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ON THE CLASS OF COMPLEX DOUGLAS-KROPINA SPACES

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Abstract. In this paper, considering the class of complex Kropina metrics we obtain the necessary and sufficient conditions that these are generalized Berwald and complex Douglas metrics, respectively. Special attention is devoted to a class of complex Douglas-Kropina spaces, in complex dimension 2. Also, some examples of complex Douglas-Kropina metrics are pointed out. Finally, the complex Douglas-Kropina metrics are characterized through the theory of projectively related complex Finsler metrics.

1. Introduction

Complex Douglas spaces are much more general than Hermitian and locally Minkowski spaces and have recently been broached by authors in [4,7], through the medium of some complex projective curvature invariants, emerging from the subject of projective real Finsler spaces, [8–11,13,15,19].

A relation $\tilde{G}^i = G^i + B^i + P\eta^i$, (called *projective change,* where P is a smooth function on $T'M$ with complex values and $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$), between the spray coefficients G^i and $\tilde{G}^i$, corresponding to the complex Finsler spaces (M,F) and $(M,\tilde{F})$ respectively, is necessary and sufficient for F and $\tilde{F}$ to be projectively related. This means that any complex geodesic curve, in [1]'s sense, of the first is also complex geodesic curve for the second as point sets, and the other way around. The exploration of the projective change leads us to projective curvature invariants: three of Douglas type and two of Weyl type. The vanishing of the projective curvature invariants of Douglas type defines the complex Douglas spaces and a projective curvature invariant of Weyl type characterizes the complex Berwald spaces, (for more see [4]).

More characterizations for the complex Douglas spaces are pointed out in [7] and there, such complex metrics are exemplified by the complex Randers

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metrics. In the present paper, the general theory on complex Douglas spaces is applied to the class of complex Kropina spaces.

Subsequently, we make an overview of the paper's content.

In §2, some preliminary properties of the n-dimensional complex Finsler spaces are stated, ([1, 2, 12, 14, 16–18, 20, 21]).

Beginning with §3 the general theory on complex Kropina spaces [3, 6] is supplied with some special outcomes related the conditions under which these spaces are generalized Berwald, complex Berwald and complex Douglas. In contrast to the complex Randers spaces [6], where $A := (\overset{a}{\delta}_k|\beta|)\eta^k = 0$ is necessary and sufficient condition for the generalized Berwald property, for the class of complex Kropina spaces this is only necessary. An improvement is brought by Theorem 3.1 which establishes the necessary and sufficient conditions for generalized Berwald property of the complex Kropina metrics. Then, the complex Douglas property is highlighted by some results, (Theorems 3.2, 3.3 and Corollary 3.2). Moreover, we find that a complex Douglas-Kropina space of complex dimension two, with some additional assumptions, is a complex Berwald space, (Theorem 3.4). Through an example, it is shown that this is not valid for complex dimension $n \geq 3$. By default, bearing Figure 1 (from [7]) in mind, there are own complex Douglas spaces, which are not complex Berwald. This is the main motivation for which here we make a systematic study of complex Kropina metrics.

The last part of the paper (§4) is devoted to the projectiveness of the complex Kropina metric $F := \frac{\alpha^2}{|\beta|}$, $|\beta| \neq 0$. The necessary and sufficient conditions for which the metrics F and α are projectively related are contained in Theorems 4.1 and 4.2. Also, under assumption of $A = 0$, we prove that the complex Kropina metric F on a domain D from $\mathbb{C}^n$ is projectively related to the complex Euclidean metric ε on D if and only if α is projectively related to the Euclidean metric ε and, F is a complex Berwald metric with $(\overset{a}{\delta}_k|\beta|)\eta^k = 0$, (Theorem 4.3).

2. Preliminaries

In this section we briefly set the necessary ideas for the next sections. Let M be an n-dimensional complex manifold and $z = (z^k)_{k=\overline{1,n}}$ be the complex coordinates in a local chart. The complexified $T_C M$ of the real tangent bundle $T_R M$, splits into the sum of the holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is itself a complex manifold and the local coordinates in a local chart will be denoted by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. These are changed into $(z'^k, \eta'^k)_{k=\overline{1,n}}$ by the rules $z'^k = z'^k(z)$ and $\eta'^k = \frac{\partial z'^k}{\partial z^l}\eta^l$.

