3(iii) that the Galois conjugates of $f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)$ over H_K are given by

$$\begin{cases} f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 0 \pmod{12},\\ f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 9 \pmod{12},\\ f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 8 \text{ or } 5 \pmod{12}. \end{cases}$$

Let

$$F = K\left(h_E(\varphi_E(1/3))\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)\right)$$

which is an abelian extension of K. Since

$$f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau) + f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau) = 0$$

by Lemma 3.7, F contains all four distinct values $f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)$, $f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)$, $f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)$ and $f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)$. We then deduce that

$$F \ni \frac{\left(f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)\right)^{24} \left(f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)\right)^{12}}{\left(f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)\right)^{24} \left(f_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)\right)^{12}}$$

$$= \frac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{108}}{g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{36} g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{36} g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{36}}$$

by Propositions 2.3(i) and 2.4(ii)

$$= 3^{-18} g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{144} \quad \text{by Lemma 3.11.}$$

Therefore, we conclude by Proposition 4.2(iv) that $F = K_{\mathfrak{m}}$ as desired. □

Lemma 6.2. *Let* $u + v\mathrm{i} \in \mathbb{H}$ *with* $t = e^{-2\pi v}$*, and let* $\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right] \in (1/3)\mathbb{Z}^2 \setminus \mathbb{Z}^2$*. If* $t \leq e^{-\pi\sqrt{3}}$*, then we have*

$$\left| g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(u+v\mathrm{i}) \right| < \begin{cases} 1.75t^{1/12} & \text{if } \langle r \rangle = 0, \\ 1.25t^{-1/36} & \text{if } \langle r \rangle \neq 0 \end{cases}$$
$$\leq 1.25t^{-1/36}.$$

Proof. We may assume that $\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right] \in \left\{ \left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right], \left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right], \left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right] \right\}$ by Proposition 2.3(i). Then we derive that

$$\left| g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(u+v\mathrm{i}) \right| \leq \begin{cases} \sqrt{3}\, t^{(1/2)\mathbf{B}_2(0)} \prod_{n=1}^{\infty} (1+t^n)^2 & \text{if } r = 0, \\ t^{(1/2)\mathbf{B}_2(1/3)} (1+t^{1/3}) \prod_{n=1}^{\infty} (1+t^{n-1/3})^2 & \text{if } r = 1/3, \end{cases}$$

by the definition (5)

$$\leq \begin{cases} \sqrt{3}\, t^{1/12} \prod_{n=1}^{\infty} e^{2t^n} & \text{if } r = 0, \\ t^{-1/36} e^{t^{1/3}} \prod_{n=1}^{\infty} e^{2t^{n-1/3}} & \text{if } r = 1/3, \end{cases}$$

since $1 + x < e^x$ for all $x > 0$

$$= \begin{cases} \sqrt{3}\, t^{1/12} e^{2t/(1-t)} & \text{if } r = 0, \\ t^{-1/36} e^{t^{1/3} + 2t^{2/3}/(1-t)} & \text{if } r = 1/3. \end{cases}$$

Moreover, now that $t \leq e^{-\pi\sqrt{3}}$, we get that

$$\left| g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(u+v\mathrm{i}) \right| < \begin{cases} 1.75t^{1/12} & \text{if } r = 0, \\ 1.25t^{-1/36} & \text{if } r = 1/3 \end{cases}$$
$$\leq 1.25t^{-1/36}.$$
□

Lemma 6.3. *Suppose that* $h_K \geq 2$ *and* $d_K \leq -20$*. Let* $Q = ax^2 + bxy + cy^2$ *be a nonidentity reduced form of discriminant* d_K*, and let* $\gamma \in \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$*.*

(i) $\left| g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_Q) \right| > 1.98 e^{-\pi\sqrt{|d_K|}/24}.$

(ii) $\left|\dfrac{j(\tau_Q)^2(j(\tau_Q)-1728)^3}{j(\tau_K)^2(j(\tau_K)-1728)^3}\right| < 877383e^{-5\pi\sqrt{|d_K|}/2}$.

(iii) *If* $s \in \{0,\,1/3,\,2/3\}$, *then* $\left|\dfrac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)g_{\left[\begin{smallmatrix}1/3\\s\end{smallmatrix}\right]}(\tau_K)}{g_{\gamma\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)g_{\gamma\left[\begin{smallmatrix}1/3\\s\end{smallmatrix}\right]}(\tau_Q)}\right| < 5.93e^{-\pi\sqrt{|d_K|}/36}$.

