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DENSITY OF THE HOMOTOPY MINIMAL PERIODS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R)

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Abstract. We study the homotopical minimal periods for maps on infra-solvmanifolds of type (R) using the density of the homotopical minimal period set in the natural numbers. This extends the result of [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

1. Introduction

Let $f : X \to X$ be a self-map on a topological space X. We define the following: The set of *periodic points* of f with *minimal period* n

$$P_n(f) = \mathrm{Fix}(f^n) - \bigcup_{k<n} \mathrm{Fix}(f^k)$$

and the set of *homotopy minimal periods* of f

$$\mathrm{HPer}(f) = \bigcap_{g\simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}.$$

The famous Šarkovs'kiĭ theorem characterizes the dynamics (minimal periods) of a map of interval [32]. The set of minimal periods of maps on the circle has been completely described in [2]. This led to a problem of study the set of *homotopy minimal periods* of self-maps. Such an invariant gives an information about rigid dynamics of self-maps. A fundamental question is to determine if the set $\mathrm{HPer}(f)$ of homotopy minimal periods is empty, finite or infinite. This problem was successfully studied in [18] when the space is a torus of any dimension, and this was extended in [14] (see also [15, 27]) to any nilmanifold,

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and in [12, 28] and [19] to the special solvmanifolds modeled on Sol^3 and Sol_1^4 respectively. When X is the Klein bottle, the same problem was studied in [13, 21, 23, 30], and when X is an infra-nilmanifold and f is an expanding map, it was shown in [7, 24, 26, 33] that $\mathrm{HPer}(f)$ is co-finite.

It is now natural to seek for more information when $\mathrm{HPer}(f)$ becomes infinite. When X is a flat manifold, some sufficient conditions on X and f for $\mathrm{HPer}(f)$ to be infinite were found in [10, 29]. For this purpose, the following invariant was considered: The *lower density* of the homotopy minimal periods of f is defined to be ([10, Definition 1.1] and [16, Remark 3.1.60])

$$\mathrm{DH}(f) = \liminf_{n\to\infty} \frac{\#(\mathrm{HPer}(f) \cap [0, n])}{n}.$$

From the definition, $\mathrm{DH}(f) \in [0, 1]$. If $\mathrm{HPer}(f)$ is either empty or finite, then $\mathrm{DH}(f) = 0$. So, we are interested in the case when $\mathrm{HPer}(f)$ is infinite. If one picks randomly a natural number, $\mathrm{DH}(f)$ is a lower bound for the probability of choosing number in $\mathrm{HPer}(f)$. Thus, the real number $\mathrm{DH}(f)$ will bring to us more information about the periods of given map f when $\mathrm{HPer}(f)$ is infinite.

The purpose of this paper is to study the lower density of homotopy minimal periods of maps infra-solvmanifolds of type (R). This extends the results in [10] from flat manifolds to infra-solvmanifolds of type (R). We give some examples of maps on infra-solvmanifolds of dimension three for which the corresponding density is positive.

2. Infra-solvmanifolds

Let S be a connected and simply connected solvable Lie group. A discrete subgroup Γ of S is a *lattice* of S if $\Gamma\backslash S$ is compact, and in this case, we say that the quotient space $\Gamma\backslash S$ is a *special* solvmanifold. Let $\Pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of the lattice $\Gamma = \Pi \cap S$ of S. That is, Π fits the short exact sequence:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S & \longrightarrow & \mathrm{Aff}(S) & \longrightarrow & \mathrm{Aut}(S) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Pi/\Gamma & \longrightarrow & 1.
\end{array}$$

Then Π acts freely on S and the manifold $\Pi\backslash S$ is called an *infra-solvmanifold*. The finite group $\Phi = \Pi/\Gamma$ is the *holonomy group* of Π or $\Pi\backslash S$. It sits naturally in $\mathrm{Aut}(S)$. Thus every infra-solvmanifold $\Pi\backslash S$ is finitely covered by the special solvmanifold $\Gamma\backslash S$. An infra-solvmanifold $\Pi\backslash S$ is of type (R) if S is of *type* (R) or *completely solvable*, i.e., if $\mathrm{ad}\, X : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for all X in the Lie algebra $\mathfrak{S}$ of S. It is known that if S is of type (R), then $\exp : \mathfrak{S} \to S$ is surjective. The abelian groups $\mathbb{R}^n$ and the connected, simply connected nilpotent Lie groups are of type (R). Hence the flat manifolds and the infra-nilmanifolds are examples of infra-solvmanifolds of type (R).

We first recall the following:

Lemma 2.1 ([25, Lemma 2.1])**.** *Let S and S' be simply connected solvable Lie groups, and let $\Pi \subset \mathrm{Aff}(S)$ and $\Pi' \subset \mathrm{Aff}(S')$ be finite extensions of lattices $\Gamma = \Pi \cap S$ of S and $\Gamma' = \Pi' \cap S'$ of S', respectively. Then there exist fully invariant subgroups $\Lambda \subset \Gamma$ and $\Lambda' \subset \Gamma'$ of Π and Π' respectively, which are of finite index, so that any homomorphism $\theta : \Pi \to \Pi'$ restricts to a homomorphism $\Lambda \to \Lambda'$.*

When the infra-solvmanifolds are of type (R), we have the following second Bieberbach type result.

Theorem 2.2 ([20, Theorem 2.3])**.**
(1) *Any continuous map $f : \Pi\backslash S \to \Pi'\backslash S'$ between infra-solvmanifolds of type* (R) *has an affine map $(d, D) : S \to S'$ as a homotopy lift.*
(2) *Any continuous map $f : \Gamma\backslash S \to \Gamma'\backslash S'$ between special solvmanifolds of type* (R) *has a Lie group homomorphism $D : S \to S'$ as a homotopy lift.*

When f is a homeomorphism, D can be chosen to be invertible.

Let $f : \Pi\backslash S \to \Pi\backslash S$ be a self-map on the infra-solvmanifold $\Pi\backslash S$ of type (R) with affine homotopy lift $(d, D) : S \to S$. Since $\mathrm{HPer}(f)$ is a homotopy invariant, we may assume that f is induced by the affine map (d, D). The map f induces a homomorphism $\varphi : \Pi \to \Pi$ on the group Π of covering transformations of the covering projection $S \to \Pi\backslash S$, which is given by

$$(*) \qquad \varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

For any $(a, A) \in \Phi$, let $\varphi(a, A) = (a', A')$; then $A'D = DA$. Thus the homomorphism φ induces a function $\bar{\varphi} : \Phi \to \Phi$ given by $\bar{\varphi}(A) = A'$ and this function satisfies $\bar{\varphi}(A)D = DA$ for all $A \in \Phi$. However, in general, $\bar{\varphi}$ is not necessarily a homomorphism.