A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \backslash \{0\}$;

ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\forall \lambda \in \mathbb{C}$;

iv) the Hermitian matrix $\big(g_{i\bar{j}}(z,\eta)\big)$ is positive definite, where $g_{i\bar{j}} := \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor. Equivalently, this means that the indicatrix is strongly pseudo-convex.

Consequently, from iii) we have $\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L$, $\frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0$ and $L = g_{i\bar{j}}\eta^i\bar{\eta}^j$.

Considering the sections of the complexified tangent bundle of $T'M$, $VT'M \subset T'(T'M)$ is the vertical bundle, and $VT''M$ is its conjugate. The vertical distribution $V_uT'M$ is locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$. The complex nonlinear connection, briefly (*c.n.c.*), is an instrument in 'linearization' of the geometry of the manifold $T'M$. A (*c.n.c.*) is a supplementary complex subbundle to $VT'M$ in $T'(T'M)$, i.e., $T'(T'M) = HT'M \oplus VT'M$. The horizontal distribution $H_uT'M$ is locally spanned by $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where $N_k^j(z,\eta)$ are the coefficients of the (*c.n.c.*). The pair $\{\delta_k := \frac{\delta}{\delta z^k}, \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\}$ is called the adapted frame of the (*c.n.c.*), which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we obtain an adapted frame $\{\delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted frames are $\{dz^k, \delta\eta^k := d\eta^k + N_j^k dz^j\}$ and $\{d\bar{z}^k, \delta\bar{\eta}^k\}$.

Let $S \in T'(T'M)$ be a complex spray. Locally, it can be expressed as follows

$$S = \eta^k \frac{\partial}{\partial z^k} - 2G^k(z,\eta)\frac{\partial}{\partial \eta^k}, \tag{2.1}$$

where G^k are the spray coefficients, [16]. Between the notions of complex spray and (*c.n.c.*) there exists an interdependence, one determining the other, (for more details see [16]).

A (*c.n.c.*) related only to the fundamental function of the complex Finsler space (M, F) is the so-called Chern-Finsler (*c.n.c.*), (see [1]), with the local coefficients $N_j^i := g^{\overline{m}i}\frac{\partial g_{l\overline{m}}}{\partial z^j}\eta^l$. Subsequently, δ_k is the adapted frame with respect to the Chern-Finsler (*c.n.c.*). A Hermitian connection D, of $(1,0)$-type, which satisfies in addition $D_{JX}Y = JD_XY$, for all X horizontal vectors and J the natural complex structure of the manifold, is the Chern-Finsler connection, [1]. It is locally given by the following coefficients (see [16]):

$$L^i_{jk} := g^{\bar{l}i}\delta_k g_{j\bar{l}} = \dot{\partial}_j N^i_k \ ; \ C^i_{jk} := g^{\bar{l}i}\dot{\partial}_k g_{j\bar{l}}. \tag{2.2}$$

Note that the spray coefficients perform $2G^i = N^i_j\eta^j = L^i_{jk}\eta^j\eta^k$.

In [1]'s terminology, the complex Finsler space (M, F) is *strongly Kähler* if and only if $T^i_{jk} = 0$, *Kähler* if and only if $T^i_{jk}\eta^j = 0$ and *weakly Kähler* if and only if $g_{i\bar{l}}T^i_{jk}\eta^j\bar{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. In [12] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of the complex Finsler metrics which come from Hermitian metrics on M, so-called *purely Hermitian metrics* in [16], (i.e., $g_{i\bar{j}} = g_{i\bar{j}}(z)$), all those nuances of Kähler are same. On the other hand, as in Aikou's work [2], a complex

Finsler space which is Kähler and $L^i_{jk} = L^i_{jk}(z)$ is called a *complex Berwald* space.