Proof. (i) Since Q is a nonidentity, we have $2 \le a \le \sqrt{|d_K|/3}$ by Remark 4.6(ii). We then establish that

$$\begin{aligned}
&\left|g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_Q)\right|\\
&\ge 2t^{(1/2)\mathbf{B}_2(0)}\prod_{n=1}^{\infty}(1-t^n)^2 \quad \text{by the definition (5), where } t=e^{-\pi\sqrt{|d_K|}/a}\\
&\ge 2e^{-\pi\sqrt{|d_K|}/24}\prod_{n=1}^{\infty}(1-e^{-\pi\sqrt{3}n})^2 \quad \text{because } e^{-\pi\sqrt{|d_K|}/2}\le t\le e^{-\pi\sqrt{3}}\\
&\ge 2e^{-\pi\sqrt{|d_K|}/24}\prod_{n=1}^{\infty}(1+e^{-\pi 99\sqrt{3}n/100})^{-2}\\
&\quad \text{since } 1-x>1/(1+x^{99/100}) \text{ for } 0<x\le e^{-\pi\sqrt{3}}\\
&\ge 2e^{-\pi\sqrt{|d_K|}/24}\prod_{n=1}^{\infty}e^{-2e^{-\pi 99\sqrt{3}n/100}} \quad \text{owing to the fact } 1+x<e^x \text{ for all } x>0\\
&= 2e^{-\pi\sqrt{|d_K|}/24}e^{-2e^{-\pi 99\sqrt{3}/100}/(1-e^{-\pi 99\sqrt{3}/100})}\\
&> 1.98e^{-\pi\sqrt{|d_K|}/24}.
\end{aligned}$$

(ii) For convenience, let $g(\tau)=g_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau)^{12}$. By Lemma 5.2(i) and the fact $d_K \le -20$, we see that

$$|g(\tau_K)| \le 2^{12}e^{-\pi\sqrt{|d_K|}}e^{-24+24/(1-e^{-\pi\sqrt{20}})} < 4097e^{-\pi\sqrt{|d_K|}}, \tag{14}$$

and similarly

$$\begin{aligned}|g(\tau_Q)| &\le 2^{12}e^{-\pi\sqrt{|d_K|}/a}e^{-24+24/(1-e^{-\pi\sqrt{|d_K|}/a})}\\ &< 2^{12}e^{-\pi\sqrt{3}}e^{-24+24/(1-e^{-\pi\sqrt{3}})} < 19.71\end{aligned} \tag{15}$$

by Remark 4.6(ii). It then follows from (14) and the fact $d_K \le -20$ that

$$|g(\tau_K)| < 0.0033. \tag{16}$$

Thus we achieve that

$$\begin{aligned}&\left|\frac{j(\tau_Q)^2(j(\tau_Q)-1728)^3}{j(\tau_K)^2(j(\tau_K)-1728)^3}\right|\\ &= \left|\frac{g(\tau_K)^5}{g(\tau_Q)^5}\right|\left|\frac{(g(\tau_Q)+16)^6(g(\tau_Q)+64)^3(g(\tau_Q)-8)^6}{(g(\tau_K)+16)^6(g(\tau_K)+64)^3(g(\tau_K)-8)^6}\right|\end{aligned}$$

by Remark 3.5

$$\leq \left(\frac{4097e^{-\pi\sqrt{|d_K|}}}{1.98^{12}e^{-\pi\sqrt{|d_K|}/2}}\right)^5 \left|\frac{(16+19.71)^6(64+19.71)^3(8+19.71)^6}{(16-0.0033)^6(64-0.0033)^3(8-0.0033)^6}\right|$$

by (i) and (14)–(16)

$$< 877383e^{-5\pi\sqrt{|d_K|}/2}.$$

(iii) We get that

$$\left|\frac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)g_{\left[\begin{smallmatrix}1/3\\s\end{smallmatrix}\right]}(\tau_K)}{g_{\gamma\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)g_{\gamma\left[\begin{smallmatrix}1/3\\s\end{smallmatrix}\right]}(\tau_Q)}\right|$$

