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확장된 고정점이론을 이용한 비선형시스템의 근을 구하는 방법

A New Method of Finding Real Roots of Nonlinear System Using Extended Fixed Point Iterations

김 성 수† · 김 지 수*
(Sung-Soo Kim · Ji-Soo Kim)

Abstract – In this paper, a new numerical method of finding the roots of a nonlinear system is proposed, which extends the conventional fixed point iterative method by relaxing the constraints on it. The proposed method determines the real valued roots and expands the convergence region by relaxing the constraints on the conventional fixed point iterative method, which transforms the diverging root searching iterations into the converging iterations by employing the metric induced by the geometrical characteristics of a polynomial. A metric is set to measure the distance between a point of a real-valued function and its corresponding image point of its inverse function. The proposed scheme provides the convenience in finding not only the real roots of polynomials but also the roots of the nonlinear systems in the various application areas of science and engineering.

Key Words : Geometrical property of an inverse function, Conventional Fixed Point Iteration Methods, Extended Fixed Point Iteration Methods, Polynomial Roots

1. Introduction

The fixed-point theorem is one of the most powerful mathematical tools that have been widely applied in various fields, not only in pure mathematics, but also in the areas of engineering. For over a century, many researchers have demonstrated the existence and the properties of fixed point theory through analysis, topology, geometry, and numerical analysis. After Banach proved that a contraction mapping in the field of complete metric space possesses a unique point in 1922 [1], the contractive mapping was developed by many researchers as fixed point theory [2, 3]. In recent decades, the various numerical methods including the fixed point theory have been employed in fixed point theory to approximate the roots of polynomials and to demonstrate the convergence property [4, 5].

In this research, a new numerical method is proposed to find the real roots of the nonlinear systems by constructing a scheme that builds a converging sequence to a root of a function even in the case that the conventional fixed point theorem (CFPT) does not guarantee the convergence.

The CFPT is defined as follows. Let $f(x)$ be a real-valued function with an initial point x_0 in the domain of $f(x)$. Then the fixed point iterations $x_{n+1}=f(x_n)$, $n\in N$, the natural numbers, build a sequence $\{x_n\}$ that may converge to a point x_r. If $f(x)$ is continuous, then the obtained x_r is a fixed point of $f(x)$ such as $f(x)=x$. Analytically, for a complete metric space (X, d) with $f:X\to X$, if f is continuous and satisfies $d(f(x)), f(x_0)) \leqq \lambda d(x, x_0)$ with $x, x_0 \in X$, $0 \leqq \lambda < 1$ then f has a unique fixed point x_r in X [7].

In this paper, an extended fixed point iterative method finds the real-valued roots of a polynomial by relaxing the constraint $0 \leqq \lambda < 1$ of CFPT, such that the regions of convergence to a root may be expanded. The metric defined by the proposed method converges to a fixed point, a root of $f(x)$, without the constraint $0 \leqq \lambda < 1$. The proposed iterative method provides the various good properties compared to CFPT besides relaxing the constraint. However, further research on the extended contractive iterative method (ECIM) is needed

On the extended contractive iterative method on the extended contractive iteration method (ECIM) for the applications in various fields, such as the linear and nonlinear system analysis in discrete and continuous control

† Corresponding Author : Department of Electrical Engineering, Chungbuk National University, Korea.
E-mail: sungkim@chungbuk.ac.kr
* Department of Earth and Environmental Science, Chungbuk National University, Korea.
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theory.

The remainder of this paper is organized as follows. In Section 2, the problems involved in CFPT are discussed. Section 3 presents the ECIM and the extended fixed point theorem (EFPT) with proofs. In Section 4, some numerical simulations are presented to demonstrate the advantages of the proposed method. Finally, Section 5 draws the conclusions of the present study.

2. Problems Involved in CFPT

Let $f : \mathrm{X} \rightarrow \mathrm{X}$ be a mapping from a set X to itself. We call a point $x \in X$ a fixed point of f when $f(x) = x$. We will discuss here the most basic fixed-point theorem in analysis to relax the constraints on it. Let (X, d) be a complete metric space and $f : X \rightarrow X$ be a map, such that $d(f(x), f(x_0)) \leqq \lambda d(x, x_0)$ for some $0 \leqq \lambda < 1$ and all $x,\ x_0 \in X$. Then f has a unique fixed point x_r in X. Moreover, for any $x_0 \in X$, the sequence $x_{0,}\ f(x_0),\ f(f(x_0)), \cdots$ converges to a fixed point $x_r \in X$ [4, 6, 7]. A function $f(x)$ is called a contraction that shrinks the metric by a uniform factor λ for all pairs of points, where the contraction does not hold for some $\lambda > 1$.