In [16] it is proved that the Chern-Finsler (*c.n.c.*) does not generally come from a complex spray. But, its local coefficients N^i_j always determine a complex spray with coefficients $G^i = \frac{1}{2}N^i_j\eta^j$. Further, G^i induce a (*c.n.c.*) denoted by $\overset{c}{N}{}^i_j := \dot{\partial}_j G^i$ which is called *canonical* in [16]. It is proved that it coincides with Chern-Finsler (*c.n.c.*) if and only if the complex Finsler metric is Kähler. With respect to the canonical (*c.n.c.*), we consider the frame $\{\overset{c}{\delta}_k, \dot{\partial}_k\}$, where $\overset{c}{\delta}_k := \frac{\partial}{\partial z^k} - \overset{c}{N}{}^j_k\dot{\partial}_j$, and its dual coframe $\{dz^k, \overset{c}{\delta}\eta^k\}$, where $\overset{c}{\delta}\eta^k := d\eta^k + \overset{c}{N}{}^k_j dz^j$.

An extension of the complex Berwald space notion is that of *generalized Berwald* spaces, studied by the authors in [6]. It is with $\dot{\partial}_{\bar{h}} G^i = 0$. Any complex Berwald space is generalized Berwald. Moreover, in [4] we proved that any generalized Berwald space, which is weakly Kähler, is a complex Berwald space.

Theorem 2.1 ([5]). *Let F and $\tilde{F}$ be complex Finsler metrics on the manifold M. Then F and $\tilde{F}$ are projectively related if and only if there is a smooth function P on $T'M$ with complex values, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i;\ i = \overline{1,n}, \tag{2.3}$$

where $B^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$, where $\theta^{*i} = 2g^{\bar{j}i}\overset{c}{\delta}_{\bar{j}} L$.

Note that θ^{*i} is vanishing if and only if the space is weakly Kähler.

Corollary 2.1 ([5]). *Let F be a generalized Berwald metric on the manifold M and $\tilde{F}$ another complex Finsler metric on M. Then, F and $\tilde{F}$ are projectively related if and only if*

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k(\dot{\partial}_{\bar{r}}\tilde{F})\ ;\ B^r = -\frac{1}{\tilde{F}}\theta^{*l}(\dot{\partial}_l\tilde{F})\eta^r; \tag{2.4}$$
$$P = \frac{1}{\tilde{F}}[(\delta_k\tilde{F})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{F})]$$

for any $r = \overline{1,n}$. Moreover, the projective change is $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ and $\tilde{F}$ is also generalized Berwald.

A complex Finsler space (M, F) is called *complex Douglas* space if and only if all complex curvature invariants of Douglas type are vanishing, which is equivalent with the conditions: (M, F) is a generalized Berwald space and $K^i := \theta^{*i} - \frac{1}{n}\theta^{*l}_l\eta^i = \varphi^i_{\bar{r}s}\bar{\eta}^r\eta^s$, where $\varphi^i_{\bar{r}s}$ are smooth functions which depend only on z and $\bar{z}$. Note that any complex Berwald space is a complex Douglas space. Purely Hermitian metrics give trivial examples of complex Douglas and generalized Berwald metrics.

Theorem 2.2 ([7]). *Let F and $\tilde{F}$ be projectively related complex Finsler metrics on the manifold M. Then, F is a Douglas metric if and only if $\tilde{F}$ is also a Douglas metric.*

3. Generalized Berwald spaces with Kropina metric

We consider $z \in M$, $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$, $\tilde{a} := a_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ a Hermitian metric on M and $b = b_i(z) dz^i$ a differential $(1,0)$-form. By these objects we have defined on $\{(z,\eta) \in T'M \mid b_i(z)\eta^i \neq 0\}$ the complex Kropina metric

$$F := \frac{\alpha^2}{|\beta|},\ |\beta| \neq 0, \tag{3.1}$$

where $\alpha^2(z,\eta) := a_{i\bar{j}}(z)\eta^i\bar{\eta}^j$, $|\beta(z,\eta)| = \sqrt{\beta(z,\eta)\overline{\beta(z,\eta)}}$ and $\beta(z,\eta) = b_i(z)\eta^i$, (for more details see [3,6]).