$$= \sqrt{3}\left|g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)g_{\left[\begin{smallmatrix}1/3\\s\end{smallmatrix}\right]}(\tau_K)g_{\gamma\left[\begin{smallmatrix}1/3\\s'\end{smallmatrix}\right]}(\tau_Q)g_{\gamma\left[\begin{smallmatrix}1/3\\s''\end{smallmatrix}\right]}(\tau_Q)\right|$$

with $\{s,\, s',\, s''\} = \{0,\, 1/3,\, 2/3\}$ by Lemma 3.11

$$< \sqrt{3}\left(1.75e^{-\pi\sqrt{|d_K|/12}}\right)\left(1.25e^{\pi\sqrt{|d_K|/36}}\right)$$

$$\times \left(1.25e^{\pi\sqrt{|d_K|/36}}\right)\left(1.25e^{\pi\sqrt{|d_K|/36}}\right)$$

by Lemma 6.2 and the fact $2 \leq a \leq \sqrt{|d_K|/3}$

$$< 5.93e^{-\pi\sqrt{|d_K|/36}}.$$

□

Remark 6.4. In particular, suppose that $d_K = -20$, -23 or -24. In this case, we have

$$\mathrm{C}(d_K) = \begin{cases} \{x^2+5y^2,\, 2x^2+2xy+3y^2\} & \text{if } d_K = -20, \\ \{x^2+xy+6y^2,\, 2x^2 \pm xy+3y^2\} & \text{if } d_K = -23, \\ \{x^2+6y^2,\, 2x^2+3y^2\} & \text{if } d_K \equiv -24. \end{cases}$$

Thus, if $Q = ax^2+bxy+cy^2$ is a nonidentity reduced form of discriminant d_K, then we obtain by Lemma 5.2(i) that

$$\begin{aligned}|g(\tau_Q)| &\leq 2^{12}e^{-\pi\sqrt{|d_K|}/a}e^{-24+24/(1-e^{-\pi\sqrt{|d_K|}/a})} \\ &\leq 2^{12}e^{-\pi\sqrt{20}/2}e^{-24+24/(1-e^{-\pi\sqrt{20}/2})} < 3.73.\end{aligned}$$

In like manner as Lemma 6.3(ii) one can derive the inequality

$$\left|\frac{j(\tau_Q)^2(j(\tau_Q)-1728)^3}{j(\tau_K)^2(j(\tau_K)-1728)^3}\right| < 77e^{-5\pi\sqrt{|d_K|}/2}.$$

Theorem 6.5. *If* 3 *splits in* K*, then* $h_E(\varphi_E(1/3))$ *generates* $K_{\mathfrak{m}}$ *over* K*.*

Proof. Since 3 splits in K, we have $d_K \equiv 1 \pmod 3$. If $h_K = 1$, then the assertion is immediate by Proposition 4.2(iii). So, let $h_K \geq 2$, and hence

$d_K \leq -20$ by Remark 4.6(iii). By (13) and Proposition 2.3(iii) we know that there are two Galois conjugates of $f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)$ over H_K, namely,

$$\xi_1 = f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K) \quad \text{and} \quad \xi_2 = \begin{cases} f_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 4 \pmod{12},\\ f_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K) & \text{if } d_K \equiv 1 \pmod{12}.\end{cases}$$

Let $\xi = (\xi_1 - \xi_2)^{12}$. Since ξ is fixed by every element of $\mathrm{Gal}(K_{\mathfrak{m}}/H_K)$, it lies in H_K. We then derive that

$$\begin{aligned}\xi &= \frac{j(\tau_K)^4(j(\tau_K)-1728)^6}{2^{60}\cdot 3^{48}}\\ &\quad\times\begin{cases}\dfrac{g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{12}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{24}g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{24}} & \text{if } d_K \equiv 4 \pmod{12},\\ \dfrac{g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{12}g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{24}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{24}} & \text{if } d_K \equiv 1 \pmod{12},\end{cases}\\ &\quad\text{by Propositions 2.3(i) and 2.4(ii)}\\ &= \frac{j(\tau_K)^4(j(\tau_K)-1728)^6}{2^{60}\cdot 3^{42}}\\ &\quad\times\begin{cases}\dfrac{1}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{36}g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{36}} & \text{if } d_K \equiv 4 \pmod{12},\\ \dfrac{1}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{36}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{36}} & \text{if } d_K \equiv 1 \pmod{12},\end{cases}\\ &\quad\text{by Lemma 3.11.}\end{aligned}$$