The first problem to be considered is that it is not easy to choose a proper form of $g(x)$ with convergence property for a given function $f(x) = g(x) - x = 0$. Secondly, even the several different forms of $g(x)$ converge to the same fixed point, the convergence regions of different forms of $g(x)$ may differ one from another due to the convergence constraint $|g'(x_0)| < 1.0$, which we call the convergence region problems.

For example, suppose that a polynomial $f(x) = x^2 - x - 1 = 0$ is given in a form of $f(x) = g(x) - x = 0$. Several different forms of $g(x)$ maybe obtained as shown in Table 1, such as $g_a(x) = 1 + x^{-1}$, $g_b(x) = (1+x)^{-1}$, $g_c(x) = x^2 - 1$ and $g_d(x) = \pm\sqrt{(x+1)}$. The functions, $g_a(x)$, $g_b(x)$ and $g_c(x)$ have the different regions of convergence (ROC) which satisfy the constraint $|g'(x_0)| < 1.0 \quad x,\ x_0 \in X$, while the function $g_d(x)$ is not in the proper form of convergence as presented in Table 1.

As shown in Table 1, a function $f(x)$ may be expressed in several different forms of $g(x)'s$ each of which has its own ROC, while $g_d(x)$ is in improper form for finding a root using the fixed point iterations. The example $f(x)$ in Table 1 has the two roots, x_1 and x_2, but none of the forms presents the two contractive fixed points as the roots of $f(x)$simultaneously.

The function $g_a(x)$ has only x_2 as a fixed point and $g_b(x)$ constructs a contractive iterations converging to only x_1 while the sequence generated by the iterations of $g_c(x)$ converges to -1 and 0 alternatively not to the roots, x_1, x_2 since $x_1 = -0.6183$ and $x_2 = 1.6180$ are not in the region of ROC $(-0.5,\ 0.5)$ as shown in Table 1. Based on the observations described, it is possible that those two roots of $f(x)$ can be obtained algebraically, which is not true in general for a higher order polynomial. However, the contractive iterations which converge to the roots may not be obtained by employing the fixed point theorem only. Those roots which are in ROC, where the area of ROC is derived by the constraint$|g'(x_0)| < 1.0$, are obtained through the contractive property of fixed point theorem. Otherwise the contractive property cannot be kept if the roots are outside of ROC. Convergence to a fixed point requires that the root of a function must be inside of ROC which is derived from $|g'(x_0)| < 1.0$ where x_0 is a point in the domain of the function $g(x)$.

From the above observations, we may conclude two facts. Firstly it is not easy to choose a proper form of $g(x)$ with the convergence property for a given function $f(x) = g(x) - x = 0$ from all possible forms $g(x)'s$ since we do not have any information on $g(x)$ in advance. Secondly, choosing the proper initial value in the domain of $g(x)$ for

Table 1 An example : $f(x) = x^2 - x - 1$ where $f(x) = g(x) - x = 0$, $x_1 = -0.61803$, $x_2 = 1.6180$

$f(x) = x^2 - x - 1 = 0$	$g(x)$	$\lvert g'(x)\rvert < 1$	ROC	The roots of $f(x) = 0$
$x = 1 + \frac{1}{x}$	$g_a(x) = 1 + \frac{1}{x}$	$\lvert \frac{-1}{x_0^2} \rvert < 1.0$	$x_0 < -1.0,\ x_0 > 1.0$	$\lvert g(x_1)\rvert = \lvert -2.61803\rvert > 1$ $\lvert g'(x_2)\rvert = \lvert -0.38196\rvert < 1.0$
$x = \frac{1}{x-1}$	$g_b(x) = \frac{1}{x-1}$	$\lvert \frac{-1}{(x_0-1)^2} \rvert < 1.0$	$x_0 < 0.0,\ x_0 > 2.0$	$\lvert g'(x_1)\rvert = \lvert -0.38196\rvert < 1.0$ $\lvert g'(x_2)\rvert = \lvert -2.61803\rvert > 1.0$
$x = x^2 - 1$	$g_c(x) = x^2 - 1$	$\lvert 2x_0\rvert < 1.0$	$x_0 < 0.5,\ x_0 > -0.5$	$\lvert g'(x_1)\rvert = \lvert -1.2360\rvert > 1.0$ $\lvert g'(x_2)\rvert = \lvert -3.2360\rvert > 1.0$
$x = \pm\sqrt{x+1}$	$g_d(x) = \pm\sqrt{x+1}$	Improper form		

the fixed point iteration is not easy because the choice of $g(x)$ constructs the region of convergence differently. However, it is not necessary for the initial values to be in the ROC when ECIM is used. It is true that if a root of a function $f(x)$ is in ROC, then the iterative sequence converges to a root of $f(x)$. In addition to this, in order to find the roots in ROC, the initial value also needs to be in ROC for the contraction property. When we choose an initial value outside of ROC, the iterations may and may not converge to the roots of a function $f(x)$. Even we choose an initial value in ROC, if a root of $f(x)$ locates in the outside of ROC, the generated contractive sequence does not converge to a root. Therefore we may conclude that the form of $g(x)$ determines the property of contractive convergence.