Complex Kropina metrics are important in complex Finsler geometry, too. Like complex Randers metrics, they represent a medium where Hermitian geometry interacts with complex Finsler geometry properly. Nevertheless, a complex Kropina metric can be purely Hermitian if $n = 1$. Thus, $n \geq 2$ is necessary condition for which $F := \frac{\alpha^2}{|\beta|}$ is non-purely Hermitian, that is its fundamental tensor metric $g_{i\bar{j}}$ simultaneously depends on z and η.

Since any purely Hermitian metric is a complex Douglas one, our next study is focused on the non-purely Hermitian complex Kropina metrics.

Corresponding to a complex Kropina metric we recall the main tools:

$$g_{i\bar{j}} = 2q^2 a_{i\bar{j}} - \frac{2}{|\beta|^2} l_i l_{\bar{j}} + \frac{1}{q^4|\beta|^2}\eta_i\eta_{\bar{j}} \ \ ; \ \ \det(g_{i\bar{j}}) = 2^{n-1}q^{2n}\det(a_{i\bar{j}}). \tag{3.2}$$

$$g^{\bar{j}i} = \frac{1}{2q^2}a^{\bar{j}i} - \frac{2 - q^2||b||^2}{2q^4|\beta|^2}\eta^i\bar{\eta}^j + \frac{1}{2q^2|\beta|^2}\left(\beta b^i\bar{\eta}^j + \bar{\beta}\eta^i\bar{b}^j\right),$$

alongside the local coefficients of Chern-Finsler *c.n.c.*

$$N^i_j = \overset{a}{N^i_j} - \frac{\bar{\beta}}{|\beta|^2} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j}\eta^i - \frac{q^2\beta}{2} t^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j},$$

where $t^{\bar{r}i} := a^{\bar{r}i} + \frac{2-q^2||b||^2}{q^2|\beta|^2}\eta^i\bar{\eta}^r + \frac{1}{|\beta|^2}(\bar{\beta}\eta^i\bar{b}^r - \beta b^i\bar{\eta}^r)$; $\overset{a}{N^k_j} := a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l$, with the settings

$$l_i := 2\alpha\frac{\partial \alpha}{\partial \eta^i} \ ; \ b_i := \frac{2\beta}{|\beta|}\frac{\partial|\beta|}{\partial\eta^i} \ ; \ \eta_i := \frac{\partial L}{\partial \eta^i} = 2q^2 l_i - q^4\bar{\beta}b_i; \tag{3.3}$$

$$q := \frac{\alpha}{|\beta|}; b^i := a^{\bar{j}i}b_{\bar{j}} \ ; \ ||b||^2 := a^{\bar{j}i}b_i b_{\bar{j}} \ ; \ b^{\bar{i}} := \bar{b}^i.$$

Therefore, the spray coefficients are

$$G^i = \overset{a}{G^i} - \frac{\bar{\beta}}{2|\beta|^2} l_{\bar{r}}\frac{\partial \bar{b}^r}{\partial z^j}\eta^i\eta^j - \frac{q^2\beta}{4}t^{\bar{r}i}\frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j \tag{3.4}$$

and so for the generalized Berwald Kropina spaces we prove the following.

Theorem 3.1. *Let (M,F) be a connected non-purely Hermitian complex Kropina space. Then, (M,F) is generalized Berwald if and only if*

$$G^i = \overset{a}{G^i} - \frac{1}{2|\beta|^2}A\eta^i, \tag{3.5}$$

with $A := (\bar{\beta} l_{\bar{r}} \frac{\partial b^{\bar{r}}}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r)\eta^j$. Moreover, any of these assertions implies

$$B = 0; \quad \frac{\partial b_{\bar{m}}}{\partial z^j} = \frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r b_{\bar{m}}; \quad \dot{\partial}_{\bar{m}} A = \frac{\beta}{|\beta|^2} A b_{\bar{m}}, \tag{3.6}$$

where $B := \frac{\partial b_{\bar{r}}}{\partial z^j}\eta^j (b^{\bar{r}} - \frac{\beta}{|\beta|^2}||b||^2 \bar{\eta}^r)$.