Recall further that:

Theorem 2.3 ([11, Theorem 6.1])**.** *Let $f : M \to M$ be a self-map on a compact PL-manifold of dimension ≥ 3. Then f is homotopic to a map g with $P_n(g) = \emptyset$ if and only if $NP_n(f) = 0$.*

The infra-solvmanifolds of dimension 1 or 2 are the circle, the torus and the Klein bottle. Theorem 2.3 for dimensions 1 and 2 is verified respectively in [1], [2] and [13, 21, 30]. Immediately we have for any self-map f on an infra-solvmanifold of any dimension,

$$\mathrm{HPer}(f) = \{k \mid NP_k(f) \neq 0\}.$$

Recalling from [17] that

$$NP_n(f) = \text{(number of irreducible essential orbits of Reidemeister classes of } f^n) \times n,$$

we have

$$\mathrm{HPer}(f) = \{k \mid \exists \text{ an irreducible essential fixed point class of } f^k\}.$$

Recall from [6, Propositions 9.1 and 9.3] the following: Let f be a map on an infra-solvmanifold $\Pi\backslash S$ of type (R) induced by an affine map $(d,D) : S \to S$. For any $\alpha \in \Pi$, $\mathrm{Fix}(\alpha(d,D))$ is an empty set or path connected. Hence every nonempty fixed point class of f is path connected, and every isolated fixed point class forms an essential fixed point class with index $\pm\det(I - A_* D_*)$ where $\alpha = (a, A)$. When the infra-solvmanifold $\Pi\backslash S$ is of type (R), the converse also holds. Namely, every essential fixed point class of f consists of a single element. Remark that $(d,D)^k$ induces the map f^k. Any fixed point class of f^k is of the form $p(\mathrm{Fix}(\alpha(d,D)^k))$ for some $\alpha = (a, A) \in \Pi$. It is essential if and only if it consists of a single element with index $\pm\det(I - A_* D_*^k)$. Note further that it is reducible to ℓ if and only if $\ell \mid k$ and there exists $\beta \in \Pi$ such that $p(\mathrm{Fix}(\beta(d,D)^\ell)) \subset p(\mathrm{Fix}(\alpha(d,D)^k))$, or equivalently, the Reidemeister class $[\beta]$ of f^ℓ is boosted up to the Reidemeister class $[\alpha]$ of f^k. This means that $[\alpha] = [\beta\varphi^\ell(\beta)\varphi^{2\ell}(\beta)\cdots\varphi^{k-\ell}(\beta)]$ as the Reidemeister class of f^k. For some $\gamma \in \Pi$, we thus have $\alpha = \gamma(\beta\varphi^\ell(\beta)\varphi^{2\ell}(\beta)\cdots\varphi^{k-\ell}(\beta))\varphi^k(\gamma)^{-1}$. Hence

$$\begin{aligned}\alpha &= (\gamma\beta\varphi^\ell(\gamma)^{-1})(\varphi^\ell(\gamma)\varphi^\ell(\beta)\varphi^{2\ell}(\gamma)^{-1})\cdots(\varphi^{k-\ell}(\gamma)\varphi^{k-\ell}(\beta)\varphi^k(\gamma)^{-1})\\ &= \beta'\varphi^\ell(\beta')\varphi^{2\ell}(\beta')\cdots\varphi^{k-\ell}(\beta')\end{aligned}$$

with $\beta' = \gamma\beta\varphi^\ell(\gamma)^{-1}$. Consequently, the fixed point class $p(\mathrm{Fix}(\alpha(d,D)^k))$ is irreducible if and only if for any $\beta \in \Pi$ and for any $\ell < k$ with $\ell \mid k$,

$$\alpha(d,D)^k \neq (\beta(d,D)^\ell)^{k/\ell}$$

or

$$\alpha \neq \beta\varphi^\ell(\beta)\varphi^{2\ell}(\beta)\cdots\varphi^{k-\ell}(\beta).$$

For any endomorphism D on S, we denote the differential of $D : S \to S$ by $D_* : \mathfrak{S} \to \mathfrak{S}$. Now, in conclusion, we can summarize the above observation as follows:

Theorem 2.4. *Let $f : \Pi\backslash S \to \Pi\backslash S$ be a self-map on the infra-solvmanifold $\Pi\backslash S$ of type* (R) *with an affine homotopy lift $(d,D) : S \to S$. Let $\varphi : \Pi \to \Pi$ be the homomorphism induced by (d,D), i.e., $\varphi(\alpha)(d,D) = (d,D)\alpha\ \forall \alpha \in \Pi$. Then*

$$\begin{aligned}\mathrm{HPer}(f) &= \left\{ k \;\middle|\; \begin{array}{c} \exists\alpha = (a,A) \in \Pi \text{ such that } \det(I - A_*D_*^k) \neq 0 \text{ and} \\ \forall\ \ell < k \text{ with } \ell \mid k,\ \forall\ \beta \in \Pi, \\ \alpha(d,D)^k \neq (\beta(d,D)^\ell)^{k/\ell} \end{array} \right\}\\ &= \left\{ k \;\middle|\; \begin{array}{c} \exists\alpha = (a,A) \in \Pi \text{ such that } \det(I - A_*D_*^k) \neq 0 \text{ and} \\ \forall\ \ell < k \text{ with } \ell \mid k,\ \forall\ \beta \in \Pi, \\ \alpha \neq \beta\varphi^\ell(\beta)\varphi^{2\ell}(\beta)\cdots\varphi^{k-\ell}(\beta) \end{array} \right\}.\end{aligned}$$

In order to generalize the results of [10] from flat manifolds to infra-solvmanifolds of type (R), we need the following observation which is crucial in our discussion.

Lemma 2.5. *Let Λ be a lattice of a connected, simply connected solvable Lie group S of type* (R)*, and let $K : S \to S$ be a Lie group homomorphism such that $K(\Lambda) \subset \Lambda$. For some choice of a linear basis in the Lie algebra $\mathfrak{S}$ of S, K_* is an upper block triangular matrix with diagonal blocks integer matrices; in particular $\det K_*$ is an integer.*

Proof. First we assume that S is nilpotent and thus Λ is a finitely generated torsion-free nilpotent group. The lower central series of Λ is defined inductively via $\gamma_1(\Lambda) = \Lambda$ and $\gamma_{i+1}(\Lambda) = [\Lambda, \gamma_i(\Lambda)]$. The isolator of a subgroup H of Λ is defined by

$$\sqrt[\Lambda]{H} = \{x \in \Lambda \mid x^k \in H \text{ for some } k \geq 1\}.$$

It is known ([31, p. 473], [4, Chap. 1] or [22]) that the sequence

$$\Lambda = \Lambda_1 \supset \Lambda_2 = \sqrt[\Lambda]{\gamma_2(\Lambda)} \supset \cdots \supset \Lambda_c = \sqrt[\Lambda]{\gamma_c(\Lambda)} \supset \Lambda_{c+1} = 1$$

forms a central series with $\Lambda_i/\Lambda_{i+1} \cong \mathbb{Z}^{k_i}$. Now we can choose a generating set

$$\mathbf{a} = \{\mathbf{a}_1, \ldots, \mathbf{a}_c\}$$

in such a way that Λ_i is the group generated by Λ_{i+1} and $\mathbf{a}_i = \{a_{i1}, \ldots, a_{in_i}\}$. We refer to $\mathbf{a} = \{\mathbf{a}_1, \ldots, \mathbf{a}_c\}$ as a preferred basis of Λ. Under the diffeomorphism $\log : S \to \mathfrak{S}$, the image $\log \mathbf{a}$ of $\mathbf{a}$ is a basis of the vector space $\mathfrak{S}$. Note also that $\Lambda_i = \Lambda \cap \gamma_i(S)$ is a lattice of $\gamma_i(S)$ and a fully invariant subgroup of Λ. Since $K(\Lambda) \subset \Lambda$, it follows that $K(\Lambda_i) \subset \Lambda_i$ and the differential of K is expressed as a rational matrix with respect to the basis $\log \mathbf{a}$ of the form

$$\begin{bmatrix} K_{c*} & * & \cdots & * \\ 0 & K_{c-1_*} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{1*} \end{bmatrix},$$

where each square matrix K_{i*} is an integer matrix, see also [24, Lemma 4.2].

