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A HOMOTOPY CONTINUATION METHOD FOR SOLVING A MATRIX EQUATION

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Abstract. In this paper, a homotopy continuation method for obtaining the unique Hermitian positive definite solution of the nonlinear matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I$ with $p_i > 1$ is proposed, which does not depend on a good initial approximation to the solution of matrix equation.

1. Introduction

In this paper we investigate the Hermitian positive definite (HPD) solutions of the nonlinear matrix equation

$$X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I, \tag{1.1}$$

where $p_i > 1$ $(i = 1, 2, \dots, m)$, $A_1, A_2, \dots, A_m$ are $n \times n$ nonsingular complex matrices, I is an $n \times n$ identity matrix and m is a positive integer. Here, A_i^* denotes the conjugate transpose of the matrix A_i.

When $m = 1$, this type of nonlinear matrix equations arises in Nano research, the analysis of ladder networks, dynamic programming, control theory, stochastic filtering, statistics and many other applications (see [1,5,6,8,18,19]). There were some contributions in the literature to the solvability, numerical solutions, and perturbation analysis (see [7,9,10,13,14,17] and therein).

When $m > 1$, Eq. (1.1) is recognized as playing an important role in solving a system of linear equations. For example, in many physical calculations, one must solve the system of linear equation

$$Mx = f,$$

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where x and f are column vectors, and

$$M = \begin{pmatrix} I & 0 & \cdots & 0 & A_1 \\ 0 & I & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -I \end{pmatrix}$$

arises in a finite difference approximation to an elliptic partial differential equation (for more information, refer to [3]). We can rewrite M as $M = \widetilde{M} + D$, where

$$\widetilde{M} = \begin{pmatrix} X^{p_1} & 0 & \cdots & 0 & A_1 \\ 0 & X^{p_2} & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X^{p_m} & A_m \\ A_1^* & A_2^* & \cdots & A_m^* & -I \end{pmatrix}, \quad D = \begin{pmatrix} I-X^{p_1} & 0 & \cdots & 0 & 0 \\ 0 & I-X^{p_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I-X^{p_m} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

$\widetilde{M}$ can be factored as

$$\widetilde{M} = \begin{pmatrix} I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \\ A_1^*X^{-p_1} & A_2^*X^{-p_2} & \cdots & A_m^*X^{-p_m} & I \end{pmatrix} \begin{pmatrix} X^{p_1} & 0 & \cdots & 0 & A_1 \\ 0 & X^{p_2} & \cdots & 0 & A_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & X^{p_m} & A_m \\ 0 & 0 & \cdots & 0 & -X \end{pmatrix}$$

if and only if X is a solution of equation $X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i = I$. In the last few years, matrix equation (1.1) was investigated in some special cases. When $0 < |p_i| < 1$, Duan-Liao-Tang [4] obtained the existence of a unique HPD solution by fixed point theorems for monotone and mixed monotone operators in a normal cone. Lim [12] derived the existence of a unique HPD solution by using a strict contraction for the Thompson metric on the open convex cone of positive definite matrices. When $p_i > 0$, Li-Zhang [11] derived a sufficient condition for the existence of a unique HPD solution by Banach contraction mapping principle.

The convergence of many iterative methods for the solution of matrix equations usually depends on a good initial approximation to the solution. Correspondingly, these convergence results only guarantee the existence of a well-defined convergent sequence of iterates for very restricted sets of starting matrices. To overcome the local convergence of iterative processes, we will use the homotopy continuation method (see [2] for more details) and the technique developed in [14] for obtaining the unique HPD solution of (1.1) with $p_i > 1$ in this paper. Note that the matrix equation (1.1) does not always have unique Hermitian positive definite solution in the case $p_i > 1$. We will derived some necessary conditions and sufficient conditions for the existence and uniqueness of Hermitian positive definite solutions to the matrix equation (1.1) in the case $p_i > 1$, which differ from the results in [11].

The rest of the paper is organized as follows. In Section 2, we give some preliminary knowledge that will be used to develop this work. In Section 3, we derive some necessary conditions and sufficient conditions for the existence and uniqueness of Hermitian positive definite solutions to the equation (1.1). In Section 4, we discuss the homotopy continuation methods for obtaining the unique Hermitian positive definite solution to the equation (1.1).

The following notations are used throughout this paper. We denote by $\mathcal{C}^{n\times n}$, $\mathcal{H}^{n\times n}$, $\mathcal{P}^{n\times n}$ and $\mathcal{U}^{n\times n}$ the set of all $n\times n$ complex matrices, Hermitian matrices, Hermitian positive definite matrix and unitary matrices, respectively. For column vectors $a_1, a_2, \ldots, a_n$, $A = (a_1, \ldots, a_n) = (a_{ij}) \in \mathcal{C}^{n\times n}$ and a matrix B, $A \otimes B = (a_{ij}B)$ is a Kronecker product, and $\mathrm{vec}A$ is a vector defined by $\mathrm{vec}A = (a_1^T, \ldots, a_n^T)^T$. The symbol $\|\cdot\|$ stands for the spectral norm. We denote by $\lambda_1(M)$ and $\lambda_n(M)$ the maximal and minimal eigenvalues of M, respectively. For $X, Y \in \mathcal{H}^{n\times n}$, we write $X \geq Y (X > Y)$ if $X - Y$ is a Hermitian positive semi-definite (definite) matrix. For $A, B \in \mathcal{H}^{n\times n}$, the sets $[A,B]$, (A,B) and $(A,B]$ are defined by $[A,B] = \{X \in \mathcal{H}^{n\times n} | A \leq X \leq B\}$, $(A,B) = \{X \in \mathcal{H}^{n\times n} | A < X < B\}$ and $(A,B] = \{X \in \mathcal{H}^{n\times n} | A < X \leq B\}$, respectively.

2. Preliminaries

In this section, we present some lemmas that will be needed to develop this paper.