In this work, we propose a new contractive iteration method to relax the constraint of convergence, $|g'(x_0)| < 1.0$ by constructing a scheme that enhances the possibility of convergence to the roots of a function. The proposed method simplifies not only the conventional fixed point iteration but also the algorithm that generates a converging sequence instead of diverging sequence. Consequently, the proposed method simplifies the searching process for a unique form of $g(x)$ and expanding ROC by constructing the proposed scheme that transforms a diverging sequence to a converging sequence.

3. Extended Contractive Iteration Method (ECIM)

Practically, the fixed point theorem in a continuous interval is constrained to satisfy $|g'(x_0)| < 1.0$ to find the roots of the function, such that a sequence of points $\{x_k\}$ for $k \in N$ where N is the natural number, converges to a fixed point $x_r \in X$. The fixed point is a root of a given polynomial $f(x) = 0$ when the constraint $|g'(x_0)| < 1.0$ is satisfied, where $g(x)$ is obtained from the relationship $f(x) = g(x) - x = 0$. Otherwise, the sequence $g_k(g_{k-1}(\cdots(g_0(x_0))))$ diverges for $k \in N$ the natural number [6]. In the proposed scheme, for a given arbitrary function $f(x)$, a unique $g(x)$ is generated by setting $g(x) = f(x) + x$, we present a new method of generating a sequence which converges to a root of a function whether the constraint of CFPT is satisfied or not. At the same time, the problem of finding a proper $g(x)$ has been relaxed simultaneously.

The choice of initial value is set to be free in the proposed method since the sequence built by an arbitrary initial value is set to converge to roots or may diverge, which implies that the choice of initial value does not need to be in ROC for finding the roots of polynomial in the proposed scheme. However, a root of $f(x)$ needs to be in ROC if the fixed point generated by CFPT is set to be a root of the given polynomial.

The proposed method expands the CFPT by generating a sequence of metrics defined in the opposite direction of divergence as described in Fig. 1. The sequence of metrics obtained by the proposed scheme is set to converge to a real-valued root of a function. For a chosen initial point x_0, the corresponding function value $g(x_0)$ is determined, such that a metric is defined as shown in Eq. (1) [6]. Since a point $Q(x_0, x_0)$ is on a line $p(x) = x$, a metric can be generated between $Q(x_0, x_0)$ and $P(x_0, g(x_0))$ in the form of l_2-norm such as

$$d = d(P, Q) = |g(x_0) - x_0| \tag{1}$$

A new point x_n in the domain of a function $g(x)$ is generated by Eq. (2)

$$x_n = x_0 - x^*d \tag{2}$$

The parameter c is defined $c = sign^*dif$ as presented in

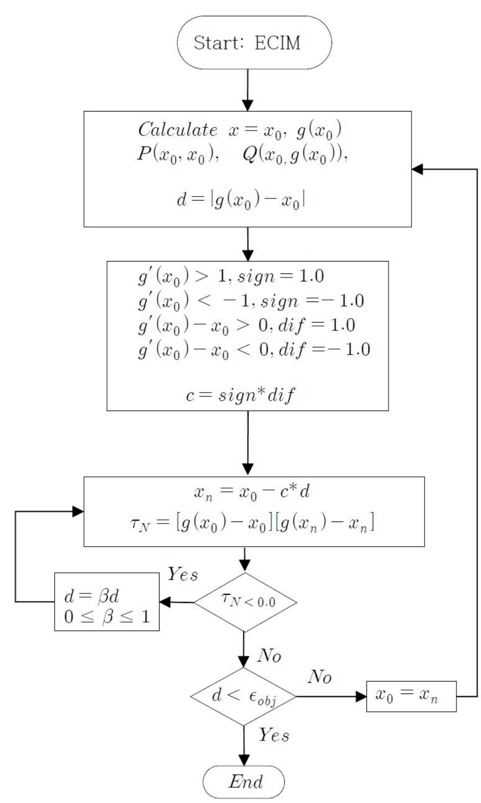

Fig. 1 A scheme of finding the root of a function using ECIM

Table 2 A method of finding the next following point in the domain of function $g(x)$

$sign$	dif	$c = sign*dif$	$x_n = x_0 - c*d$
+1	+1	+1	$x_n = x_0 - d$
+1	-1	-1	$x_n = x_0 + d$
-1	-1	+1	$x_n = x_0 - d$
-1	+1	-1	$x_n = x_0 + d$