Now we go back to the cases where S is solvable of type (R). According to [34, Remark 8.2], Λ is a positive polycyclic group and S is its supersolvable completion. Let N be the maximal connected nilpotent normal subgroup of S. Then $\Lambda \cap N$ is the nilradical $\mathrm{nil}(\Lambda)$ of Λ, which is a lattice of N, see [34, Proposition 5.1]. Hence we have the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & S & \longrightarrow & S/N \cong \mathbb{R}^s & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathrm{nil}(\Lambda) & \longrightarrow & \Lambda & \longrightarrow & \Lambda/\mathrm{nil}(\Lambda) \cong \mathbb{Z}^s & \longrightarrow & 1. \end{array}$$

By the assumption on K, K restricts to a homomorphism $\kappa : \Lambda \to \Lambda$. Thus κ and hence K in turn restricts to $\kappa' : \mathrm{nil}(\Lambda) \to \mathrm{nil}(\Lambda)$ and then induces a homomorphism $\bar{\kappa} : \mathbb{Z}^s \to \mathbb{Z}^s$. We choose a preferred basis of $\mathrm{nil}(\Lambda)$ under which $K' : N \to N$ yields a rational matrix K'_* with diagonal blocks integer matrices as above. Now we can complete the set of generators of $\mathrm{nil}(\Lambda)$ to a set of generators $\mathbf{a} = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_c\}$, called a *preferred basis*, of Λ so that $\bar\kappa$ induces an integer matrix $\bar K_*$ and so κ induces an upper block triangular matrix

$$K_* = \begin{bmatrix} K'_* & * \\ 0 & \bar K_* \end{bmatrix}$$

so that all diagonal blocks are integer matrices and hence $\det K_*$ is an integer. □

Remark 2.6. Let Λ be a lattice of a connected, simply connected solvable Lie group S of type (R). In the proof of the above lemma, we can choose a preferred basis (generator) $\mathbf{a}$ of Λ so that $\log \mathbf{a}$ is a (linear) basis of the Lie algebra $\mathfrak{S}$ of S and if K is a homomorphism on S such that $K(\Lambda) \subset \Lambda$, then K_* is an upper block triangular matrix with diagonal blocks integer matrices with respect to the ordered basis $\log \mathbf{a}$. We also refer to $\log \mathbf{a}$ as a *preferred basis* of Λ or $\mathfrak{S}$.

3. Density of homotopy minimal periods

In this section, we will generalize the main result of [10] from flat manifolds to infra-solvmanifolds of type (R).

Let f be a self-map on an infra-solvmanifold $\Pi\backslash S$ of type (R) with holonomy group Φ. Let f have an affine homotopy lift (d, D). Recall that f induces a homomorphism $\varphi : \Pi \to \Pi$ satisfying the identity $(*)$: $\varphi(\alpha)(d, D) = (d, D)\alpha$, $\forall \alpha \in \Pi \subset S \rtimes \mathrm{Aut}(S)$. Let $\Gamma = \Pi \cap S$. It is not necessarily true that $\varphi(\Gamma) \subset \Gamma$. Using Lemma 2.1, we can choose a lattice $\Lambda \subset \Gamma$ of S so that $\varphi(\Lambda) \subset \Lambda$. Thus for any $\lambda = (\lambda, I) \in \Lambda$, we have $\varphi(\lambda) = (\varphi(\lambda), I)$ and so

$$(\varphi(\lambda), I)(d, D) = (d, D)(\lambda, I).$$

Evaluating at the identity 1 of S, we obtain that $\varphi(\lambda) \cdot d = d \cdot D(\lambda)$. Consequently, we have that

$$\varphi|_\Lambda = \mu(d)D.$$

Furthermore, for any $(a, A) \in \Pi$, since Γ is a normal subgroup of Π, we have $(a, A)(\gamma, I)(a, A)^{-1} \in \Gamma$; this implies $(\mu(a)A)(\Gamma) \subset \Gamma$ and $(\mu(a)A)(\Lambda) \subset \Lambda$. Consequently, we have homomorphisms $\mu(d)D, \mu(a)A : S \to S$ such that $(\mu(d)D)(\Lambda) \subset \Lambda$ and $(\mu(a)A)(\Lambda) \subset \Lambda$. We have to notice here that it is not necessary to have that $D(\Lambda), A(\Lambda) \subset \Lambda$. By Remark 2.6, we can choose a preferred basis $\mathbf{a}$ of Λ so that $(\mu(d)D)_* = \mathrm{Ad}(d)D_*$ and $(\mu(a)A)_* = \mathrm{Ad}(a)A_*$ are upper block triangular rational matrices with diagonal blocks integer matrices with respect to the basis $\log \mathbf{a}$ of $\mathfrak{S}$.

In what follows, we shall denote $\mu(d)D$ and $\mu(a)A$ by $\mathbb{D}$ and $\mathbb{A}$, respectively. By Lemma 2.5, the differentials of $\mathbb{D}$ and $\mathbb{A}$ induce rational matrices with integer blocks on the diagonal. By considering only integer blocks on the diagonal, we obtain integer matrices, denoted by $\mathbb{D}_*$ and $\mathbb{A}_*$. Hence,

$$\mathbb{D}_* = \begin{bmatrix} \mathbb{D}_{c*} & 0 & \cdots & 0 \\ 0 & \mathbb{D}_{c-1_*} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbb{D}_{0*} \end{bmatrix}.$$

This does not change the determinant and the eigenvalue of the differentials of $\mathbb{D}$ and $\mathbb{A}$. We can call $\mathbb{D}_*$ and $\mathbb{A}_*$ *linearizations* of D, (d, D) or f, and $\alpha = (a, A) \in \Pi$, respectively. We denote the free abelian group of all integer linear combinations of the basis vectors in $\log \mathbf{a} = \{\log \mathbf{a}_c, \log \mathbf{a}_{c-1}, \dots, \log \mathbf{a}_0\}$ by simply $\mathcal{Z} = \mathcal{Z}_c \oplus \mathcal{Z}_{c-1} \oplus \cdots \oplus \mathcal{Z}_0$. Then we have $\mathbb{D}_{i*}(\mathcal{Z}_i) \subset \mathcal{Z}_i$, $\mathbb{D}_*(\mathcal{Z}) \subset \mathcal{Z}$ and $\mathbb{A}_*(\mathcal{Z}) \subset \mathcal{Z}$.

In the following, we provide three lemmas that generalize [10, Lemmas 4.3 and 4.5, Proposition 4.6] from flat manifolds to infra-solvmanifolds of type (R). These are essential in proving our main results.

Lemma 3.1. *Let $M = \Pi\backslash S$ be an infra-solvmanifold of type* (R). *Let f be a self-map on M with an affine homotopy lift (d, D). Assume that*

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det \mathbb{D}_* \neq 0, \pm 1$.

Then there exists a positive integer N_0 such that

$$\Big| \det \left(\frac{I - \mathbb{D}_*^{k\ell}}{I - \mathbb{D}_*^{\ell}} \right) \Big| > 1$$

for all positive integers k and ℓ, provided their prime divisors are all greater than N_0.