Lemma 2.1 ([15, Lemma 2]). (i) *If $X \in \mathcal{P}^{n\times n}$ and $r > 0$, then $X^{-r} = \frac{1}{\Gamma(r)}\int_0^\infty e^{-sX}s^{r-1}ds$.*

(ii) *If $A, B \in \mathcal{C}^{n\times n}$, then $e^{A+B} - e^A = \int_0^1 e^{(1-t)A}Be^{t(A+B)}dt$.*

Lemma 2.2 ([16, Theorem 3.2.1]). *If $A, T \in \mathcal{C}^{n\times n}$ and T is nonsingular, then $e^{T^{-1}AT} = T^{-1}e^AT$.*

Lemma 2.3 ([16, Theorem 1.9.1]). *Let $A \in \mathcal{C}^{m\times n}, B \in \mathcal{C}^{p\times q}, C \in \mathcal{C}^{n\times k}, D \in \mathcal{C}^{q\times r}$. Then*

(i) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

(ii) $(A \otimes B)^* = A^* \otimes B^*$.

Lemma 2.4 ([16, Lemma 1.9.1]). *Let $A \in \mathcal{C}^{l\times m}, X \in \mathcal{C}^{m\times n}, B \in \mathcal{C}^{n\times k}$. Then*

$$\mathrm{vec}(AXB) = (B^T \otimes A) \cdot \mathrm{vec}X.$$

Lemma 2.5 ([20, Theorem 6.19]). *Let $A \in \mathcal{C}^{m\times m}$ and $B \in \mathcal{C}^{n\times n}$ with eigenvalues λ_i and μ_j, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, respectively. Then the eigenvalues of $A \otimes B$ are $\lambda_i\mu_j$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.*

Lemma 2.6 ([11, Theorem 3.1, Theorem 3.2]). *The nonlinear matrix equation $X - \sum_{i=1}^m A_i^* X^{-p_i} A_i = Q$ $(p_i > 0)$ always has Hermitian positive definite solutions. Moreover, if X is a Hermitian positive definite solution of $X - \sum_{i=1}^m A_i^* X^{-p_i} A_i = Q$ $(p_i > 0)$, then $Q \leq X \leq Q + \sum_{i=1}^m A_i^* A_i / \lambda_{\min}^{p_i}(Q)$, where Q is an $n \times n$ Hermitian positive definite matrix.*

Lemma 2.7. *Suppose that* $m \geq 1,\ p > 1$ *and* $1 < x, y < \frac{mp}{mp-1}$. *Then*

$$0 < f(x,y) = \frac{\sqrt{(x-1)(y-1)}(x^p - y^p)}{(x-y)x^{\frac{p}{2}}y^{\frac{p}{2}}} < \frac{1}{m}.$$

Proof. Let $g_1(x) = \frac{(x-1)^{1/2}}{x^{p/2}},\ 1 < x < \frac{mp}{mp-1},\ p > 1$. A calculation gives that $g_1'(x) = \frac{x^{\frac{p}{2}-1}((1-p)x+p)}{2\sqrt{x-1}x^p}$. Note that $1 < x < \frac{mp}{mp-1}$ and $p > 1$. Then $(1-p)x + p > \frac{(m-1)p}{mp-1} \geq 0$. Therefore $g_1'(x) \geq 0,\ 1 < x < \frac{mp}{mp-1}$, which implies $g_1(x)$ is monotonically increasing on $(1, \frac{mp}{mp-1})$. It follows that

$$g_1(x) < g_1\left(\frac{mp}{mp-1}\right) = \sqrt{\frac{(mp-1)^{p-1}}{(mp)^p}},\ \ 1 < x < \frac{mp}{mp-1}. \tag{2.1}$$

Let $g_2(x) = x^p,\ 1 < x < \frac{mp}{mp-1},\ p > 1$. By the mean value theorem, there exists $\xi \in (1, \frac{mp}{mp-1})$ such that

$$\frac{g_2(x) - g_2(y)}{x-y} = g_2'(\xi) < p\left(\frac{mp}{mp-1}\right)^{p-1},\ \ x, y \in (1, \frac{mp}{mp-1}). \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$0 < f(x,y) = g_1(x) \cdot g_1(y) \cdot \frac{g_2(x) - g_2(y)}{x-y} < \frac{1}{m}.$$

□

3. The existence and uniqueness of HPD solutions

In this section, some necessary conditions and sufficient conditions for the existence and uniqueness of HPD solutions of (1.1) are given.

Theorem 3.1. *Eq.* (1.1) *has a HPD solution if and only if there exist* $Q_i \in \mathcal{C}^{n\times n},\ i = 1, 2, \ldots, m,\ P \in \mathcal{U}^{n\times n}$, *and diagonal matrices* $\Gamma, \Lambda > 0$ *such that*

$$A_i = P^*\Gamma^{\frac{p_i}{2}}Q_i\Lambda P,\ i = 1, 2, \ldots, m,$$

where $\Gamma - \Lambda^2 = I$ *and* $\sum_{i=1}^m Q_i^*Q_i = I$. *In this case,* $X = P^*\Gamma P$ *is a HPD solution of Eq.* (1.1).

Proof. If Eq. (1.1) has a HPD solution X, it follows from the spectral decomposition theorem that there exist $P \in \mathcal{U}^{n\times n}$ and a diagonal matrix $\Gamma > 0$ such that $X = P^*\Gamma P$. Then Eq. (1.1) can be rewritten as

$$P^*\Gamma P - \sum_{i=1}^m A_i^*P^*\Gamma^{-p_i}PA_i = I. \tag{3.1}$$

Multiplying the left side of Eq. (3.1) by P and the right side by P^*, we have

$$\sum_{i=1}^m PA_i^*P^*\Gamma^{-p_i}PA_iP^* = \Gamma - I. \tag{3.2}$$

Note that A_i $(i = 1, 2, \ldots, m)$ are nonsingular matrices. Then $X > I$, which implies

$$\Gamma > I. \tag{3.3}$$

It follows that Eq. (3.2) will be turned into the following form

$$\sum_{i=1}^{m} (\Gamma - I)^{-\frac{1}{2}} P A_i^* P^* \Gamma^{-p_i} P A_i P^* (\Gamma - I)^{-\frac{1}{2}} = I. \tag{3.4}$$

Let $\Lambda = (\Gamma - I)^{\frac{1}{2}}$, $Q_i = \Gamma^{-\frac{p_i}{2}} P A_i P^* \Lambda^{-1}$. It is easy to verify that $\Gamma - \Lambda^2 = I$ and $A_i = P^* \Gamma^{\frac{p_i}{2}} Q_i \Lambda P$. From Eq. (3.4) it follows that $\sum_{i=1}^{m} Q_i^* Q_i = I$.