Suppose that ξ does not generate H_K over K. Then, it follows from Proposition 4.5 that there is a nonidentity reduced form Q of discriminant d_K which leaves ξ fixed. Hence, we obtain that

$$\begin{aligned}1 = \left|\frac{\xi^{Q^{-1}}}{\xi}\right| &= \left|\frac{j(\tau_Q)^4(j(\tau_Q)-1728)^6}{j(\tau_K)^4(j(\tau_K)-1728)^6}\right|\\ &\quad\times\begin{cases}\left|\dfrac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{36}g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{36}}{g_{\gamma_Q^T\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)^{36}g_{\gamma_Q^T\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_Q)^{36}}\right| & \text{if } d_K \equiv 4 \pmod{12},\\ \left|\dfrac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{36}g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{36}}{g_{\gamma_Q^T\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)^{36}g_{\gamma_Q^T\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_Q)^{36}}\right| & \text{if } d_K \equiv 1 \pmod{12},\end{cases}\\ &\quad\text{by Propositions 2.3(iii) and 4.5}\end{aligned}$$

$$< \begin{cases} \left(877383e^{-5\pi\sqrt{|d_K|}/2}\right)^2 \left(5.93e^{-\pi\sqrt{|d_K|}/36}\right)^{36} & \text{if } d_K \leq -35, \\ \left(77e^{-5\pi\sqrt{|d_K|}/2}\right)^2 \left(5.93e^{-\pi\sqrt{|d_K|}/36}\right)^{36} & \text{if } d_K = -20 \text{ or } -23, \end{cases}$$

by Lemma 6.3(ii), (iii) and Remark 6.4

$< 1,$

which yields a contradiction. Therefore, ξ generates H_K over K; and hence $h_E(\varphi_E(1/3)) = f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)$ generates $K_{\mathfrak{m}}$ over K by Proposition 4.2(i) and (iii). □

7. Generation of ray class fields modulo 4

Let $\mathfrak{m} = 4\mathcal{O}_K$. As a corollary of Theorems 5.1 and 5.3 we obtain the following.

Corollary 7.1. *The value $h_E(\varphi_E(1/4))$ generates $K_{\mathfrak{m}}$ over K.*

Proof. Let

$$F = K\left(h_E(\varphi_E(1/4))\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K)\right)$$

which is a subfield of $K_{\mathfrak{m}}$ by Proposition 4.2(ii). Let $\min(\tau_K, \mathbb{Q}) = x^2 + bx + c$. Regarding $\gamma = \left[\begin{smallmatrix}1-2b & -2c\\ 2 & 1\end{smallmatrix}\right]$ as an element of $W_{K,4}/\{\pm I_2\}$ we see that

$$\begin{aligned} F &\ni \frac{1}{2}\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K)^{\gamma}\right) \quad \text{since } F \text{ is abelian over } K \\ &= \frac{1}{2}\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}2/4\\1/4\end{smallmatrix}\right]}(\tau_K)\right) \quad \text{by Propositions 4.4 and 2.3(iii)} \\ &= f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K) \quad \text{by Lemma 3.8.} \end{aligned}$$

Thus we achieve that

$$\begin{aligned} K_{\mathfrak{m}} &\supseteq F \\ &= K\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)\right) \\ &= K_{2\mathcal{O}_K}\left(f_{\left[\begin{smallmatrix}0\\1/4\end{smallmatrix}\right]}(\tau_K)\right) \quad \text{by Theorems 5.1 and 5.3} \\ &= K_{\mathfrak{m}} \quad \text{by Proposition 4.2(iii),} \end{aligned}$$

and hence $F = K_{\mathfrak{m}}$. This completes the proof. □

8. Generation of ray class fields modulo 6

Lt $\mathfrak{m} = 6\mathcal{O}_K$. As corollaries of Theorems 5.1, 5.3, 6.1 and 6.5 we obtain the followings.