Proof. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of $\mathbb{D}_*$ counted with multiplicities. We first show that all $1 - \lambda_i^\ell$ are nonzero. In fact, if $1 - \lambda_i^\ell = 0$ for some i, then $\lambda_i^\ell = 1$ and so λ_i is a primitive ℓ_0-th root of unity for some ℓ_0 where $1 \leq \ell_0 \mid \ell$. Since $\lambda_i \neq 1$, $\ell_0 > 1$. If p is a prime divisor of ℓ_0, then it is a prime divisor of ℓ and so $p > N_0$. It follows that $[\mathbb{Q}(\lambda_i) : \mathbb{Q}] \geq p - 1 > m$. This contradicts the fact that $[\mathbb{Q}(\lambda_i) : \mathbb{Q}]$ is smaller than the size m of $\mathbb{D}_*$. Thus $I - \mathbb{D}_*^\ell$ is invertible and $(I - \mathbb{D}_*^{k\ell})/(I - \mathbb{D}_*^\ell) = I + \mathbb{D}_*^\ell + \cdots + \mathbb{D}_*^{(k-1)\ell}$ and

$$\det \left(\frac{I - \mathbb{D}_*^{k\ell}}{I - \mathbb{D}_*^{\ell}} \right) = \prod_{i=1}^{m} \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell}.$$

Let $N_0 > m+1$ and let ℓ be a positive integer all of whose prime divisors are greater than N_0. Assume $|\lambda_i| = 1$. By our assumption, $\lambda_i \neq 1$ and is a root of unity. The above argument shows that $1 - \lambda_i^\ell \neq 0$ and by the same reasoning $1 - \lambda_i^{k\ell} \neq 0$; thus $(1 - \lambda_i^{k\ell})/(1 - \lambda_i^\ell)$ is nonzero and finite for each such k and ℓ. Hence we can choose a constant $\delta > 0$ such that for all such k and ℓ

$$\Big| \frac{1 - \lambda_i^{k\ell}}{1 - \lambda_i^\ell} \Big| > \delta.$$

For λ_i with $|\lambda_i| \neq 1$, as $N_0 \to \infty$, we have

$$\Big|\frac{1-\lambda_i^{k\ell}}{1-\lambda_i^{\ell}}\Big| = |1+\lambda_i^{\ell}+\lambda_i^{2\ell}+\cdots+\lambda_i^{k-1}| \to \begin{cases} 1 & \text{when } |\lambda_i|<1 \\ \infty & \text{when } |\lambda_i|>1. \end{cases}$$

By the assumption that $|\det \mathbb{D}_*| > 1$, there exists an eigenvalue whose absolute value is bigger than 1. Hence as $N_0 \to \infty$ we have

$$\prod_{i=1}^{m}\Big|\frac{1-\lambda_i^{k\ell}}{1-\lambda_i^{\ell}}\Big| \to \infty.$$

Consequently, for N_0 large enough, the lemma is proved. □

Lemma 3.2. *Let $M = \Pi\backslash S$ be an infra-solvmanifold of type* (R). *Let f be a self-map on M with an affine homotopy lift (d, D). Assume that*

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det \mathbb{D}_* \neq 0, \pm 1$.

Then there exists a positive integer N_1 such that

$$|\det(I-\mathbb{D}_*^k)| > \sum_{1<\ell\mid k} |\det(I-\mathbb{D}_*^{k/\ell})|$$

for all positive integers k, provided all its positive prime divisors are greater than N_1.

Proof. Let $\lambda_1, \ldots, \lambda_m$ be the eigenvalues of $\mathbb{D}_*$ counted with multiplicities. From our assumptions and hence from the observations in the proof of Lemma 3.1, we have:

- Since $\det \mathbb{D}_* \neq 0$, all λ_i are nonzero.
- If $|\lambda_i| = 1$, then $\lambda_i \neq 1$ and λ_i is a root of unity, and $1-\lambda_i^k \neq 0$ for all k whose prime divisors are $> N_0$ where N_0 is a positive integer chosen in the previous lemma; hence there are constants $0 < \delta_1 < \delta_2$ such that for all λ_i with $|\lambda_i| \leq 1$, we have $\delta_1 \leq |1-\lambda_i^k| \leq \delta_2$ for all k with this property.

For those eigenvalues with $|\lambda_i| > 1$, we claim that there is a sufficiently large k such that

$$\sum_{1<\ell\mid k} |1-\lambda_i^{k/\ell}| < |1-\lambda_i^k|.$$

Suppose on the contrary that for any $K > 0$ there is $k_0 > K$ such that

$$|1-\lambda_i^{k_0}| \leq \sum_{1<\ell\mid k_0} |1-\lambda_i^{k_0/\ell}|.$$

Then

$$|1-\lambda_i^{k_0}| \leq \sum_{1<\ell\mid k_0} |1-\lambda_i^{k_0/2}| < \tau(k_0)|1-\lambda_i^{k_0/2}|,$$

where $\tau(k)$ is the number of all the divisors of k. Since $\tau(k) \leq 2\sqrt{k}$ (see [16, Exercise 3.2.17]), we have

$$2\sqrt{k_0} > |1+\lambda_i^{k_0/2}| \geq |\lambda_i|^{k_0/2} - 1,$$

which contradicts the obvious fact that $\lim_{k\to 0} \sqrt{k}/(|\lambda_i|^{k/2}-1) = 0$.

Therefore we can choose $N_1 \geq N_0$ such that if k is a positive integer whose prime divisors are $\geq N_1$, then

$$\begin{aligned} \sum_{1<\ell \mid k} |\det(I-\mathbb{D}_*^{k/\ell})| &= \sum_{1<\ell \mid k} \left(\prod_{i=1}^{m} |1-\lambda_i^{k/\ell}| \right) \\ &\leq \prod_{i=1}^{m} \left(\sum_{1<\ell \mid k} |1-\lambda_i^{k/\ell}| \right) \\ &< \prod_{i=1}^{m} |1-\lambda_i^{k}| = |\det(I-\mathbb{D}_*^{k})|. \end{aligned}$$

□

Lemma 3.3. *Let $M = \Pi\backslash S$ be an infra-solvmanifold of type* (R). *Let f be a self-map on M with an affine homotopy lift (d, D). Assume that*

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det \mathbb{D}_* \neq 0, \pm 1$.

Then there exists a positive integer N_2 such that the equality

$$\mathcal{Z} = \bigcup_{1<\ell \mid k} (I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(\mathcal{Z})$$

is impossible for all positive integers k, provided its positive prime divisors are all greater than N_2.

Proof. Remark that the proof of Lemma 3.1 shows that there exists a positive integer N_0 such that for all positive integers k whose prime divisors are greater than N_0, $I-\mathbb{D}_*^{\ell}$ has nonzero determinant if $\ell \mid k$. Since $\mathbb{D}_*(\mathcal{Z}) \subset \mathcal{Z}$, we have

$$\begin{aligned} (I-\mathbb{D}_*^{k})(\mathcal{Z}) &= (I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(I-\mathbb{D}_*^{\ell})(\mathcal{Z}) \\ &\subset (I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(\mathcal{Z}) \end{aligned}$$

for all k and ℓ with $\ell \mid k$. Thus if we had the equality

$$\mathcal{Z} = \bigcup_{\ell \mid k, 1<\ell<k} (I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(\mathcal{Z})$$

we would have

$$\begin{aligned} \mathcal{Z}/(I-\mathbb{D}_*^{k})(\mathcal{Z}) &= \left(\bigcup_{\ell \mid k, \ell\neq 1,k} (I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(\mathcal{Z}) \right) /(I-\mathbb{D}_*^{k})(\mathcal{Z}) \\ &= \bigcup_{\ell \mid k, \ell\neq 1,k} \left((I+\mathbb{D}_*^{\ell}+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(\mathcal{Z})/(I-\mathbb{D}_*^{k})(\mathcal{Z}) \right). \end{aligned}$$