Conversely, assume there exist $P \in \mathcal{U}^{n\times n}$, $Q_i \in \mathcal{C}^{n\times n}$, $\sum_{i=1}^{m} Q_i^* Q_i = I$ and diagonal matrices $\Gamma, \Lambda > 0$, $\Gamma - \Lambda^2 = I$ such that

$$A_i = P^* \Gamma^{\frac{p_i}{2}} Q_i \Lambda P, \; i = 1, 2, \ldots, m.$$

Let $X = P^* \Gamma P$, then X is a HPD matrix, and it follows that

$$\begin{aligned} X - \sum_{i=1}^{m} A_i^* X^{-p_i} A_i &= P^* \Gamma P - \sum_{i=1}^{m} P^* \Lambda^* Q^* \Gamma^{\frac{p_i}{2}} P (P^* \Gamma P)^{-p_i} P^* \Gamma^{\frac{p_i}{2}} Q_i \Lambda P \\ &= P^* \Gamma P - \sum_{i=1}^{m} P^* \Lambda Q_i^* Q_i \Lambda P = P^* (\Gamma - \Lambda^2) P = I, \end{aligned}$$

which implies X is a solution of Eq. (1.1). □

Theorem 3.2. *If Eq.* (1.1) *has HPD solutions on* $(I, \frac{mq}{mq-1} I)$, *then the HPD solution is unique, where* $q = \max_{1\le i\le m}\{p_i\}$.

Proof. Suppose that X, Y are two HPD solutions of (1.1) such that $I < X, Y < \frac{mq}{mq-1} I$, we will prove $X = Y$.

Since X is a HPD solution of (1.1), according to Theorem 3.1, there exist $P_1 \in \mathcal{U}^{n\times n}$, $Q_i \in \mathcal{C}^{n\times n}$, $i = 1, 2, \ldots, m$ and diagonal matrices $\Gamma_1, \Lambda_1 > 0$ such that

$$A_i = P_1^* \Gamma_1^{p_i/2} Q_i \Lambda_1 P_1, \;\; i = 1, 2, \ldots, m, \tag{3.5}$$

where

$$\sum_{i=1}^{m} Q_i^* Q_i = I \;\text{ and }\; \Gamma_1 - \Lambda_1^2 = I. \tag{3.6}$$

In this case, $X = P_1^* \Gamma_1 P_1$, where $\Gamma_1 = diag(\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1n})$ with $\{\lambda_{1j}\}$ the eigenvalues of X.

Similarly, since Y is a HPD solution of Eq. (1.1), there exist $P_2 \in \mathcal{U}^{n\times n}$, $U_i \in \mathcal{C}^{n\times n}$, $i = 1, 2, \ldots, m$ and diagonal matrices $\Gamma_2, \Lambda_2 > 0$ such that

$$A_i = P_2^* \Gamma_2^{p_i/2} U_i \Lambda_2 P_2, \;\; i = 1, 2, \ldots, m, \tag{3.7}$$

where

$$\sum_{i=1}^{m} U_i^* U_i = I \text{ and } \Gamma_2 - \Lambda_2^2 = I. \tag{3.8}$$

In this case, $Y = P_2^* \Gamma_2 P_2$, where $\Gamma_2 = diag(\lambda_{21}, \lambda_{22}, \ldots, \lambda_{2n})$ with $\{\lambda_{2j}\}$ the eigenvalues of Y.

According to Lemma 2.1, Lemma 2.2, (3.5) and (3.7), we have

$$\begin{aligned} X - Y &= \sum_{i=1}^{m} A_i^* (X^{-p_i} - Y^{-p_i}) A_i \\ &= -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-(1-t)sY}(X - Y)e^{-tsX} dt s^{p_i} ds A_i \\ &= -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 P_2^* \Lambda_2 U_i^* \Gamma_2^{\frac{p_i}{2}} e^{-(1-t)s\Gamma_2} \\ &\qquad P_2(X - Y)P_1^* e^{-ts\Gamma_1} \Gamma_1^{\frac{p_i}{2}} Q_i \Lambda_1 P_1 dt s^{p_i} ds. \end{aligned} \tag{3.9}$$

Let

$$W = P_2(X - Y)P_1^*. \tag{3.10}$$

Then Eq. (3.9) can be rewritten as

$$W = -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 \Lambda_2 U_i^* \Gamma_2^{p_i/2} e^{-(1-t)s\Gamma_2} W e^{-ts\Gamma_1} \Gamma_1^{p_i/2} Q_i \Lambda_1 dt s^{p_i} ds. \tag{3.11}$$

From (3.11), Lemmas 2.3 and 2.4, it follows that

$$\begin{aligned} \mathrm{vec}W = &-\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 (e^{-ts\Gamma_1} \Gamma_1^{p_i/2} Q_i \Lambda_1)^T \\ &\qquad \otimes (\Lambda_2 U_i^* \Gamma_2^{p_i/2} e^{-(1-t)s\Gamma_2}) dt s^{p_i} ds \cdot \mathrm{vec}W \\ = &-\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 (\Lambda_1 Q_i^T \Gamma_1^{p_i/2} e^{-ts\Gamma_1}) \\ &\qquad \otimes (\Lambda_2 U_i^* \Gamma_2^{p_i/2} e^{-(1-t)s\Gamma_2}) dt s^{p_i} ds \cdot \mathrm{vec}W \\ = &-\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} (\Lambda_1 \otimes \Lambda_2)(Q_i^T \otimes U_i^*)(\Gamma_1^{p_i/2} \otimes \Gamma_2^{p_i/2}) \int_0^\infty \int_0^1 e^{-ts\Gamma_1} \\ &\qquad \otimes e^{-(1-t)s\Gamma_2} dt s^{p_i} ds \cdot \mathrm{vec}W. \end{aligned} \tag{3.12}$$