Corollary 8.1. *If 2 splits in K, then $h_E(\varphi_E(1/3))$ generates $K_{\mathfrak{m}}$ over K.*

Proof. We see that

$$\begin{aligned}[K_{\mathfrak{m}} : K_{3\mathcal{O}_K}] &= \frac{[K_{\mathfrak{m}} : H_K]}{[K_{3\mathcal{O}_K} : H_K]} \\ &= \phi(2\mathcal{O}_K) \quad \text{by Lemma 4.1 and the fact } \mathcal{O}_K^* = \{\pm 1\} \\ &= 1 \quad \text{because we are assuming that 2 splits in } K,\end{aligned}$$

from which it follows that $K_{\mathfrak{m}} = K_{3\mathcal{O}_K}$. Thus, we conclude by Theorems 6.1 and 6.5 that

$$K_{\mathfrak{m}} = K_{3\mathcal{O}_K} = K\left(h_E(\varphi_E(1/3))\right). \qquad \square$$

Corollary 8.2. *Suppose that* 2 *is ramified and* 3 *is inert in* K. *Then* $h_E(\varphi_E(1/6))$ *generates* $K_{\mathfrak{m}}$ *over* K.

Proof. Let

$$F = K\left(h_E(\varphi_E(1/6))\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)\right)$$

which is a subfield of $K_{\mathfrak{m}}$ by Proposition 4.2(ii). Now that $d_K \equiv 8 \pmod{12}$, one can consider the matrices

$$\alpha = \begin{bmatrix}1 & d_K/2\\ 2 & 1\end{bmatrix}, \quad \beta = \begin{bmatrix}3 & d_K/2\\ 2 & 3\end{bmatrix}, \quad \gamma = \begin{bmatrix}5 & d_K/2\\ 2 & 5\end{bmatrix}$$

as elements of $W_{K,6}/\{\pm I_2\}$. Since F is an abelian extension of K, we obtain that

$$\begin{aligned}F &\ni \frac{1}{4}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{\alpha} + f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{\beta} + f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{\gamma}\right) \\ &= \frac{1}{4}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}2/6\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}2/6\\3/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}2/6\\5/6\end{smallmatrix}\right]}(\tau_K)\right) \\ &\quad \text{by Propositions 4.4 and 2.3(iii)} \\ &= f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K) \quad \text{by Lemma 3.9.}\end{aligned}$$

We then attain that

$$\begin{aligned}K_{\mathfrak{m}} &\supseteq F \\ &= K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}(\tau_K)\right) \\ &= K_{2\mathcal{O}_K}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)\right) \quad \text{by Theorems 5.1 and 5.3} \\ &= K_{\mathfrak{m}} \quad \text{by Proposition 4.2(iii),}\end{aligned}$$

from which it follows that $F = K_{\mathfrak{m}}$. $\square$

Corollary 8.3. *If* 2 *is inert in* K, *then* $h_E(\varphi_E(1/6))$ *generates* $K_{\mathfrak{m}}$ *over* K.

Proof. Let

$$F = K\left(h_E(\varphi_E(1/6))\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)\right)$$

which is a subfield of $K_{\mathfrak{m}}$ by Proposition 4.2(ii). Since 2 is inert in K, we have $d_K \equiv 5 \pmod 8$ and $\tau_K = (-1+\sqrt{d_K})/2$. Let

$$\alpha = \begin{bmatrix}4 & 3(d_K-1)/4\\ 3 & 1\end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix}1 & 3(d_K-1)/4\\ 3 & 4\end{bmatrix}$$

which are elements of $W_{K,\,6}/\{\pm I_2\}$. Since F is an abelian extension of K, we get that

$$\begin{aligned}
F &\ni \frac{1}{3}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{\alpha} + f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{\beta}\right)\\
&= \frac{1}{3}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}3/6\\1/6\end{smallmatrix}\right]}(\tau_K) + f_{\left[\begin{smallmatrix}3/6\\4/6\end{smallmatrix}\right]}(\tau_K)\right)\\
&\quad\ \text{by Propositions 4.4 and 2.3(iii)}\\
&= f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K) \quad \text{by Lemma 3.10.}
\end{aligned}$$

Whence we obtain that

$$\begin{aligned}
K_{\mathfrak{m}} &\supseteq F\\
&= K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K),\ f_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)\right)\\
&= K_{3\mathcal{O}_K}\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)\right) \quad \text{by Theorems 6.1 and 6.5}\\
&= K_{\mathfrak{m}} \quad \text{by Proposition 4.2(iii).}
\end{aligned}$$

Therefore, we achieve $F = K_{\mathfrak{m}}$. □

Lemma 8.4. *Suppose that $d_K \leq -20$ and $h_K \geq 2$. Let $Q = ax^2 + bxy + cy^2$ be a nonidentity reduced form of discriminant d_K, and let $\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right] \in (1/6)\mathbb{Z}^2 \setminus \mathbb{Z}^2$.*