We remark that $\mathcal{Z}$ is a free abelian group and $I+\mathbb{D}_*^\ell+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell}$ defines an injective endomorphism of $\mathcal{Z}$. In particular, we have an isomorphism

$$\begin{aligned}&\mathcal{Z}/(I-\mathbb{D}_*^\ell)(\mathcal{Z})\\ &\cong (I+\mathbb{D}_*^\ell+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})\mathcal{Z}/(I+\mathbb{D}_*^\ell+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})(I-\mathbb{D}_*^\ell)(\mathcal{Z})\\ &= (I+\mathbb{D}_*^\ell+\mathbb{D}_*^{2\ell}+\cdots+\mathbb{D}_*^{k-\ell})\mathcal{Z}/(I-\mathbb{D}_*^k)(\mathcal{Z}).\end{aligned}$$

This would imply

$$\mathcal{Z}/(I-\mathbb{D}_*^k)(\mathcal{Z})\cong \bigcup_{\ell\,\mid\, k,\ell\neq 1,k}\big(\mathcal{Z}/(I-\mathbb{D}_*^\ell)(\mathcal{Z})\big)$$

and hence we would have

$$|\det(I-\mathbb{D}_*^k)|\le \sum_{\ell\,\mid\, k,\ell\neq 1,k}|\det(I-\mathbb{D}_*^\ell)|\le \sum_{1<\ell\,\mid\, k}|\det(I-\mathbb{D}_*^{k/\ell})|.$$

This contradicts Lemma 3.2. □

Now we are ready to state and prove our main results.

Theorem 3.4. *Let $M=\Pi\backslash S$ be an infra-solvmanifold of type* (R)*. Let f be a self-map on M with an affine homotopy lift (d,D). Let $\varphi:\Pi\to\Pi$ be the homomorphism satisfying*

$$\varphi(\alpha)(d,D)=(d,D)\alpha,\quad \forall\alpha\in\Pi.$$

Assume that

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det\mathbb{D}_*\neq 0,\pm 1$;
(3) $\mathrm{fix}(\bar\varphi:\Phi\to\Phi)=\{I\}$.

Then there exists an integer N with the following property: if k is a positive integer with prime factorization $k=p_1^{n_1}\cdots p_s^{n_s}$ such that all p_i's are greater than N, then $k\in\mathrm{HPer}(f)$.

Proof. Choose an integer N so that $N\ge\max\{m+1,N_2,\ \text{order of }\bar\varphi\}$. Let $k=p_1^{n_1}\cdots p_s^{n_s}$ be a prime factorization of k such that all p_i's are greater than N. Then we have to show that $k\in\mathrm{HPer}(f)$. For this purpose, by Theorem 2.4, we need to find $\alpha=(a,A)\in\Pi$ satisfying:

- $\det(I-A_*D_*^k)\neq 0$,
- $\forall\,\ell<k$ with $\ell\mid k$, $\forall\,\beta\in\Pi$, $\alpha(d,D)^k\neq(\beta(d,D)^\ell)^{k/\ell}$.

We will show that we can choose $\alpha=(a,I)$ in $\Gamma\subset\Pi$. Recall first that the proof of Lemma 3.1 shows that there exists a positive integer N_0 such that for all positive integers k whose prime divisors are larger than N_0, $I-\mathbb{D}_*^\ell$ has nonzero determinant if $\ell\mid k$. Since $N\ge N_0$, $\det(I-\mathbb{D}_*^\ell)\neq 0$ for all $\ell\mid k$. In particular, $\det(I-\mathbb{D}_*^k)\neq 0$. By [25, Lemma 3.3] [9, Theorem 1], we have $\det(I-D_*^k)=\det(I-\mathbb{D}_*^k)\neq 0$.

It remains to prove the second condition. We assume on the contrary that for any $\alpha = (a, I) \in \Gamma$, there exists $\ell < k$ with $\ell \mid k$ and there exist $\beta = (b, B) \in \Pi$ such that $\alpha(d, D)^k = (\beta(d, D)^\ell)^{k/\ell}$, which is equivalent to

$$(\dagger) \qquad \alpha = \beta\varphi^\ell(\beta)\varphi^{2\ell}(\beta)\cdots\varphi^{k-\ell}(\beta).$$

Now we recall that since D is an automorphism, φ is the conjugation by (d, D), $\varphi|_\Gamma = \mu(d)D$ and $\bar{\varphi}$ is the conjugation by D. The matrix part (the holonomy part) of both sides of (†) yields

$$I = B\bar{\varphi}^\ell(B)\bar{\varphi}^{2\ell}(B)\cdots\bar{\varphi}^{k-\ell}(B).$$

Taking $\bar{\varphi}^\ell$, we have

$$I = \bar{\varphi}^\ell(B)\bar{\varphi}^{2\ell}(B)\cdots\bar{\varphi}^{k-\ell}(B)\bar{\varphi}^k(B).$$

Hence

$$\bar{\varphi}^k(B)^{-1} = B^{-1} = \bar{\varphi}^\ell(B)\bar{\varphi}^{2\ell}(B)\cdots\bar{\varphi}^{k-\ell}(B).$$

This gives us $\bar{\varphi}^k(B) = B$. By the choice of k, k must be relatively prime to the order p of $\bar{\varphi}$. Choose $x, y \in \mathbb{Z}$ so that $kx + py = 1$. Since $\bar{\varphi} = \bar{\varphi}^{kx+py} = (\bar{\varphi}^k)^x$, it follows that $\bar{\varphi}(B) = B$. Since $\mathrm{fix}(\bar{\varphi}) = \{I\}$ by our assumption, we have $B = I$. Plugging into (†), we have

$$a = b\varphi^\ell(b)\varphi^{2\ell}(b)\cdots\varphi^{k-\ell}(b).$$

Since $\varphi|_\Gamma = \mu(d)D = \mathbb{D}$, we have

$$a = b\mathbb{D}^\ell(b)\mathbb{D}^{2\ell}(b)\cdots\mathbb{D}^{k-\ell}(b)$$

for some $\ell < k$ with $\ell \mid k$.

Now we have to show that for any $\ell < k$ with $\ell \mid k$

$$\{e\mathbb{D}^\ell(e)\mathbb{D}^{2\ell}(e)\cdots\mathbb{D}^{k-\ell}(e) \mid e \in \Gamma\} \neq \Gamma.$$

Recall in the proof of Lemma 2.5 that Γ has a central series

$$\Gamma = \Gamma_0 \supset \mathrm{nil}(\Gamma) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_c \supset \Gamma_{c+1} = 1$$

with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$. Since $\mathbb{D}(\Gamma_i) \subset \Gamma_i$, it induces $\bar{\mathbb{D}}_i : \Gamma_i/\Gamma_{i+1} \to \Gamma_i/\Gamma_{i+1}$. Note also that

$$\mathbb{D}_* = \begin{bmatrix} \bar{\mathbb{D}}_c & 0 & \cdots & 0 \\ 0 & \bar{\mathbb{D}}_{c-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\mathbb{D}}_0 \end{bmatrix},$$

where $\bar{\mathbb{D}}_i$ are integer matrices. Hence some $\bar{\mathbb{D}}_i$ satisfies the assumptions (1) and (2). By Lemma 3.3, we have

$$\{\bar{e} + \bar{\mathbb{D}}_i^\ell(\bar{e}) + \bar{\mathbb{D}}_i^{2\ell}(\bar{e}) + \cdots + \bar{\mathbb{D}}_i^{k-\ell}(\bar{e}) \mid \bar{e} \in \Gamma_i/\Gamma_{i+1}\} \neq \Gamma_i/\Gamma_{i+1}.$$

This proves our assertion. Hence $a \in \Gamma$ can be chosen so that

$$a \neq b\varphi^\ell(b)\varphi^{2\ell}(b)\cdots\varphi^{k-\ell}(b)$$

for any $b \in \Gamma$. This contradiction proves the second condition. □

Corollary 3.5. *Let $M = \Pi\backslash S$ be an infra-solvmanifold of type* (R) *and let f be a self-map on M with an affine homotopy lift (d, D). Let $\varphi : \Pi \to \Pi$ be the homomorphism satisfying*

$$\varphi(\alpha)(d, D) = (d, D)\alpha, \quad \forall \alpha \in \Pi.$$

Assume that

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det \mathbb{D}_* \neq 0, \pm 1$;
(3) $\mathrm{fix}(\bar{\varphi} : \Phi \to \Phi) = \{I\}$.