Table 2 with two functions $sign$ and dif in Eq. (3),

$$sign = \begin{cases} +1, g'(x_0) > 1.0 \\ -1, g'(x_0) < -1.0 \end{cases}, \quad dif = \begin{cases} +1, g(x_0) - x_0 > 0 \\ -1, g(x_0) - x_0 < 0 \end{cases} \tag{3}$$

The variable $sign$ is 1.0 when $g'(x_0) > 1.0$ and -1 when $g'(x_0) < -1.0$, while the variable dif becomes 1.0 when $g(x_0) - x_0 > 0.0$ and -1 when $g(x_0) - x_0 < 0.0$. The parameter $c = sign*dif$ presents the four cases as shown in Table 2. A sequence is generated by the selecting a value in the domain of $g(x)$ using $x_n = x_0 - c*d$. As it is shown in Table 2, when $g'(x_0) > 1.0$, we need to determine whether $g(x)$ locates above or below the line $p(x) = x$ and set the direction of sampling process in the domain of $g(x)$, such that the sequence converges to a root of a polynomial. Roots of polynomial locate at the points of intersections between $g(x)$ and $p(x) = x$.

Suppose that x_0 and $g(x_0)$are the present sampled value and its corresponding $g(x)$ to x_0. Then, both the sampled value obtained from the next sampling process $x_n = x_0 - c*d$ and its corresponding $g(x_n)$ need to be in the same side with respect to $p(x) = x$ until the sequence converges to a root. Then we check whether $g(x_0)$ and $g(x_n)$ are in the same side with respect to $p(x) = x$ by the following expression, where * denotes multiplication

$$\tau_n = [g(x_0) - x_0]*[g(x_n) - x_n] \tag{4}$$

If they are in the same side with respect to $p(x) = x$, $\tau_n > 0.0$, otherwise, $\tau_n < 0.0$. When $\tau_n < 0.0$, we reduce the sampling step size by a shrinking parameter $\beta \in (0,1)$, such that a new metric

$$d = \beta * d \tag{5}$$

is revised recursively until $\tau_n > 0.0$. The next sampling point is determined by $x_n = x_0 - c*d$. The sequence sampling process is repeated until

$$d < \epsilon_{obj} \tag{6}$$

where the predetermined criteria ϵ_{obj} is set as an allowed error. When the constraint, $|g'(x_0)| < 1.0$, is not fulfilled, the sequence from the domain of $g(x)$ does not converge to a root $f(x) = 0$ in CFPT as we have found. However, the diverging sequence extracted from CFPT may be switched to a converging metric sequence via the proposed scheme.

The sequence of points $\{x_k\}$ for $k \in N$ is designed to be generated through the recursive iterations, $x_{k+1} = x_k - \beta(g(x_k) - x_k)$ where $\beta \in [0,1)$ and $k \in N$ the natural number. The variable $\beta \in [0,1)$ affects the speed of convergence to a fixed point.

Theorem 1. [Extended Contractive Iteration Theorem]

Let (X, d) be a complete metric space and $f : X \to X$ be a map, such that

$$d(f(x), f(x_0)) > \lambda d(x, x_0) \tag{7}$$

for some $\lambda > 1.0$ and all x, $x_0 \in X$. Then f has a unique fixed point x_r in X for any $x_0 \in X$, where the sequence of iterations generated by $x_{k+1} = x_k - \beta(g(x_k) - x_k)$ for $\beta \in (0,1)$, such that $\{x_k\}$ converges to a fixed point $x_r \in X$ of f for $k \in N$ the natural number, $\lim_{k \to \infty} x_k = x_r$.

[Proof] From the constraint, $d(f(x), f(x_0)) > \lambda d(x, x_0)$ where $\lambda > 1.0$ implies that the absolute value of derivative is greater than 1.0 at each point $x_0 \in X$, such that $f'(x_0) \geqq \lambda > 1.0$. Let $\beta = \lambda^{-1} \in (0,1)$ and $k = 0$, then we define the value $x_1 = x_0 - \beta(f(x_{0) - x_0})$ and $d_0 = x_0 - x_1$.

Since $f'(x_0) > \lambda > 1$, $f(x_0) - f(x_1) > x_0 - x_1$ becomes $f(x_0) - x_0 > f(x_1) - x_1$, such that $f(x_1) - x_1 = \beta(f(x_0) - x_0)$ for $\beta \in (0, 1)$ where $x_1 < x_0$. Using the obtained x_1, we set the value x_2, such that $x_2 < x_1$, $x_1 - x_2 = \beta(f(x_1) - x_1)$. Then $x_1 - x_2 = \beta(f(x_1) - x_1) = \beta(\beta(f(x_0) - x_0)) = \beta^2(f(x_0) - x_0)$ and the distance between x_1 and x_2 becomes $d_1 = \beta^2(f(x_0) - x_0) < d_0$.