Proof. Contractions with l_i and b_i in (3.4), yield the linear system

$$\begin{cases} \frac{4}{q^2}(\overset{a}{G^i} - G^i)l_i = 2A + \alpha^2 B; \\ 2\bar{\beta}(\overset{a}{G^i} - G^i)b_i = A + \alpha^2 B, \end{cases} \tag{3.7}$$

with unknowns A and B. Also, by derivation with respect to $\bar{\eta}$, the relation (3.4) implies that

$$(\dot{\partial}_{\bar{m}} G^i)\bar{b}^m b_i = -\frac{\beta^2}{|\beta|^2}(1 - q^2||b||^2)B. \tag{3.8}$$

If (M,F) is generalized Berwald, then $\dot{\partial}_{\bar{m}} G^i = 0$. Thus, using (3.8) and $1 - q^2||b||^2 \neq 0$, we obtain $B = 0$. When substituted into the system (3.7), this yields

$$(\overset{a}{G^i} - G^i)b_i = \frac{\beta}{2|\beta|^2}A, \tag{3.9}$$

$$\beta(\overset{a}{G^i} - G^i)l_i = \alpha^2(\overset{a}{G^i} - G^i)b_i.$$

By derivation with respect to $\bar{\eta}$, the second relation in (3.9) implies that $a_{i\bar{m}}\beta(\overset{a}{G^i} - G^i) = l_{\bar{m}}(\overset{a}{G^i} - G^i)b_i$ and then, $\beta(\overset{a}{G^i} - G^i) = (\overset{a}{G^r} - G^r)b_r\eta^i$. These, along with the first relation in (3.9), give (3.5).

Conversely, the condition (3.5) substituted into the relations (3.7), implies that $B = 0$. Then, after two successive differentiations of $B = 0$ with respect η and $\bar{\eta}$, we can deduce that $\frac{\partial b_{\bar{m}}}{\partial z^j} = \frac{\beta}{|\beta|^2}\frac{\partial b_{\bar{r}}}{\partial z^j}\bar{\eta}^r b_{\bar{m}}$. In addition, we have $\dot{\partial}_{\bar{m}} A = (b_{\bar{m}} l_{\bar{r}} \frac{\partial b^{\bar{r}}}{\partial z^j} + \beta \frac{\partial b_{\bar{m}}}{\partial z^j})\eta^j$, which together with the last relation, gives $\dot{\partial}_{\bar{m}} A = \frac{\beta}{|\beta|^2} A b_{\bar{m}}$ and it completes (3.6).

Since $\overset{a}{G^i}$ are always holomorphic with respect to η, the derivation with respect to $\bar{\eta}$ of (3.5) implies $\dot{\partial}_{\bar{m}} G^i = \frac{\beta}{2|\beta|^4} A b_{\bar{m}}\eta^i - \frac{1}{2|\beta|^2}(\dot{\partial}_{\bar{m}} A)\eta^i$ and, owing to (3.6), $\dot{\partial}_{\bar{m}} G^i = 0$, that is the space is generalized Berwald. □

Remark 3.1. Obviously, if (M,F) is a connected complex Kropina space with the property that $A = 0$, then it is generalized Berwald and $G^i = \overset{a}{G^i}$. Moreover, if α is Kähler and $A=0$, then the space is Berwald, (see Theorem 4.5 from [6]).

Subsequently, all reasonings will be made under assumptions of non-purely Hermitian and generalized Berwald. Since $G^k = \overset{a}{G}{}^k - \frac{1}{2|\beta|^2}A\eta^k$, then

$$\overset{c}{N}{}_j^k = \frac{1}{2}a^{\bar{m}k}\left(\frac{\partial a_{l\bar{m}}}{\partial z^j} + \frac{\partial a_{j\bar{m}}}{\partial z^l}\right)\eta^l + \frac{1}{2|\beta|^2}\left[\frac{\bar{\beta}}{|\beta|^2}Ab_j\eta^k - A\delta_j^k - (\dot{\partial}_j A)\eta^k\right],$$

which together with (3.2), leads to

$$\overset{c}{\delta}_{\bar{m}}L = -\frac{1}{2}|\beta|q^2\Gamma_{l\bar{r}\bar{m}}\tau^l\bar{\eta}^r + q^4\Omega_{\bar{m}}, \tag{3.10}$$