Then $\mathrm{DH}(f)$ *is positive.*

Proof. By Theorem 3.4, there exists an integer N with the following property: for any positive integer $k = p_1^{n_1} \cdots p_s^{n_s}$ with all p_i's distinct primes and greater than N, $k \in \mathrm{HPer}(f)$. Thus

$$\mathrm{HPer}(f) \supset \{k \mid \text{any prime divisor of } k \text{ is} > N\}.$$

Let $q_1, \ldots, q_\ell$ be the all prime numbers which are smaller than or equal to N. Then the set

$$\{k \mid k \equiv 1 \mod q_1 \cdots q_\ell\}$$

is contained in the set on the right-hand side of the above. For, if $k \equiv 1 \mod q_1 \cdots q_\ell$ and if p is a prime divisor of k with $p \leq N$, then $p = q_j$ for some j; thus $q_j \mid k$ and $q_j \mid k - 1$ and hence $q_j = 1$, a contradiction.

Furthermore, we have that $N! \mid k - 1$ implies $q_1 \cdots q_\ell \mid k - 1$. This shows that

$$\mathrm{HPer}(f) \supset \{k \mid k \equiv 1 \mod N!\}$$

and the set on the right-hand side has density $1/N!$. Consequently,

$$\mathrm{DH}(f) \geq 1/N! > 0. \qquad \square$$

A special solvmanifold is an infra-solvmanifold with the trivial holonomy group. Hence the third condition of Corollary 3.5 on such a manifold is automatically fulfilled. Immediately we have:

Corollary 3.6. *Let f be a self-map on a special solvmanifold M with a Lie group homomorphism D as a homotopy lift. Assume that*

(1) *any eigenvalue λ of $\mathbb{D}_*$ of modulus* 1 *is a root of unity, but not* 1;
(2) $\det \mathbb{D}_* \neq 0, \pm 1$.

Then $\mathrm{DH}(f)$ *is positive.*

4. Computational results

In this section, we will consider some examples on infra-solvmanifolds up to dimension three. For infra-solvmanifolds up to dimension 3, there are only three possibilities for the solvable Lie group G on which the manifold is modeled. It can be modeled on either the abelian groups $\mathbb{R}^n (n \leq 3)$, the 2-step nilpotent Heisenberg group Nil or the 2-step solvable Lie group Sol.

We can find a complete description of $\mathrm{HPer}(f)$ for maps f on tori in [2] and [18], and on the Klein bottle in [21]. The remaining infra-solvmanifolds of dimension 3 are three-dimensional flat manifolds, infra-nilmanifolds on Nil and infra-solvmanifolds of Sol.

We will give three examples, one from each remaining manifold. For any self-map f on the manifold $\Pi\backslash G$, let $\varphi : \Pi \to \Pi$ be a homomorphism induced by f. Consider an affine map (d, D) on G satisfying $(*)$. To apply Corollary 3.5, we have to consider the case where $\mathbb{D} = \mu(d)D$ is invertible. If this is the case, then $(*)$ says that φ is the conjugation by (d, D), that is, $\varphi(\alpha) = (d, D)\alpha(d, D)^{-1}$. If $\alpha = (a, I) \in \Gamma$, then

$$\varphi(\alpha) = (d, D)(a, I)(d, D)^{-1} = (dD(a)d^{-1}, I) = (\mu(d)D(a), I).$$

Here $\mu(d)$ is the automorphism on G obtained by conjugating by the element $d \in G$. Thus $\varphi(\Gamma) \subset \Gamma$ and $\varphi|_\Gamma = \mu(d)D = \mathbb{D}$, and hence $\bar{\varphi}$ is the conjugation by D. In particular $\bar{\varphi} : \Phi \to \Phi$ is an isomorphism.

We start with the following easy observation.

Lemma 4.1. *Let Φ be a group with presentation*

$$\Phi = \langle x, y \mid x^2 = y^2 = 1, xy = yx\rangle,$$

and let ψ be an isomorphism on Φ. Then $\mathrm{fix}(\psi) = 1$ *if and only if ψ satisfies one of the following*:

- $\psi(x) = y$, $\psi(y) = xy$
- $\psi(x) = xy$, $\psi(y) = x$

4.1. A flat manifold of dimension three

We have a complete classification of three-dimensional Bieberbach groups. There are six orientable ones and four nonorientable ones, see the book [35, Theorems 3.5.5 and 3.5.9]. Every group has an explicit representation into $\mathbb{R}^3 \rtimes \mathrm{GL}(4, \mathbb{Z})$ (not into $\mathbb{R}^4 \rtimes O(4)$) in this book. Of course one of them is $\mathfrak{G}_1 = \mathbb{Z}^3$. Let

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix},\ e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix},\ e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},$$

and let $t_i = (e_i, I) \in \mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$. Then t_1, t_2 and t_3 generate the subgroup Γ of $\mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$, which is isomorphic to the group of all integer vectors of $\mathbb{R}^3$.

Let $\alpha = (a, A)$, $\beta = (b, B)$ and $\gamma = (c, C)$ be elements of $\mathbb{R}^3 \rtimes \mathrm{GL}(3, \mathbb{Z})$, where

$$a = \begin{bmatrix}\frac12\\0\\0\end{bmatrix},\ b = \begin{bmatrix}0\\\frac12\\\frac12\end{bmatrix},\ c = \begin{bmatrix}\frac12\\\frac12\\\frac12\end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \ C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then A, B, C have order 2 and $AB = C = BA$, and

$$\mathfrak{G}_6 = \left\langle t_1, t_2, t_3, \alpha, \beta, \gamma \;\middle|\; \begin{array}{l} [t_i, t_j] = 1, \gamma\beta\alpha = t_1 t_3, \\ \alpha^2 = t_1, \alpha t_2 \alpha^{-1} = t_2^{-1}, \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta t_1 \beta^{-1} = t_1^{-1}, \beta^2 = t_2, \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma t_1 \gamma^{-1} = t_1^{-1}, \gamma t_2 \gamma^{-1} = t_2^{-1}, \gamma^2 = t_3 \end{array} \right\rangle.$$