Assume that

$$\Lambda_1 = diag(\sigma_{11}, \sigma_{12}, \ldots, \sigma_{1n}), \quad \Lambda_2 = diag(\sigma_{21}, \sigma_{22}, \ldots, \sigma_{2n}).$$

According to (3.3), (3.6) and (3.8) , we have

$$0 < \sigma_{1j} = \sqrt{\lambda_{1j} - 1}, \ \ 0 < \sigma_{2j} = \sqrt{\lambda_{2j} - 1}, \ \ j = 1, 2, \ldots, n. \tag{3.13}$$

Let

$$\begin{aligned} &B = \Lambda_1 \otimes \Lambda_2, \ \ J_i = Q_i^T \otimes U_i^*, \\ &C_i = \frac{(\Gamma_1^{p_i/2} \otimes \Gamma_2^{p_i/2})}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\Gamma_1} \otimes e^{-(1-t)s\Gamma_2} dts^{p_i} ds, \ i = 1, 2, \ldots, m. \end{aligned} \tag{3.14}$$

Then (3.12) can be rewritten as

$$\text{vec}W + B \sum_{i=1}^{m} J_i C_i \cdot \text{vec}W = 0. \tag{3.15}$$

By Lemma 2.5 and (3.13), we have

$$\begin{aligned} B &= (\sigma_{1l} \cdot \Lambda_2)_{n^2 \times n^2} = diag(\sigma_{1l} \cdot \sigma_{2j})_{n^2 \times n^2} \\ &= diag(\sqrt{\lambda_{1l} - 1} \cdot \sqrt{\lambda_{2j} - 1})_{n^2 \times n^2}, \ \ l, j = 1, 2, \ldots, n, \end{aligned} \tag{3.16}$$

where

$$diag(\sqrt{\lambda_{1l} - 1} \cdot \sqrt{\lambda_{2j} - 1})_{n^2 \times n^2}$$

$$= \begin{pmatrix} \sigma_{11}\sigma_{21} & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_{11}\sigma_{22} & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{11}\sigma_{2n} & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \sigma_{1n}\sigma_{21} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \vdots & 0 & \sigma_{1n}\sigma_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \sigma_{1n}\sigma_{2n} \end{pmatrix},$$

and

$$0 < \sigma_{1l} = \sqrt{\lambda_{1l} - 1}, \ \ 0 < \sigma_{2j} = \sqrt{\lambda_{2j} - 1}, \ \ l, j = 1, 2, \ldots, n.$$

$$\begin{aligned} C_i &= \frac{(\Gamma_1^{p_i/2} \otimes \Gamma_2^{p_i/2})}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\Gamma_1} \otimes e^{-(1-t)s\Gamma_2} dts^{p_i} ds, \\ &= diag\left(\lambda_{1l}^{p_i/2} \cdot \lambda_{2j}^{p_i/2} \cdot \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\lambda_{1l}} \cdot e^{-(1-t)s\lambda_{2j}} dts^{p_i} ds\right)_{n^2 \times n^2} \\ &= diag\left(\frac{\lambda_{2j}^{p_i} - \lambda_{1l}^{p_i}}{(\lambda_{2j} - \lambda_{1l})\lambda_{1l}^{p_i/2}\lambda_{2j}^{p_i/2}}\right)_{n^2 \times n^2}, \ i = 1, 2, \ldots, m, \ l, j = 1, 2, \ldots, n. \end{aligned} \tag{3.17}$$

Note that B is nonsingular. Multiplying the left side of Eq. (3.15) by B^{-1}, we have

$$B^{-1} \cdot \mathrm{vec}W + \sum_{i=1}^{m} J_i C_i \cdot \mathrm{vec}W = (I + \sum_{i=1}^{m} J_i C_i B\,)B^{-1} \cdot \mathrm{vec}W = 0. \tag{3.18}$$

A combination of (3.14) and Lemma 2.3 gives

$$\begin{aligned} J_i^* J_i &= (Q_i^T \otimes U_i^*)^*(Q_i^T \otimes U_i^*) = (\overline{Q}_i \otimes U_i)(Q_i^T \otimes U_i^*) \\ &= (\overline{Q}_i Q_i^T) \otimes (U_i U_i^*) = (\overline{Q_i Q_i^*}) \otimes (U_i U_i^*). \end{aligned}$$

It follows (3.6), (3.8), and Lemma 2.5 that $0 < ||J_i|| \le 1$. Then

$$\left\| \sum_{i=1}^{m} J_i C_i B \right\| \le \sum_{i=1}^{m} \|C_i B\|. \tag{3.19}$$

By the hypothesis of the theorem, we have $I < X, Y < \frac{mq}{mq-1} I$, which implies that $1 < \lambda_{1l}, \lambda_{2j} < \frac{mq}{mq-1} < \frac{mp_i}{mp_i-1}$, $i = 1, 2, \ldots, m$, $l, j = 1, 2, \ldots, n$. Then it follows from (3.17) and (3.16) that

$$\begin{aligned} \sum_{i=1}^{m} \| \, C_i B \, \| &= \sum_{i=1}^{m} \max_{l,j} \left\{ \frac{\sqrt{\lambda_{1l} - 1}\sqrt{\lambda_{2j} - 1}(\lambda_{2j}^{p_i} - \lambda_{1l}^{p_i})}{(\lambda_{2j} - \lambda_{1l})\lambda_{1l}^{\frac{p_i}{2}} \lambda_{2j}^{\frac{p_i}{2}}} \right\} \\ &= \sum_{i=1}^{m} \max_{l,j} \{ f(\lambda_{1l}, \lambda_{2j}) \}, \end{aligned} \tag{3.20}$$

where $f(x, y)$ is defined in Lemma 2.7. A combination of Lemma 2.7, (3.19) and (3.20) gives that

$$\left\| \sum_{i=1}^{m} J_i C_i B \right\| < m \cdot \frac{1}{m} = 1,$$

which implies $I + \sum_{i=1}^{m} J_i C_i B$ is nonsingular. It follows (3.18) that $\mathrm{vec}W = 0$. By (3.10), we have $X = Y$. $\square$

Theorem 3.3. *If* $\lambda_1(\sum_{i=1}^{m} A_i^* A_i) < \frac{1}{mq-1}$, *then Eq.* (1.1) *has a unique HPD solution* $X \in (I, \frac{mq}{mq-1} I)$, *where* $q = \max_{1 \le i \le m}\{p_i\}$.