For the obtained values x_1, x_2, we can determine x_3, $d_2 = x_2 - x_3 = \beta(f(x_2) - x_2) = \beta(\beta^2(f(x_0) - x_0)) = \beta^3(f(x_0) - x_0)$ such that $d_2 < d_1 < d_0$ for $\beta \in (0, 1)$. Then for an arbitrary k, we obtain $d_k = x_k - x_{k+1} = \beta^{k+1}(f(x_0) - x_0) < d_{k-1}$ and $x_k < x_{k-1}$ for all $k \in N$, $d_k = x_k - x_{k+1} = \beta^{k+1}(f(x_0) - x_0) < d_{k-1}$ such that $\lim_{k \to \infty} d_k = 0$. Thus for any $x_0 \in X$ the iterations $x_{k+1} = x_k - \beta(g(x_k) - x_k)$ for $k \in Z$, where $Z = \{0, 1, 2, \cdots\}$, $0 < \beta < 1$, generates the sequence $\{x_k\}$ where x_k converges to the fixed point $x_r \in X$ of f, $\lim_{k \to \infty} x_k = x_r$.

Now, for an arbitrary function $f(x)=g(x)-x$, if it has a real root, then the root locates at the point where $g(x)$ crosses $p(x)=x$ whether the constraints in CFPT is satisfied or not, since for a root $x_r \in X$, the root searching iteration converges to x_r from an initial value x_0 by using both CFPT and ECIM. Hence, whenever $g(x)$ crosses $p(x)=x$, a real valued root can be obtained by the contractive iterations. It turns out to be sufficient to get the conclusion of the contraction mapping theorem for the functions with the real-valued roots by the following Theorem 2.

Theorem 2. [Extended Fixed Point Theorem (EFPT)]
Let (X,d) be a complete metric space and $f: \mathrm{X} \rightarrow \mathrm{X}$ be a map, such that

$$d(f(x),\, f(x_0)) = \lambda d(x,\, x_0) \tag{8}$$

for some $\lambda \neq 1.0$ and all $x,\ x_0 \in X$. Then f has a unique fixed point x_r in X whenever f has a real valued root using the iteration method.

[Proof] Firstly, for $f(x)$ with $\lambda \in (0,1)$ and any $x_0 \in X$, the sequence obtained from iterations x_0, $g(x_0)$, $g(g(x_0))$, $g(g(g(x_0)))$, $\cdots$, converges to a unique fixed point of $f(x)=g(x)-x=0$ for $x_r,\, x_0 \in X$ if $d(f(x),f(x_0)) < \lambda d(x,x_0)$ by CFPT. Secondly, for $\lambda > 1$, the sequence extracted with $0 \leqq \beta < 1.0$

$$x_{k+1} = x_k - \beta(g(x_k) - x_k) \tag{9}$$

converges to x_r, $\lim_{k \to \infty} x_k = x_r$, $k \in N$ the natural number by the Theorem 1. Therefore, the statements in Theorem 2 hold.

4. Numerical Simulations

For a given function $f(x)$, the proposed method ECIM is applied to find the roots of the function $f(x)=x^2-x-1$ whose roots cannot be obtained using the fixed points of the conventional fixed point theorem. In order to use the proposed scheme, a unique polynomial $g(x)$ is obtained by the relation $f(x)=g(x)-x=0$. As it is shown in Table 1, for $g(x)=x^2-1$, the conventional fixed point iterations do not converge to the roots of $f(x)$ when the roots of $f(x)$ are not in ROC $(-0.5, 0.5)$ obtained from the constraints $|g'(x)|<1$. However, the proposed scheme builds a sequence of contractive iterations, where a fixed point becomes a root of the function $f(x)$.

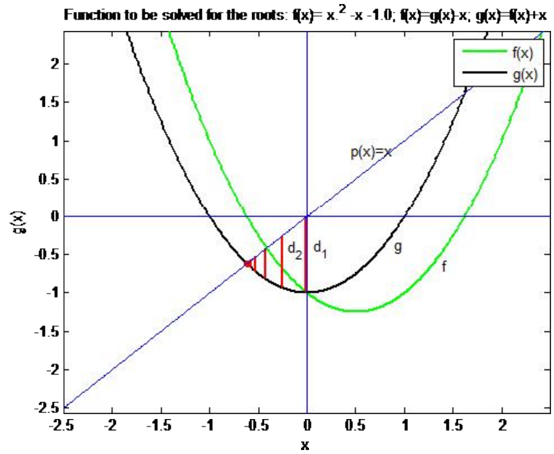

Fig. 2 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 22, a fixed point $x_r = -0.618033970$, error$=-6.8738122e-08$, the initial value $x_0 = -0.01$