where $\Gamma_{l\bar{r}\bar{m}} := \frac{\partial a_{l\bar{m}}}{\partial \bar{z}^r} - \frac{\partial a_{l\bar{r}}}{\partial \bar{z}^m}$; $\tau^l := \frac{1}{F}a^{\bar{m}l}\bar{\eta}_m = \frac{2}{|\beta|}\eta^l - q^2\frac{\beta}{|\beta|}b^l$, $\Omega_{\bar{m}} := \beta\overset{a}{N}{}_{\bar{m}}^{\bar{s}}b_{\bar{s}} - \beta\frac{\partial b_{\bar{r}}}{\partial \bar{z}^m}\bar{\eta}^r - \bar{\beta}\frac{\partial b_l}{\partial \bar{z}^m}\eta^l + \frac{1}{2}(\dot{\partial}_{\bar{m}}\bar{A}) + \bar{A}\chi_{\bar{m}}$, $\chi_{\bar{m}} := q^2(\dot{\partial}_{\bar{m}}q^2) = \frac{1}{\alpha^2}l_{\bar{m}} - \frac{\beta}{|\beta|^2}b_{\bar{m}}$, which satisfy the properties:

Lemma 3.1. *Let (M,F) be a connected non-purely Hermitian complex Kropina space. Then,*

$$\begin{gathered}\Gamma_{l\bar{r}\bar{m}}\bar{\eta}^r\bar{\eta}^m = 0\ ;\quad \Omega_{\bar{m}}\bar{\eta}^m = 0\ ;\\
(\dot{\partial}_{\bar{m}}A)\bar{\eta}^m = A\ ;\quad (\dot{\partial}_l A)\eta^l = 2A\ ;\quad \dot{\partial}_l(\dot{\partial}_{\bar{m}}A) = \frac{\beta}{|\beta|^2}(\dot{\partial}_l A)b_{\bar{m}}\ ;\\
\dot{\partial}_l(\dot{\partial}_s A)\eta^l = \dot{\partial}_s A\ ;\quad \dot{\partial}_{\bar{m}}(\dot{\partial}_s A)\bar{\eta}^m = \dot{\partial}_s A\ ;\\
\dot{\partial}_j\chi_{\bar{m}} = \frac{1}{\alpha^2}\left(a_{j\bar{m}} - \frac{1}{\alpha^2}l_j l_{\bar{m}}\right)\ ;\quad \dot{\partial}_j\Omega_{\bar{m}} = \frac{1}{\beta}b_j\Omega_{\bar{m}} + \bar{A}(\dot{\partial}_j\chi_{\bar{m}})\ ;\\
(\dot{\partial}_l\Omega_{\bar{m}})\eta^l = \Omega_{\bar{m}}\ ;\quad (\dot{\partial}_{\bar{r}}\Omega_{\bar{m}})\bar{\eta}^r = \Omega_{\bar{m}}\ ;\quad (\dot{\partial}_{\bar{r}}\Omega_{\bar{m}})\bar{\eta}^m = -\Omega_{\bar{r}}\ ;\\
(\dot{\partial}_l\Omega_{\bar{m}})b^l = \frac{1}{\beta}||b||^2\Omega_{\bar{m}} - \frac{\bar{\beta}\bar{A}}{\alpha^2}\chi_{\bar{m}}\ ;\quad (\dot{\partial}_{\bar{r}}\Omega_{\bar{m}})b^{\bar{r}}b^{\bar{m}} = -\frac{\beta(1-q^2||b||^2)}{\alpha^2}Y,\end{gathered} \tag{3.11}$$

where $Y := -(\dot{\partial}_{\bar{m}}\bar{A})b^{\bar{m}} + \frac{\beta}{\alpha^2}(1+q^2||b||^2)\bar{A}$.