Proof. It follows from Lemma 2.6 that Eq. (1.1) always has HPD solutions. Moreover, for any HPD solution X of Eq. (1.1), we have $I \le X \le I + \sum_{i=1}^{m} A_i^* A_i \le (1 + \lambda_1(\sum\limits_{i=1}^{m} A_i^* A_i))I$. Therefore, if $\lambda_1(\sum_{i=1}^{m} A_i^* A_i) < \frac{1}{mq-1}$, it follows that $I < X < \frac{mq}{mq-1} I$. By Theorem 3.2, we have X is the unique HPD solution of Eq. (1.1). $\square$

4. The homotopy continuation method

In this section, by means of the homotopy continuation method (see [2] for more details) and the technique developed in [14], we derive a numerical iterative process for solving the matrix equation (1.1).

Define the nonlinear map $F : \mathcal{P}^{n\times n} \subset \mathcal{C}^{n\times n} \to \mathcal{C}^{n\times n}$ by

$$F(X) = I + \sum_{i=1}^{m} A_i^* X^{-p_i} A_i. \tag{4.1}$$

The idea of homotopy continuation method for solving the matrix equation $F(X) = X$ is to consider a homotopy $H : [0,1]\times\mathcal{P}^{n\times n} \to \mathcal{P}^{n\times n}$ such that there exists a continuous solution curve $X : [0,1] \to \mathcal{P}^{n\times n}$ of $H(t,X) = 0,\ t \in [0,1]$, starting at a known point $X_0 = X(0)$ and ending at a solution of $F(X) = X$.

In this section, we define the homotopy $H : [0,1] \times \mathcal{P}^{n\times n} \to \mathcal{P}^{n\times n}$ by

$$H(t,X) = I + t\sum_{i=1}^{m} A_i^* X^{-p_i} A_i - X. \tag{4.2}$$

Then at $t = 0$, the solution of $H(t,X) = 0$ is a known matrix I, while at $t = 1$, the solution X of $H(t,X) = 0$ also solves $F(X) = X$. To discuss the numerical method for solving the homotopy equation $H(t,X) = 0$, we rewrite the homotopy equation $H(t,X) = 0$ as the following fixed point form.

Assume that $G : [0,1] \times \mathcal{P}^{n\times n} \subset [0,1] \times \mathcal{C}^{n\times n} \to \mathcal{P}^{n\times n}$ is a map such that

$$X(t) = G(t, X(t)),\ \ t \in [0,1],$$

where $X : [0,1] \to \mathcal{P}^{n\times n}$ denotes the solution of $H(t,X) = 0$. Then for each t, we can consider the iterative process

$$X_{n+1} = G(t, X_n). \tag{4.3}$$

Since for a fixed t, this process will converge to $X(t)$ only for starting values near that point, to overcome the local convergence of iterative process, we consider the following numerical continuation process:

A partition of $J = [0,1]$:

$$0 = t_0 < t_1 < \cdots < t_N = 1, \tag{4.4}$$

and a sequence of integers $\{j_k\}$, $k = 1,\ldots,N-1$, is chosen with the property that the points

$$\begin{cases} X_{k,j+1} = G(t_k, X_{k,j}), & j = 0,\ldots,j_k-1,\ k = 1,\ldots,N-1; \\ X_{k+1,0} = X_{k,j_k}, & X_{1,0} = X_0, \end{cases} \tag{4.5}$$

are well-defined and such that

$$X_{N,j+1} = G(1, X_{N,j}) \tag{4.6}$$

converges to $X(1)$ as $j \to \infty$.

The main idea is to choose the partition (4.4) so that $X(t_k)$ lies in some domain of attraction $D_{t_{k+1}}$ for each k, $1 \le k \le N$. Then, if $X_{k,0} \in D_{t_{k+1}}$, the sequence generated by (4.3) for $t = t_k$ must produce an iterate $X_{k,j_k} \in D_{t_{k+1}}$, which in turn can be taken as the starting point $X_{k+1,0} = X_{k,j_k}$ for the next iteration involving t_{k+1}. Thus with $X_{1,0} = X_0$ as initial point, the entire process can be carried out until finally $t_k = t_N = 1$ is reached. For $t = 1$, $X_{N,0} = X_{N,j_{N-1}}$ is then in D_1 which ensure that (4.5) converges to $X(1)$ as $j \to \infty$.

To discuss the feasibility of the above numerical continuation process, we will use the following definition and lemmas which can be found in [2].

Definition 4.1 ([2]). If a partition (4.4) exists so that with some sequence of integers $\{j_k\}$ the entire process (4.5)-(4.6) is well-defined and so that (4.6) converges to $X(1)$, then the numerical continuation process (4.5)-(4.6) is called feasible.

Definition 4.2 ([2]). Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a given mapping. Then any nonempty set $D_0 \subset D$ is a domain of attraction of the iterative process

$$x_{n+1} = G(x_n), \quad n = 0, 1, \ldots, \tag{4.7}$$

with respect to the point x_* if for any $x_0 \in D_0$ we have $\{x_n\} \subset D$ and $\lim_{n\to\infty} x_n = x_*$.

If $x_* \in int(D_0)$ for some domain of attraction D_0, then x_* is a point of attraction of (4.7).

Lemma 4.1 ([2]). *Let $G : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be Fréchet differentiable at the fixed point $x_* \in int(D)$ of G. If $\rho(G'(x_*)) < 1$, then x_* is a point of attraction of* (4.7) *and, more precisely, there is an open ball $S(x_*, r)$ with center x_* and radius $r > 0$ which is a domain of attraction of* (4.7) *with respect to x_*. Here $\rho(\cdot)$ denotes the spectral radius of $G'(x_*)$.*

Lemma 4.2 ([2]). *Let $G : [0,1] \times D \subset [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, where D is open and assume that $x : [0,1] \to D$ is continuous and satisfies $x(t) = G(t, x(t))$. Let G have a Fréchet derivative with respect to x at $(t, x(t))$ for every $t \in [0,1]$. If $G_x(t, x(t))$ is continuous on $[0,1] \times D$ and $\rho(G_x(t, x(t))) < 1$ for all $t \in [0,1]$, then the numerical continuation process* (4.5)-(4.6) *is feasible.*

In what follows, we derive a sufficient condition for the existence of a unique HPD solution of the homotopy equation $H(t, X) = 0$ for all $t \in [0,1]$.