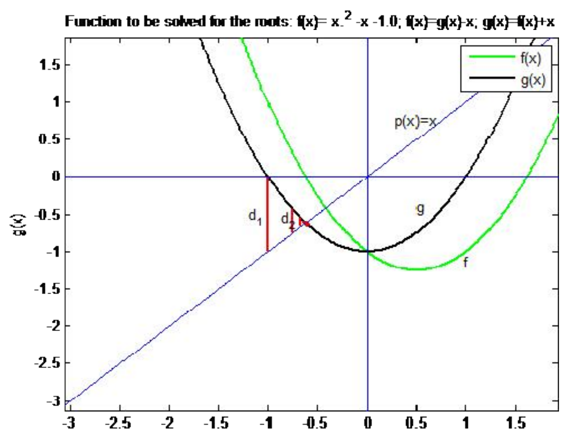

Fig. 3 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 20, a fixed point $x_r = -0.618034008$, error$=7.713588e-8$, the initial value $x_0 = -1.0$

Table 3 Experimental results for a function $f(x)=x^2-x-1$ using the proposed scheme

Fig.	Initial point: x_0	Num. of iter.: N	Converging point: x_r	Error: ϵ_N
Fig. 2	-0.01	22	-0.61803397	-6.873922e-8
Fig. 3	-1.0	20	-0.61803400	7.713588e-8
Fig. 4	0.0	26	1.61803397	-7.699750e-8
Fig. 5	3.0	21	1.61803399	5.669268e-8
Fig. 6	4.0	23	1.61803399	-5.200781e-8
Fig. 7	-2.5	19	-0.61803396	-8.889898e-8

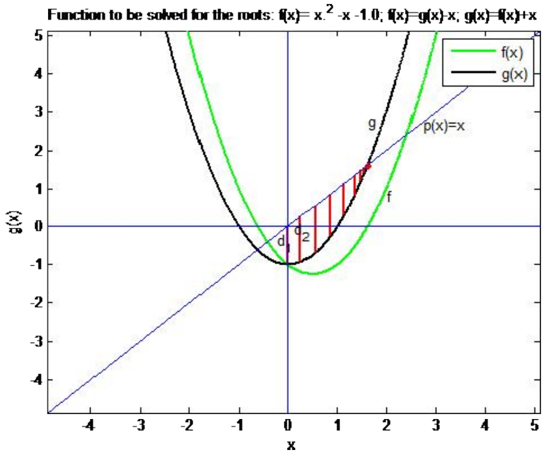

Fig. 4 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 26, a fixed point $x_r=1.618033976$, error=$-7.699750e-8$, the initial value $x_0=0.0$

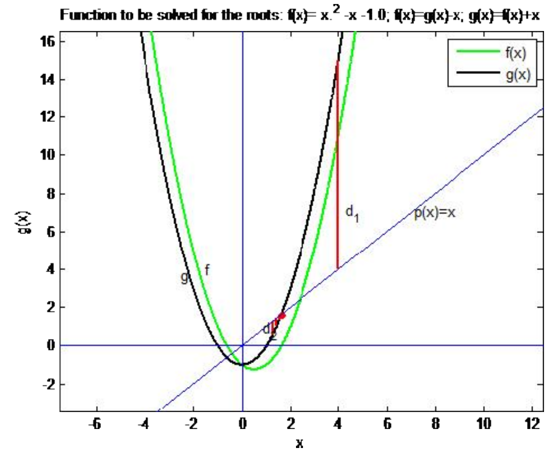

Fig. 6 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 23, a fixed point $x_r=1.618033997$, error=$-5.2007811e-8$, the initial value $x_0=4.0$

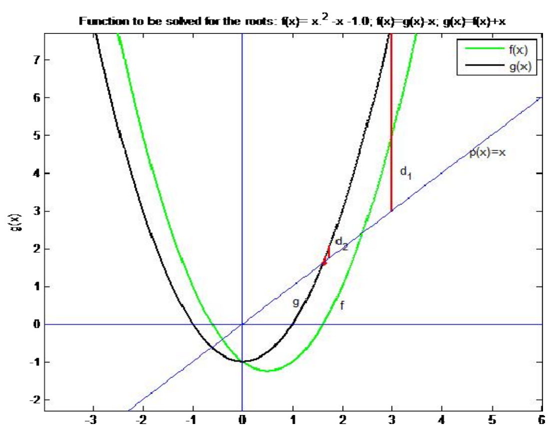

Fig. 5 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 21, a fixed point $x_r=1.618033997$, error=$5.6692688e-8$, the initial value $x_0=3.0$