Thus $\mathfrak{G}_6$ fits the short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathfrak{G}_6 \longrightarrow \Phi \longrightarrow 1,$$

where $\Phi = \langle A, B \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Every element of $\mathfrak{G}_6$ can be written uniquely in the form $\alpha^k \beta^m t_3^n$. We first observe the following: Since $\gamma\beta\alpha = t_1 t_3$, we have $\gamma = \alpha\beta^{-1} t_3$, and

$$\beta^m \alpha^k = \begin{cases} \alpha^k \beta^m & \text{when } (k, m) = (e, e) \\ \alpha^{-k} \beta^m & \text{when } (k, m) = (e, o) \\ \alpha^k \beta^{-m} & \text{when } (k, m) = (o, e) \\ \alpha^{-k} \beta^{-m} t_3 & \text{when } (k, m) = (o, o) \end{cases}$$

and

$$\begin{aligned} (\alpha^k \beta^m t_3^n)^2 &= \alpha^k (\beta^m \alpha^k) \beta^m t_3^{((-1)^{k+m}+1)n} \\ &= \begin{cases} \alpha^{2k} \beta^{2m} t_3^{2n} & \text{when } (k, m) = (e, e) \\ \beta^{2m} & \text{when } (k, m) = (e, o) \\ \alpha^{2k} & \text{when } (k, m) = (o, e) \\ t_3^{2n-1} & \text{when } (k, m) = (o, o). \end{cases} \end{aligned}$$

Let $\varphi : \mathfrak{G}_6 \to \mathfrak{G}_6$ be any homomorphism that induces an isomorphism $\bar{\varphi}$ on Φ satisfying $\mathrm{fix}(\bar{\varphi}) = \{I\}$. By Lemma 4.1, we have either $\bar{\varphi}(\bar{\alpha}) = \bar{\beta}$, $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}\bar{\beta}$ or $\bar{\varphi}(\bar{\alpha}) = \bar{\alpha}\bar{\beta}$, $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}$.

In general, φ has the form

$$\varphi(\alpha) = \alpha^{k_1} \beta^{m_1} t_3^{n_1}, \ \varphi(\beta) = \alpha^{k_2} \beta^{m_2} t_3^{n_2}, \ \varphi(t_3) = \alpha^{k_3} \beta^{m_3} t_3^{n_3}.$$

Since $\gamma = \alpha\beta^{-1} t_3$, a simple calculation shows that

$$\varphi(\gamma) = \alpha^{k_1} \beta^{m_1 - m_2} \alpha^{-k_2 + k_3} \beta^{m_3} t_3^{(-1)^{k_2 + k_3 + m_2 + m_3}(n_1 - n_2) + n_3}.$$

Case $\bar{\varphi}(\bar{\alpha}) = \bar{\beta}$, $\bar{\varphi}(\bar{\beta}) = \bar{\alpha}\bar{\beta}$.
Then k_1 is even and k_2, m_1, m_2 are odd. So, we have

$$\varphi(\gamma) = \alpha^{k_1 - k_2 + k_3} \beta^{-m_1 + m_2 + m_3} t_3^{n_1 - n_2 + n_3}.$$

Since $\alpha^2 = t_1, \beta^2 = t_2$ and $\gamma^2 = t_3$, a simple calculation shows that

$$\alpha^2 = t_1 \Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^{k_1} \beta^{m_1} t_3^{n_1})^2 = \beta^{2m_1} = t_2^{m_1};$$

$$\beta^2 = t_2 \Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2}\beta^{m_2}t_3^{n_2})^2 = t_3^{2n_2-1};$$
$$\gamma^2 = t_3 \Rightarrow k_3 = 2(k_1 - k_2 + k_3), m_3 = 0, n_3 = 0.$$

Hence $\varphi(t_3) = \alpha^{k_3} = t_1^{k_3/2}$, and it follows that $\varphi(\Gamma) \subset \Gamma$ and so $\mathbb{D} = \varphi|_\Gamma$ and

$$\mathbb{D}_* = \begin{bmatrix} 0 & 0 & \frac{k_3}{2} \\ m_1 & 0 & 0 \\ 0 & 2n_2 - 1 & 0 \end{bmatrix}.$$

Thus $\det \mathbb{D}_* = \frac{k_3}{2} m_1(2n_2 - 1) \neq 0, \pm 1$ if and only if either $k_3 \neq 0$ or $k_3 = \pm 2, m_1 = \pm 1, 2n_2 - 1 = \pm 1$. If $\mathbb{D}_*$ has an eigenvalue of modulus 1, then $\det \mathbb{D}_* = \pm 1$. This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when k_1, k_3, m_3 are even and k_2, m_1, m_2 are odd, if $k_3 \neq 0$ or

$$(k_3, m_1, n_2) \notin \{(2, 1, 0), (2, -1, 0), (-2, 1, 0), (-2, -1, 0), \\ (2, 1, 1), (2, -1, 1), (-2, 1, 1), (-2, -1, 1)\},$$

then $\mathrm{DH}(f) > 0$.

Case $\bar\varphi(\bar\alpha) = \bar\alpha\bar\beta$, $\bar\varphi(\bar\beta) = \bar\alpha$.
Then k_1, k_2, m_1 are odd and m_2 is even. So, we have

$$\varphi(\gamma) = \alpha^{k_1+k_2-k_3}\beta^{-m_1+m_2+m_3}t_3^{-n_1+n_2+n_3}.$$

Since $\alpha^2 = t_1, \beta^2 = t_2$ and $\gamma^2 = t_3$, a simple calculation shows that

$$\alpha^2 = t_1 \Rightarrow \varphi(t_1) = \varphi(\alpha)^2 = (\alpha^{k_1}\beta^{m_1}t_3^{n_1})^2 = t_3^{2n_1-1};$$
$$\beta^2 = t_2 \Rightarrow \varphi(t_2) = \varphi(\beta)^2 = (\alpha^{k_2}\beta^{m_2}t_3^{n_2})^2 = \alpha^{2k_2} = t_1^{k_2};$$
$$\gamma^2 = t_3 \Rightarrow k_3 = 0, m_3 = 2(-m_1 + m_2 + m_3), n_3 = 0.$$

Hence $\varphi(t_3) = \beta^{m_3} = t_2^{m_3/2}$, and it follows that $\varphi(\Gamma) \subset \Gamma$ and so $\mathbb{D} = \varphi|_\Gamma$ and

$$\mathbb{D}_* = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & 0 & \frac{m_3}{2} \\ 2n_1 - 1 & 0 & 0 \end{bmatrix}.$$

Hence $\det \mathbb{D}_* = k_2 \frac{m_3}{2}(2n_1 - 1) \neq 0, \pm 1$ if and only if either $m_3 \neq 0$ or $k_2 = \pm 1, m_3 = \pm 2, 2n_1 - 1 = \pm 1$. If $\mathbb{D}_*$ has an eigenvalue of modulus 1, then $\det \mathbb{D}_* = \pm 1$. This shows that the condition (2) of Corollary 3.5 implies the condition (1). Consequently, when k_1, k_2, m_1 are odd and k_3, m_2, m_3 are even, if $m_3 \neq 0$ or

$$(k_3, m_1, n_2) \notin \{(1, 2, 0), (1, -2, 0), (-1, 2, 0), (-1, -2, 0), \\ (1, 2, 1), (1, -2, 1), (-1, 2, 1), (-1, -2, 1)\},$$

then $\mathrm{DH}(f) > 0$.

4.2. An infra-nilmanifold modeled on Nil

We will consider a three-dimensional infra-nilmanifold modeled on the Heisenberg group Nil. Recall that

$$\mathrm{Nil}=\left\{\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix}\;\middle|\; x,y,z\in\mathbb{R}\right\}.$$

For all integers $k>0$, we consider the subgroups Γ_k of Nil:

$$\Gamma_k=\left\{\begin{bmatrix}1&m&-\frac{\ell}{k}\\0&1&n\\0&0&1\end{bmatrix}\;\middle|\; \ell,m,n\in\mathbb{Z}\right\}.$$

These are lattices of Nil and every lattice of Nil is isomorphic to some Γ_k. Letting

$$s_1=\begin{bmatrix}1&1&0\\0&1&0\\0&0&1\end{bmatrix},\; s_2=\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix},\; s_3=\begin{bmatrix}1&0&-\frac{1}{k}\\0&1&0\\0&0&1\end{bmatrix},$$

we obtain a presentation of Γ_k

$$\Gamma_k=\langle s_1,s_2,s_3\mid [s_3,s_1]=[s_3,s_2]=1,[s_2,s_1]=s_3^k\rangle.$$

Every element of Γ_k can be written uniquely as the form

$$s_2^n s_1^m s_3^\ell=\begin{bmatrix}1&m&-\frac{\ell}{k}\\0&1&n\\0&0&1\end{bmatrix}.$$

Remark that $s_1^m s_2^n=s_2^n s_1^m s_3^{-kmn}$. All possible almost-Bieberbach groups can be found in [3, pp. 799–801] or [5].