Theorem 4.1. *If $\lambda_1(\sum_{i=1}^m A_i^* A_i) < \frac{1}{mq-1}$, then for arbitrary $t \in [0,1]$, the homotopy equation $H(t, X) = 0$ has an unique HPD solution on $[I, \frac{mq}{mq-1} I]$, where $q = \max_{1\le i\le m}\{p_i\}$.*

Proof. Since $t > 0$, then the homotopy equation $H(t, X) = 0$ can be rewritten as

$$X - \sum_{i=1}^{m} (\sqrt{t} A_i)^* X^{-p_i} (\sqrt{t} A_i) = I.$$

By the hypothesis of the theorem, we have

$$\lambda_1(\sum_{i=1}^{m}(\sqrt{t}A_i)^*(\sqrt{t}A_i)) = \lambda_1(\sum_{i=1}^{m} tA_i^*A_i) \leq \lambda_1(\sum_{i=1}^{m} A_i^*A_i) < \frac{1}{mq-1}.$$

It follows from Theorem 3.3 that the homotopy equation $H(t, X) = 0$ has a unique HPD solution on $[I, \frac{mq}{mq-1}I]$. □

In next theorem, the local convergence of the iterative process (4.3) is obtained.

Theorem 4.2. *If $F(X) = X$ has a unique HPD solution X_* on $(I, \frac{mq}{mq-1}I)$, then there exist an open ball $N(X_*, \delta)$ with center X_* and radius $\delta > 0$ such that, for any starting value $X_0 \in N(X_*, \delta)$, $X_n = F(X_{n-1})$ converges to X_* as $n \to \infty$, where $q = \max_{1\leq i\leq m}\{p_i\}$.*

Proof. According to $F(X) = I + \sum_{i=1}^{m} A_i^*X^{-p_i}A_i$ and Lemma 2.1, for any $h \in \mathcal{P}^{n\times n}$, we have

$$\begin{aligned} F(X_* + h) - F(X_*) &= \sum_{i=1}^{m} A_i^*[(X_* + h)^{-p_i} - X_*^{-p_i}]A_i \quad (4.8)\\ &= \sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty} (e^{-s(X_*+h)} - e^{-sX_*})s^{p_i-1}dsA_i \\ &= -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty}\int_0^{1} e^{-(1-t)sX_*}he^{-st(X_*+h)}dts^{p_i}dsA_i. \end{aligned}$$

By the definition of Fréchet derivative, we obtain

$$F'(X_*)h = -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty}\int_0^{1} e^{-(1-t)sX_*}he^{-stX_*}dts^{p_i}dsA_i.$$

Let λ be any eigenvalue of $F'(X_*)$. Then there exists nonzero matrix h_* such that $F'(X_*)h_* = \lambda h_*$, that is

$$F'(X_*)h_* = -\sum_{i=1}^{m} \frac{A_i^*}{\Gamma(p_i)} \int_0^{\infty}\int_0^{1} e^{-(1-t)sX_*}h_*e^{-stX_*}dts^{p_i}dsA_i = \lambda h_*. \tag{4.9}$$

Since X_* is the unique HPD solution of $F(X) = X$, then by Theorem 3.1, there exist $P \in \mathcal{U}^{n\times n}$, $Q_i \in \mathcal{C}^{n\times n}$, $i = 1, 2, \ldots, m$ and diagonal matrices $\Gamma, \Lambda > 0$ such that

$$A_i = P^*\Gamma^{p_i/2}Q_i\Lambda P, \quad i = 1, 2, \ldots, m,$$

where

$$\sum_{i=1}^{m} Q_i^*Q_i = I \text{ and } \Gamma - \Lambda^2 = I. \tag{4.10}$$

In this case, $X_* = P^*\Gamma P$, where $\Gamma = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\{\lambda_j\}$ the eigenvalues of X. Therefore (4.9) can be rewritten as

(4.11)
$$\begin{aligned} &F'(X_*)h_* \\ =& -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 P^*\Lambda Q_i^*\Gamma^{\frac{p_i}{2}} e^{-(1-t)s\Gamma} Ph_*P^* e^{-st\Gamma} dt s^{p_i} ds \Gamma^{\frac{p_i}{2}} Q_i\Lambda P \\ =&\lambda h_*, \end{aligned}$$

which implies

$$\begin{aligned} &P(F'(X_*)h)P^* \\ =& -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 \Lambda Q_i^*\Gamma^{\frac{p_i}{2}} e^{-(1-t)s\Gamma} PhP^* e^{-st\Gamma} dt s^{p_i} ds \Gamma^{\frac{p_i}{2}} Q_i\Lambda \\ =& \lambda Ph_*P^*. \end{aligned}$$

Let $z = PhP^*$. It follows that

$$-\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 \Lambda Q_i^*\Gamma^{\frac{p_i}{2}} e^{-(1-t)s\Gamma} z e^{-st\Gamma} dt s^{p_i} ds \Gamma^{\frac{p_i}{2}} Q_i\Lambda = \lambda z.$$

Define the operator $\mathbf{L} : \mathcal{C}^{n\times n} \to \mathcal{C}^{n\times n}$ by

$$\mathbf{L}z = -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 \Lambda Q_i^*\Gamma^{\frac{p_i}{2}} e^{-(1-t)s\Gamma} z e^{-st\Gamma} \Gamma^{\frac{p_i}{2}} Q_i\Lambda dt s^{p_i} ds. \tag{4.12}$$