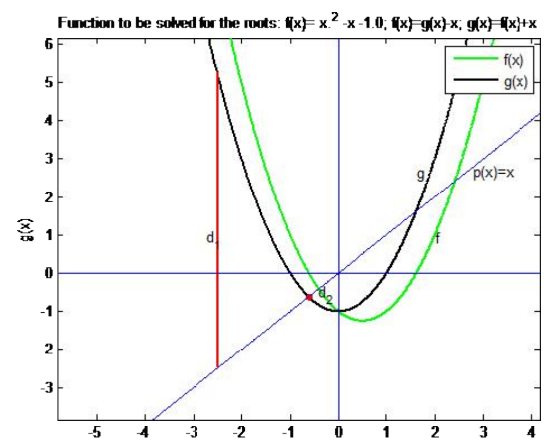

Fig. 7 Function $f(x)=x^2-x-1$ with $g(x)=f(x)+x$, the number of iterations 19, a fixed point $x_r=-0.618033965$, error=$-8.88989890e-8$, the initial value $x_0=-2.50$

Figure 2, 3, 4, 5 illustrate the behaviors of contractive iterations $f(x)$ which converge to the roots, the fixed points, with initial values -0.01, -1.0, 0.0, 3.0 respectively. Table 3 presents the results of numerical simulations using the proposed scheme with variable $\beta=0.25$, which is chosen as a constant for convenience. The variable β needs to be studied further since it affects the speed of convergence to a fixed point. The numerical simulation results present the contractive property of the proposed method by Theorem 1. The chosen sequences obtained by Theorem 1 converge to the roots of the polynomial while the sequences may diverge in CFPT.

Figure 6 and 7 represent two cases where the sequences converge to the fixed points in the proposed iterations with alternating with respect to the fixed points. In both cases, the metric d_1 is greater than the distance to an adjacent root from the initial value. Even though there are some swings around a root, the sequence eventually converges to the root which attracts the iterations to a fixed point.

The method of generating a convergence sequence is demonstrated in Figures 8, 9 using the proposed scheme for the cases that $|g'(x)|>1.0$ for all $x \in X$, the domain of $g(x)$. Figures 8, 9 describe graphically the function $g(x)$ obtained from $f(x)$ using $g(x)=f(x)+x$ where $g(x)=x^3+1.5x^2+1.95x+1.0$ as they are illustrated in Table 4. The experimental results demonstrate the property of contractive iterations to the roots of the polynomial $f(x)=x^3+1.5x^2+0.95x+1.0$ with variable $\beta=0.25$ chosen as a constant for convenience. The variable β affects the speed of convergence to the fixed point.

Table 4 A case of $g(x)=f(x)+x$ where $g(x)=x^3+1.5x^2+1.95x+1.0$ with $|g'(x)|>1.0$ for all $x\in X$, the domain of $g(x)$

Fig.	Initial point: x_0	Number of iter.: N	Converging point: x_r	Error: ϵ_N
Fig.8	1.5	23	-1.3461239	6.4729278e-8
Fig.9	-2.5	20	-1.3461239	6.8319431e-8

With the function, $g(x)=f(x)+x$, we check the derivation of $g(x)$ at x_0 a chosen initial point. If the slope at the chosen point is less than 1.0, we use the conventional fixed point iteration method using CFPT. When the absolute slope is greater than 1.0 at the chosen point x_0 such as $|g'(x)|>1.0$, we apply the specified sequence generation process using ECIM.

In Figures 8, 9, for $f(x)=x^3+1.5x^2+0.95x+1.0$, a unique function $g(x)$ is generated. The function $f(x)$ crosses the horizontal axis at the points where the roots of the function $f(x)$ locate while $g(x)$ crosses a line $p(x)=x$ at the locations of roots of $f(x)$. The sequences of metrics in Figures 8, 9 illustrate the convergence to the two roots for the initial value $x_0=1.5$ and the initial value $x_0=-2.5$ respectively.

As illustrated in Figures 2-9, the experimental results show that a sequence of metrics $\{d_k\}$ converges to a fixed point x_r even the absolute value of the slope of $g(x_0)$, $|g'(x_0)|$, is greater than 1.0. As shown in Figures, a set of metrics $\{d_k\}$ is generated by connecting the points of $g(x)$ and the corresponding points of $p(x)=x$ at each chosen point of the domain of a function $g(x)$. In Figures 8, the converging metric sequence $\{d_k\}$ to the fixed point $x_r=-1.34612960$ is illustrated for the initial value $x_0=1.5$ with the number of iterations yields 23, the error $6.47292788e-08$, and the parameter $\beta=0.25$. Figure 9 shows the case of the initial value $x_0=-2.5$ where a sequence $\{x_k\}$, which is a set of the chosen domain values of a function $f(x)$, converges to the fixed point $x_r=-1.34612960$ with error $6.83194318895e-08$. The number of iterations with the given conditions yields 24 with the process $x_{k+1}=x_k-\beta(g(x_k)-x_k)$ where the variable$\beta=0.25$ affects the speed of convergence to a fixed point. In experiments, we have demonstrated the convergence for the case and