(i) *If $\langle r\rangle \neq 0$ and $t = \left|e^{2\pi\mathrm{i}\tau_K}\right|$, then*

$$\left|g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(\tau_K)\right| < 1.1t^{(1/2)\mathbf{B}_2(\langle r\rangle)}.$$

(ii) *If $t_Q = \left|e^{2\pi\mathrm{i}\tau_Q}\right|$, then*

$$\left|g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(\tau_Q)\right| > 0.75t_Q^{1/12}.$$

(iii) *Furthermore, we have*

$$\left|\frac{g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)}{g_{\left[\begin{smallmatrix}r\\s\end{smallmatrix}\right]}(\tau_Q)}\right| < 1.$$

Proof. (i) We deduce that

$$\begin{aligned}
\left|g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau_K)\right| &\leq t^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1+t^{1/6})\prod_{n=1}^{\infty}(1+t^{n-1/2})^2 \\
&\quad \text{by the definition (5) and Proposition 2.3(i)} \\
&\leq t^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1+t^{1/6})\prod_{n=1}^{\infty}e^{2t^{n-1/2}} \quad \text{since } 1+x<e^x \text{ for all } x>0 \\
&= t^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1+t^{1/6})e^{2t^{1/2}/(1-t)} \\
&< 1.1t^{(1/2)\mathbf{B}_2(\langle r\rangle)} \quad \text{because } t=e^{-\pi\sqrt{|d_K|}}\leq e^{-\pi\sqrt{20}}.
\end{aligned}$$

(ii) We further derive that

$$\begin{aligned}
\left|g_{\left[\begin{smallmatrix} r \\ s \end{smallmatrix}\right]}(\tau_Q)\right| &\geq \begin{cases} |1-\zeta_6|t_Q^{(1/2)\mathbf{B}_2(0)}\prod_{n=1}^{\infty}(1-t_Q^n)^2 & \text{if } \langle r\rangle=0, \\ t_Q^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1-t_Q^{1/6})\prod_{n=1}^{\infty}(1-t_Q^{n-1/2})^2 & \text{if } \langle r\rangle\neq 0, \end{cases} \\
&\quad \text{by the definition (5) and Proposition 2.3(i)} \\
&\geq \begin{cases} t_Q^{1/12}\prod_{n=1}^{\infty}(1-e^{-\pi\sqrt{3}n})^2 & \text{if } \langle r\rangle=0, \\ t_Q^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1-e^{-\pi\sqrt{3}/6})(1-e^{-\pi\sqrt{3}/2})^2 & \\ \times\prod_{n=2}^{\infty}(1-e^{-\pi\sqrt{3}(n-1/2)})^2 & \text{if } \langle r\rangle\neq 0, \end{cases} \\
&\quad \text{because } t_Q=e^{-\pi\sqrt{|d_K|}/a}\leq e^{-\pi\sqrt{3}} \text{ by Remark 4.6(ii)} \\
&\geq \begin{cases} t_Q^{1/12}\prod_{n=1}^{\infty}(1+e^{-\pi 99\sqrt{3}n/100})^{-2} & \text{if } \langle r\rangle=0, \\ t_Q^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1-e^{-\pi\sqrt{3}/6})(1-e^{-\pi\sqrt{3}/2})^2 & \\ \times\prod_{n=2}^{\infty}(1+e^{-\pi 99\sqrt{3}(n-1/2)/100})^{-2} & \text{if } \langle r\rangle\neq 0, \end{cases} \\
&\quad \text{by the fact } 1-x>1/(1+x^{99/100}) \text{ for } 0<x\leq e^{-\pi\sqrt{3}} \\
&\geq \begin{cases} t_Q^{1/12}\prod_{n=1}^{\infty}e^{-2e^{-\pi 99\sqrt{3}n/100}} & \text{if } \langle r\rangle=0, \\ t_Q^{(1/2)\mathbf{B}_2(\langle r\rangle)}(1-e^{-\pi\sqrt{3}/6})(1-e^{-\pi\sqrt{3}/2})^2 & \\ \times\prod_{n=2}^{\infty}e^{-2e^{-\pi 99\sqrt{3}(n-1/2)/100}} & \text{if } \langle r\rangle\neq 0, \end{cases} \\
&\quad \text{since } 1+x<e^x \text{ for all } x>0
\end{aligned}$$