Then

$$\mathbf{L}z = \lambda z. \tag{4.13}$$

Using (4.13), Lemmas 2.3 and 2.4, we can rewrite (4.12) as

$$\begin{aligned} \mathrm{vec}(\mathbf{L}z) =& -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 (e^{-st\Gamma}\Gamma^{\frac{p_i}{2}} Q_i\Lambda)^T \\ &\otimes (\Lambda Q_i^*\Gamma^{\frac{p_i}{2}} e^{-(1-t)s\Gamma}) dt s^{p_i} ds \cdot \mathrm{vec}z \\ =& -\sum_{i=1}^{m} \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 (\Lambda\otimes\Lambda)(Q_i^T\otimes Q_i^*)(\Gamma^{\frac{p_i}{2}}\otimes\Gamma^{\frac{p_i}{2}}) \\ &(e^{-st\Gamma}\otimes e^{-(1-t)s\Gamma}) dt s^{p_i} ds \cdot \mathrm{vec}z \\ =& \lambda \mathrm{vec}z. \end{aligned} \tag{4.14}$$

Let

$$\begin{aligned} &B = \Lambda\otimes\Lambda, \quad J_i = Q_i^T\otimes Q_i^*, \\ &C_i = \frac{(\Gamma^{p_i/2}\otimes\Gamma^{p_i/2})}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\Gamma}\otimes e^{-(1-t)s\Gamma} dt s^{p_i} ds, \ i = 1, 2, \ldots, m. \end{aligned} \tag{4.15}$$

Then (4.14) can be rewritten as

$$\operatorname{vec}(\mathbf{L}z) = -\sum_{i=1}^{m} BJ_iC_i \cdot \operatorname{vec}z = \lambda \cdot \operatorname{vec}z. \tag{4.16}$$

A combination of (4.11), (4.12), (4.14) and (4.16) gives that

$$\rho(F'(X_*)) = \max\{|\lambda|\} = \rho(-\sum_{i=1}^{m} BJ_iC_i). \tag{4.17}$$

It is easy to verify that

$$\rho(-\sum_{i=1}^{m} BJ_iC_i) \le \|\sum_{i=1}^{m} BJ_iC_i\| \le \sum_{i=1}^{m} \|J_iBC_i\| \le \sum_{i=1}^{m} \|BC_i\|.$$

Therefore

$$\rho(F'(X_*)) \le \sum_{i=1}^{m} \|BC_i\|. \tag{4.18}$$

In what follows, we will estimate the upper bound of $\sum_{i=1}^{m} \|BC_i\|$.

Assume that

$$\Lambda = diag(\sigma_1, \sigma_2, \ldots, \sigma_n).$$

By (4.10), we have

$$0 < \sigma_j = \sqrt{\lambda_j - 1}, \;\; j = 1, 2, \ldots, n. \tag{4.19}$$

According to Lemma 2.5 and (4.19), we have

$$B = diag(\sigma_l \cdot \sigma_j)_{n^2 \times n^2} = diag(\sqrt{\lambda_l - 1} \cdot \sqrt{\lambda_j - 1})_{n^2 \times n^2}, \tag{4.20}$$

where

$$diag(\sqrt{\lambda_l - 1} \cdot \sqrt{\lambda_j - 1})_{n^2 \times n^2}$$
$$= \begin{pmatrix} \sigma_1\sigma_1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_1\sigma_2 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_1\sigma_n & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \sigma_n\sigma_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \vdots & 0 & \sigma_n\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \sigma_n\sigma_n \end{pmatrix},$$

and

$$0 < \sigma_j = \sqrt{\lambda_j - 1}, \;\; j = 1, 2, \ldots, n.$$

(4.21)

$$\begin{aligned} C_i &= \frac{(\Gamma^{p_i/2} \otimes \Gamma^{p_i/2})}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\Gamma} \otimes e^{-(1-t)s\Gamma} dt s^{p_i} ds, \\ &= diag\left(\lambda_l^{p_i/2} \cdot \lambda^{p_i/2} \cdot \frac{1}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-ts\lambda_l} \cdot e^{-(1-t)s\lambda_j} dt s^{p_i} ds \right)_{n^2 \times n^2} \\ &= diag\left(\frac{\lambda_j^{p_i} - \lambda_l^{p_i}}{(\lambda_j - \lambda_l)\lambda_l^{p_i/2}\lambda_l^{p_i/2}} \right)_{n^2 \times n^2}, \ i = 1, 2, \ldots, m, \ l, j = 1, 2, \ldots, n. \end{aligned}$$

Since $I < X_* < \frac{mq}{mq-1} I$, then $1 < \lambda_l, \lambda_j < \frac{mq}{mq-1} < \frac{mp_i}{mp_i - 1}$, $i = 1, 2, \ldots, m$, $l, j = 1, 2, \ldots, n$. It follows from (4.21) that

(4.22)

$$\sum_{i=1}^m \| C_i B \| = \sum_{i=1}^m \max_{l,j} \left\{ \frac{\sqrt{\lambda_l - 1}\sqrt{\lambda_j - 1}(\lambda_j^{p_i} - \lambda_l^{p_i})}{(\lambda_j - \lambda_l)\lambda_l^{\frac{p_i}{2}}\lambda_j^{\frac{p_i}{2}}} \right\} = \sum_{i=1}^m \max_{l,j}\{f(\lambda_l, \lambda_j)\},$$

where $f(x, y)$ is defined in Lemma 2.7.

Combining Lemma 2.7, (4.18) with (4.22) gives that

$$\rho(F'(X_*)) \le \sum_{i=1}^m \|BC_i\| = \sum_{i=1}^m \max_{l,j}\{f(\lambda_l, \lambda_j)\} < m \cdot \frac{1}{m} = 1.$$

According to Lemma 4.1, there exist an open ball $N(X_*, \delta)$ with center X_* and radius $\delta > 0$ such that, for any starting value $X_0 \in N(X_*, \delta)$, $X_n = F(X_{n-1})$ converges to X_* as $n \to \infty$. □

The next theorem is the main result of this section.

Theorem 4.3. *If* $\lambda_1(\sum_{i=1}^m A_i^* A_i) < \frac{1}{mq-1}$, *then the numerical continuation process* (4.5)-(4.6) *is feasible, where* $q = \max_{1 \le i \le m}\{p_i\}$.