Consider an almost Bieberbach group Π given by

$$\Pi=\left\langle s_1,s_2,s_3,\alpha\;\middle|\;\begin{array}{l}[s_3,s_1]=[s_3,s_2]=1,\;[s_2,s_1]=s_3^k,\\ \alpha s_1\alpha^{-1}=s_2,\;\alpha s_2\alpha^{-1}=s_1^{-1}s_2^{-1},\;\alpha^3=s_3^2\end{array}\right\rangle.$$

This is a 3-dimensional almost Bieberbach group $\pi_{6,2}$ or $\pi_{6,3}$ with Seifert bundle type 6.

Let $\varphi:\Pi\to\Pi$ be a homomorphism. Every element of Π is of the form $s_2^n s_1^m s_3^\ell$, $s_2^n s_1^m s_3^\ell\alpha$ or $s_2^n s_1^m s_3^\ell\alpha^2$. In order to have an isomorphism $\bar\varphi:\Phi\to\Phi$ such that $\mathrm{fix}(\varphi)=\{I\}$, we must have that $\bar\varphi(\bar\alpha)=\bar\alpha^2$. This implies that φ has the form

$$\varphi(s_1)=s_2^{n_1}s_1^{m_1}s_3^{\ell_1},\;\varphi(s_2)=s_2^{n_2}s_1^{m_2}s_3^{\ell_2},\;\varphi(\alpha)=s_2^{n_3}s_1^{m_3}\alpha^{3\ell_3+2}.$$

Then it can be seen as before that

$$\varphi(s_3^2)=\varphi(\alpha)^3=s_3^{(3\ell_3+2)-\frac{m_3(m_3+1)}{2}k+(m_3^2+m_3n_3+n_3^2)k-\frac{n_3(n_3+1)}{2}k}.$$

Since $\varphi(s_3)\in\Gamma_k$, $\varphi(s_3)$ is of the form $s_2^n s_1^m s_3^\ell$ and so

$$\varphi(s_3^2)=(s_2^n s_1^m s_3^\ell)^2=s_2^{2n}s_1^{2m}s_3^{2\ell-kmn}.$$

Hence $\varphi(s_3) = s_3^\ell$. Furthermore, the relations $\alpha s_1 \alpha^{-1} = s_2$ and $\alpha s_2 \alpha^{-1} = s_1^{-1} s_2^{-1}$ are preserved by φ. This induces the conditions $n_1 = n_2 = -m_2$ and $m_1 = -2m_2$. The relation $[s_2, s_1] = s_3^k$ yields that $\ell = m_1 n_2 - m_2 n_1 = 3m_2^2$. Consequently, the integral differential of $\mathbb{D} = \varphi|_{\Gamma_k}$ with respect to the basis $\{\log(s_1), \log(s_2), \log(s_3)\}$ of $\mathfrak{nil}$ is

$$\mathbb{D}_* = \begin{bmatrix} -2m_2 & m_2 & 0 \\ -m_2 & -m_2 & 0 \\ 0 & 0 & 3m_2^2 \end{bmatrix}.$$

Hence $\det \mathbb{D}_* = (3m_2^2)^2$ and the eigenvalues of $\mathbb{D}_*$ are ℓ and $\frac{-3\pm\sqrt{3}i}{2} m_2$. No eigenvalues of $\mathbb{D}_*$ are of modulus 1, and $\det \mathbb{D}_* = 0$ (i.e., $m_2 = 0$) or $\det \mathbb{D}_* \geq 9$. Consequently if $m_2 \neq 0$ then $\mathrm{DH}(f) > 0$.

4.3. Infra-solvmanifolds modeled on Sol

Next we will consider a closed 3-manifold with Sol-geometry. Recall that $\mathrm{Sol} = \mathbb{R}^2 \rtimes_\phi \mathbb{R}$ where

$$\phi(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Then Sol is a connected and simply connected unimodular 2-step solvable Lie group of type (R). It has a faithful representation into $\mathrm{Aff}(\mathbb{R}^3)$ as follows:

$$\mathrm{Sol} = \left\{ \begin{bmatrix} e^t & 0 & 0 & x \\ 0 & e^{-t} & 0 & y \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \;\middle|\; x, y, t \in \mathbb{R} \right\}.$$

Let M be a closed 3-manifold with Sol-geometry. Then the fundamental group Π of M is a Bieberbach group of Sol, and $M = \Pi\backslash\mathrm{Sol}$. Further, Π can be embedded into $\mathrm{Aff}(\mathrm{Sol}) = \mathrm{Sol} \rtimes \mathrm{Aut}(\mathrm{Sol})$ so that there is an exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Pi/\Gamma \longrightarrow 1,$$

where $\Gamma = \Pi \cap \mathrm{Sol}$ is a lattice of Sol and $\Phi = \Pi/\Gamma$ is a finite group, called the holonomy group of Π or M, which sits naturally into $\mathrm{Aut}(\mathrm{Sol})$, see [8]. The lattices Γ of Sol are determined by 2×2-integer matrices A

$$A = \begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}$$

of determinant 1 and trace > 2, see for example [29, Lemma 2.1]. Namely,

$$\Gamma = \Gamma_A = \langle a_1, a_2, \tau \mid [a_1, a_2] = 1, \tau a_i \tau^{-1} = A(a_i) \rangle = \mathbb{Z}^2 \rtimes_A \mathbb{Z}.$$

Let f be a self-map on $\Gamma_A\backslash\mathrm{Sol}$. By [29, Theorem 2.4], the homomorphism $\varphi : \Gamma_A \to \Gamma_A$ induced by f is determined by

$$\varphi(a_i) = \mathbf{a}^{\mathbf{u}_i},\ \varphi(\tau) = \mathbf{a}^{\mathbf{p}} \tau^{\zeta}$$

for some $\mathbf{u}_i, \mathbf{p} \in \mathbb{Z}^2$ and $\zeta \in \mathbb{Z}$. Note that φ extends uniquely to a Lie group homomorphism on Sol. It follows easily that all the possible (integer) matrices $\mathbb{D}_*$ are of the form

$$\mathbb{D}_* = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{0} \\ 0 & 0 & \zeta \end{bmatrix}.$$

We say that φ is of type (I) if $\zeta = 1$; of type (II) if $\zeta = -1$; of type (III) if $\zeta \neq \pm 1$. When φ is of type (III), we have $\varphi(a_i) = 1$.

Now we consider the conditions of Corollary 3.6. These eliminate φ of type (I) and (III). If φ of type (II) satisfies the conditions of Corollary 3.6, then $\mathrm{DH}(f) > 0$. In fact, it is shown in [28, Theorem 5.1] that such a map has $\mathrm{HPer}(f) = \mathbb{N} - 2\mathbb{N}$, and so $\mathrm{DH}(f) = 1/2$.

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