$$= \begin{cases} t_Q^{1/12} e^{-2e^{-\pi 99\sqrt{3}/100}/(1-e^{-\pi 99\sqrt{3}/100})} & \text{if } \langle r \rangle = 0, \\ t_Q^{(1/2)\mathbf{B}_2(\langle r \rangle)}(1-e^{-\pi\sqrt{3}/6})(1-e^{-\pi\sqrt{3}/2})^2 & \\ \times e^{-2e^{-\pi 297\sqrt{3}/200}/(1-e^{-\pi 99\sqrt{3}/100})} & \text{if } \langle r \rangle \neq 0 \end{cases}$$

$$> \begin{cases} 0.99 t_Q^{1/12} & \text{if } \langle r \rangle = 0, \\ 0.519 t_Q^{(1/2)\mathbf{B}_2(\langle r \rangle)} & \text{if } \langle r \rangle \neq 0 \end{cases}$$

$$\geq \begin{cases} 0.99 t_Q^{1/12} & \text{if } \langle r \rangle = 0, \\ 0.519 t_Q^{1/72} & \text{if } \langle r \rangle \neq 0 \end{cases}$$

$$> 0.75 t_Q^{1/12} \quad \text{due to the fact } t_Q \leq e^{-\pi\sqrt{3}}.$$

(iii) See [8, Lemma 4.1]. □

Theorem 8.5. *If 2 is ramified in K and 3 is not inert in K, then $h_E(\varphi_E(1/6))$ generates $K_{\mathfrak{m}}$ over K.*

Proof. If $h_K = 1$, then the assertion is straightforward by Proposition 4.2(iii). Now, let $h_K \geq 2$. Since 2 is ramified in K and 3 is not inert in K, we have $d_K \equiv 0$ or 4 (mod 12), and hence $d_K \leq -20$ by Remark 4.6(iii). Let

$$F = K\left(h_E(\varphi_E(1/6))\right) = K\left(f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau_K)\right)$$

which is an abelian extension of K as a subfield of $K_{\mathfrak{m}}$ by Proposition 4.2(ii). Let α be the element of $W_{K,6}/\{\pm I_2\}$ given by

$$\alpha = \begin{cases} \begin{bmatrix} 1 & d_K \\ 4 & 1 \end{bmatrix} & \text{if } d_K \equiv 0 \pmod{12}, \\ \begin{bmatrix} 3 & d_K/2 \\ 2 & 3 \end{bmatrix} & \text{if } d_K \equiv 4 \pmod{12}, \end{cases}$$

and set

$$\xi = \left(f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau_K)^\alpha - f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau_K)\right)^{12}$$

which belongs to F. We then derive that

$$\xi = \begin{cases} \left(f_{\left[\begin{smallmatrix} 4/6 \\ 1/6 \end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau_K)\right)^{12} & \text{if } d_K \equiv 0 \pmod{12}, \\ \left(f_{\left[\begin{smallmatrix} 2/6 \\ 3/6 \end{smallmatrix}\right]}(\tau_K) - f_{\left[\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right]}(\tau_K)\right)^{12} & \text{if } d_K \equiv 4 \pmod{12}, \end{cases}$$

by Propositions 4.4 and 2.3(iii)

$$= \frac{j(\tau_K)^4 (j(\tau_K) - 1728)^6}{2^{60} \cdot 3^{48}}$$

$$\times \begin{cases} \dfrac{g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\left[\begin{smallmatrix}1/3\\5/6\end{smallmatrix}\right]}(\tau_K)^{24} g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{24}} & \text{if } d_K \equiv 0 \pmod{12}, \\ \dfrac{g_{\left[\begin{smallmatrix}1/3\\2/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\left[\begin{smallmatrix}1/3\\1/2\end{smallmatrix}\right]}(\tau_K)^{24} g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{24}} & \text{if } d_K \equiv 4 \pmod{12}, \end{cases}$$

by Proposition 2.4(ii)

$$= \frac{j(\tau_K)^4 (j(\tau_K) - 1728)^6}{2^{60} \cdot 3^{48}}$$

$$\times \begin{cases} \dfrac{3^6}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\5/6\end{smallmatrix}\right]}(\tau_K)^{24} g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{24}} \\ \text{if } d_K \equiv 0 \pmod{12}, \\ \dfrac{3^6}{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\1/2\end{smallmatrix}\right]}(\tau_K)^{24} g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{24}} \\ \text{if } d_K \equiv 4 \pmod{12}, \end{cases}$$

by Lemma 3.11.