Proof. Define the map $G : [0, 1] \times \mathcal{P}^{n\times n} \to \mathcal{P}^{n\times n}$ by

$$G(t, X(t)) = I + t \sum_{i=1}^m A_i^* X^{-p_i} A_i.$$

In the following, we will prove the numerical continuation process (4.5)-(4.6) is feasible.

By Lemma 2.1, for any $h \in \mathcal{P}^{n\times n}$, we have

$$\begin{aligned} G(t, X(t) + h) - G(t, X(t)) &= t \sum_{i=1}^m A_i^* [(X(t) + h)^{-p_i} - X(t)^{-p_i}] A_i \qquad (4.23) \\ &= t \sum_{i=1}^m \frac{A_i^*}{\Gamma(p_i)} \int_0^\infty (e^{-s(X(t)+h)} - e^{-sX(t)}) s^{p_i - 1} ds A_i \\ &= -t \sum_{i=1}^m \frac{A_i^*}{\Gamma(p_i)} \int_0^\infty \int_0^1 e^{-(1-v)sX(t)} h e^{-sv(X(t)+h)} dv s^{p_i} ds A_i. \end{aligned}$$

By the definition of Fréchet derivative, we obtain

$$G'(t,X(t))h = -t\sum_{i=1}^{m}\frac{A_i^*}{\Gamma(p_i)}\int_0^{\infty}\int_0^1 e^{-(1-v)sX(t)}he^{-svX(t)}dvs^{p_i}dsA_i.$$

Using the same technique described in Theorem 4.2, we have that

$$\begin{aligned}\rho(G'(t,X(t))) &< t\sum_{i=1}^{m}\max_{l,j}\left\{\frac{\sqrt{\lambda_l-1}\sqrt{\lambda_j-1}(\lambda_j^{p_i}-\lambda_l^{p_i})}{(\lambda_j-\lambda_l)\lambda_l^{\frac{p_i}{2}}\lambda_j^{\frac{p_i}{2}}}\right\}\\ &= t\sum_{i=1}^{m}\max_{l,j}\{f(\lambda_l,\lambda_j)\} < t\cdot m\cdot\frac{1}{m}\le 1,\end{aligned}$$

where $0<\lambda_l,\lambda_j<\frac{mq}{mq-1}<\frac{mp_i}{mp_i-1}$, $i=1,2,\ldots,m$, $l,j=1,2,\ldots,n$, and $f(x,y)$ is defined in Lemma 2.7.

According to Lemma 4.2, the numerical continuation process (4.5)-(4.6) is feasible. □

References

[1] W. N. Anderson, Jr., T. D. Morley, and G. E. Trapp, *Positive solutions to* $X=A-BX^{-1}B^*$, Linear Algebra Appl. **134** (1990), 53–62.

[2] J. H. Avila, Jr., *The feasibility of continuation methods for nonlinear equations*, SIAM J. Numer. Anal. **11** (1974), 102–122.

[3] B. L. Buzbee, G. H. Golub, and C. W. Nielson, *On direct methods for solving Poisson's equations*, SIAM J. Numer. Anal. **7** (1970), 627–656.

[4] X. Duan, A. Liao, and B. Tang, *On the nonlinear matrix equation* $X-\sum_{i=1}^m A_i^*X^{\delta_i}A_i=Q$, Linear Algebra Appl. **429** (2008), no. 1, 110–121.

[5] J. C. Engwerda, *On the existence of a positive definite solution of the matrix equation* $X+A^TX^{-1}A=I$, Linear Algebra Appl. **194** (1993), 91–108.

[6] C.-H. Guo and P. Lancaster, *Iterative solution of two matrix equations*, Math. Comp. **68** (1999), no. 228, 1589–1603.

[7] V. I. Hasanov, *Positive definite solutions of the matrix equations* $X\pm A^*X^{-q}A=Q$, Linear Algebra Appl. **404** (2005), 166–182.

[8] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[9] J. Li, *Solutions and improved perturbation analysis for the matrix equation* $X-A^*X^{-p}A=Q$ $(p>0)$, Abatr. Appl. Anal. **2013** (2013); DOI 10.1155/2013/575964.

[10] J. Li and Y. Zhang, *Perturbation analysis of the matrix equation* $X-A^*X^{-p}A=Q$, Linear Algebra Appl. **431** (2009), no. 9, 1489–1501.

[11] ______, *On the existence of positive definite solutions of a nonlinear matrix equation*, Taiwanese J. Math. **18** (2014), no. 5, 1345–1364.

[12] Y. Lim, *Solving the nonlinear matrix equation* $X=Q+\sum_{i=1}^m M_iX^{\delta_i}M_i^*$ *via a contraction principle*, Linear Algebra Appl. **430** (2009), no. 4, 1380–1383.

[13] X.-G. Liu and H. Gao, *On the positive definite solutions of the matrix equations* $X^s\pm A^TX^{-t}A=I_n$, Linear Algebra Appl. **368** (2003), 83–97.

[14] Y. C. Mei, *Uniqueness and numerical method for Hermitian positive definite solutions of nonlinear matrix equation* $X-A^*X^{-q}A=I$ $(q>1)$, Master's Thesis, Shandong Universiteit, Jinan, 2013.

[15] J. F. Wang, Y. H. Zhang, and B. R. Zhu, *The Hermitian positive definite solutions of the matrix equation* $X + A^*X^{-q}A = I(q > 0)$, Chinese J. Numer. Math. Appl. **26** (2004), no. 2, 14–27; translated from Math. Numer. Sin. **26** (2004), no. 1, 61–72.
[16] S. F. Xu, *Matrix computation from control system*, Higher education press, Beijing, 2011
[17] X. Y. Yin and S. Y. Liu, *Positive definite solutions of the matrix equations* $X \pm A^*X^{-q}A = Q$ $(q \geq 1)$, Comput. Math. Appl. **59** (2010), 3727–3739.
[18] X. Zhan, *Computing the extremal positive definite solutions of a matrix equation*, SIAM J. Sci. Comput. **17** (1996), no. 5, 1167–1174.
[19] X. Zhan and J. Xie, *On the matrix equation* $X + A^TX^{-1}A = I$, Linear Algebra Appl. **247** (1996), 337–345.
[20] F. Zhang, *Matrix Theory*, Universitext, Springer-Verlag, New York, 1999.

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