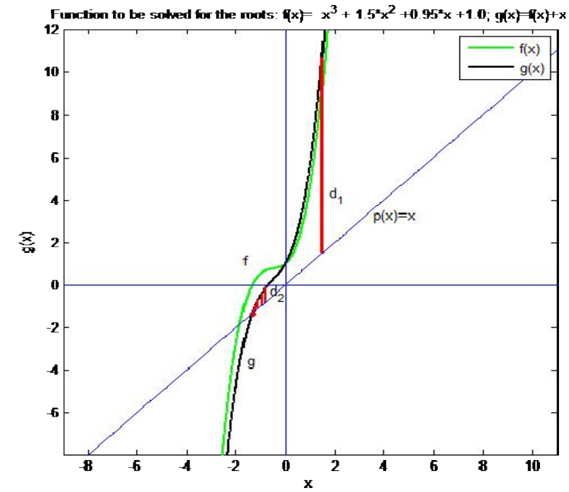

Fig. 8 Convergence of $\{x_k\}$ where $k\in N$ to the fixed point $x_r=-1.34612960$, with the initial value $x_0=1.5$, error= $6.4729788e-8$, the number of iterations=23, and $\beta=0.25$

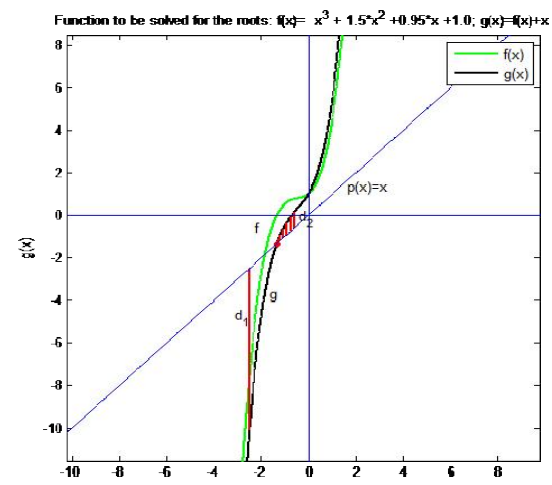

Fig. 9 With the initial value $x_0=-2.5$, the sequence $\{x_k\}$ where $k\in N$ converges to the fixed point $x_r=-1.34612960$, error=$6.831943188e-8$, the number of iterations=24, and $\beta=0.25$

using Theorems 1 and 2. Further reaches is expected to take advantage of the good properties of the proposed method in various fields.

5. Conclusions

In this paper, EFPT is proposed using the metric constructed by an extended fixed point iterative method ECIM, such that any real-valued roots of a nonlinear system may be obtained. The proposed method relaxes the constraints on the numerical fixed point iteration method for finding the roots of a nonlinear systems with the two good properties.

Firstly, EFPT lets an initial value for a system converge to a real-valued root via the relaxation on the constraint by minimizing the constraint from $|g'(x)|<1.0$ to $|g'(x)|\neq 1.0$. Secondly, a unique form of $g(x)$ from $f(x)=g(x)-x=0$ is able to be determined without the efforts for finding a proper form of $g(x)$ with trial and error. In the conventional fixed point theorem, CFPT, it is necessary to check several forms of $g(x)'s$ for $f(x)$ since a chosen $g(x)$ may diverge or converge depending on the form of $g(x)$ with the various regions of convergence. However, the proposed method, EFPT, improves the convergence to the real roots of a nonlinear system by expanding ROC with the relaxation on the constraints of $g(x)$.

The proposed method is useful to establish the local existence and uniqueness of solutions of the ordinary differential equations. As some practical applications, the contraction mappings are useful to develop the simple numerical methods for solving nonlinear equations, such as the fuzzy logic programming. For example, the proposed fixed point theory may be applied for finding equilibrium points in the area of nuclear reactor analysis. In general, the applications of the fixed point theory can be focused on any kind of differential equations related to stability of dynamic systems of continuous-time, discrete-time, hybrid of both of them. Further researches are expected to be carried out for taking advantage of the good properties of EFPT for various applications in science and engineering fields.

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저 자 소 개

김 성 수 (Sung-Soo Kim)

2001년~현재 충북대학교 전기공학과 교수, 1999년~2001년 우석대학교 전기공학과 교수, 1997년 Univ. of Central Florida (박사), 1989년 Univ. of Ark. (석사), 1983년 충북대학교 (학사)

연구분야: 인공지능, 패턴인식, 신호처리, 이동통신, 함수해석학 등.

E-mail : sungkim@chungbuk.ac.kr

김 지 수 (Ji-Soo Kim)

1995년~현재 충북대학교 지구환경학과 교수, 1993년 Univ. of Manitoba (박사), 1982년 서울대학교 (석사), 1980년 서울대학교 (학사).

연구분야: 지구물리학, 탄성파자료처리해석 등.

E-mail : geop22@cbnu.ac.kr