Suppose on the contrary that $F = K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)\right) = K\left(f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K), \xi\right)$ is properly contained in $K_{\mathfrak{m}}$. Then, by Propositions 4.4 and 4.5 there is a pair $(\gamma, Q) \in W_{K,6}/\{\pm I_2\} \times \mathrm{C}(d_K)$ for which γQ^{-1} leaves both $f_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)$ and ξ fixed. Observe that Proposition 4.2(iii) implies that $Q = ax^2 + bxy + cy^2$ must be a nonidentity reduced form. Let

$$t = |\tau_K| = e^{-\pi\sqrt{|d_K|}} \quad \text{and} \quad t_Q = |\tau_Q| = e^{-\pi\sqrt{|d_K|}/a}.$$

Then we achieve that

$$1 = \left|\frac{\xi^{\gamma Q^{-1}}}{\xi}\right| = \left|\frac{j(\tau_Q)^4 (j(\tau_Q) - 1728)^6}{j(\tau_K)^4 (j(\tau_K) - 1728)^6}\right| \left|\frac{g_{\left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_K)^{24}}{g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}0\\1/6\end{smallmatrix}\right]}(\tau_Q)^{24}}\right|$$

$$\times \begin{cases} \left|\dfrac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)^{12} g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}1/3\\1/3\end{smallmatrix}\right]}(\tau_Q)^{12}}\right| \left|\dfrac{g_{\left[\begin{smallmatrix}1/3\\5/6\end{smallmatrix}\right]}(\tau_K)^{24}}{g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}1/3\\5/6\end{smallmatrix}\right]}(\tau_Q)^{24}}\right| \\ \text{if } d_K \equiv 0 \pmod{12}, \\ \left|\dfrac{g_{\left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_K)^{12} g_{\left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_K)^{12}}{g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}0\\1/3\end{smallmatrix}\right]}(\tau_Q)^{12} g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}1/3\\0\end{smallmatrix}\right]}(\tau_Q)^{12}}\right| \left|\dfrac{g_{\left[\begin{smallmatrix}1/3\\1/2\end{smallmatrix}\right]}(\tau_K)^{24}}{g_{\gamma_Q^T \gamma^T \left[\begin{smallmatrix}1/3\\1/2\end{smallmatrix}\right]}(\tau_Q)^{24}}\right| \\ \text{if } d_K \equiv 4 \pmod{12}, \end{cases}$$

by Propositions 4.4 and 2.3(iii)

$$< \begin{cases} \left(77e^{-5\pi\sqrt{|d_K|}/2}\right)^2 \\ \times\left(5.93e^{-\pi\sqrt{|d_K|}/36}\right)^{12}\left(1.1t^{(1/2)\mathbf{B}_2(1/3)}\right)^{24}\left(0.75t_Q^{1/12}\right)^{-24} \\ \text{if } d_K=-20 \text{ or } -24, \\ \left(877383e^{-5\pi\sqrt{|d_K|}/2}\right)^2 \\ \times\left(5.93e^{-\pi\sqrt{|d_K|}/36}\right)^{12}\left(1.1t^{(1/2)\mathbf{B}_2(1/3)}\right)^{24}\left(0.75t_Q^{1/12}\right)^{-24} \\ \text{if } d_K\leq -56, \end{cases}$$

by Lemma 6.3(ii), (iii), Remarks 6.4 and Lemma 8.4

$$< e^{-(14/3-2/a)\pi\sqrt{|d_K|}}\times\begin{cases} 1.1006\cdot 10^{17} & \text{if } d_K=-20 \text{ or } -24, \\ 1.429\cdot 10^{25} & \text{if } d_K\leq -56 \end{cases}$$

$$\leq e^{-(11/3)\pi\sqrt{|d_K|}}\times\begin{cases} 1.1006\cdot 10^{17} & \text{if } d_K=-20 \text{ or } -24, \\ 1.429\cdot 10^{25} & \text{if } d_K\leq -56, \end{cases} \quad \text{since } a\geq 2$$

$$< 1,$$

which yields a contradiction. Therefore, we conclude that $F = K_{\mathfrak{m}}$. □

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