Proof. All relations result by straightforward computation. □

Due to (3.10), (3.2), and having in mind $\theta^{*i} = 2g^{\bar{m}i}(\overset{c}{\delta}_{\bar{m}}L)$, it follows

$$\begin{aligned}\overset{a}{\theta}{}^{*i} &= -\Gamma_{l\bar{r}\bar{m}}a^{\bar{m}i}\eta^l\bar{\eta}^r;\\
\theta^{*i} &= -\left(\frac{1}{2}|\beta|\Gamma_{l\bar{r}\bar{m}}\tau^l\bar{\eta}^r - q^2\Omega_{\bar{m}}\right)\left(a^{\bar{m}i} + \frac{\bar{\beta}}{|\beta|^2}b^{\bar{m}}\eta^i\right),\end{aligned} \tag{3.12}$$

related by the formula

$$\begin{aligned}\theta^{*i} = \overset{a}{\theta}{}^{*i} &- \frac{\bar{\beta}}{|\beta|^2}\Gamma_{l\bar{r}\bar{m}}\eta^l b^{\bar{m}}\bar{\eta}^r\eta^i\\
&+ q^2\left(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}}\right)\left(a^{\bar{m}i} + \frac{\bar{\beta}}{|\beta|^2}b^{\bar{m}}\eta^i\right).\end{aligned} \tag{3.13}$$

Once obtained θ^{*i} and $\overset{a}{\theta}{}^{*i}$, and taking into account that $K^i := \theta^{*i} - \frac{1}{n}\theta^{*l}_l \eta^i$, it is a technical computation to give the expressions for K^i and $\overset{a}{K}{}^i$. Certainly, it involves some trivial calculus which lead to

$$\overset{a}{K}{}^i = -\Gamma_{l\bar{r}\bar{m}}(a^{\bar{m}i}\eta^l - \frac{1}{n}a^{\bar{m}l}\eta^i)\bar{\eta}^r; \tag{3.14}$$
$$K^i = \overset{a}{K}{}^i + q^2(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})a^{\bar{m}i} - \frac{n-1}{n|\beta|^2}\bar{A}\eta^i.$$

Theorem 3.2. *Let (M, F) be a connected non-purely Hermitian complex Kropina space. If (M, F) is a complex Douglas space, then*

$$(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}} = -(\dot{\partial}_{\bar{m}}\bar{A})b^{\bar{m}} + \frac{\beta}{\alpha^2}(1 + q^2||b||^2)\bar{A}; \tag{3.15}$$
$$[\dot{\partial}_{\bar{r}}(\dot{\partial}_{\bar{m}}\bar{A})]b^{\bar{m}}b^{\bar{r}} = \frac{2\beta||b||^2}{|\beta|^2}[(\dot{\partial}_{\bar{m}}\bar{A})b^{\bar{m}} - \frac{\beta||b||^2}{|\beta|^2}\bar{A}].$$

Proof. If (M, F) is complex Douglas, then it is generalized Berwald, (i.e., $G^i = \overset{a}{G}{}^i - \frac{1}{2|\beta|^2}A\eta^i$), and $K^i = \varphi^i_{\bar{r}s}(z, \bar{z})\bar{\eta}^r\eta^s$, which means that K^i are homogeneous polynomials in η and in $\bar{\eta}$ of first degree. Thus, after two differentiations with respect to η, and with respect to $\bar{\eta}$ respectively, in the second formula (3.14) the resulting expression must identically vanish. Because $\dot{\partial}_r(\dot{\partial}_s \overset{a}{K}{}^i) = 0$ and $\dot{\partial}_{\bar{r}}(\dot{\partial}_{\bar{s}}\overset{a}{K}{}^i) = 0$, it remains that $W^i_{rs} = 0$ and $W^i_{\bar{r}\bar{s}} = 0$, where

$$W^i_{rs} := \dot{\partial}_r(\dot{\partial}_s W^i);\ W^i_{\bar{r}\bar{s}} := \dot{\partial}_{\bar{r}}(\dot{\partial}_{\bar{s}} W^i);$$
$$W^i := q^2(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})a^{\bar{m}i} - \frac{n-1}{n|\beta|^2}\bar{A}\eta^i.$$

Developing the calculations for $W^i_{rs}b^r b^s$ and $W^i_{rs}b^{\bar{r}}b^{\bar{s}}$ and taking into account the properties (3.11), we obtain that

$$\frac{-2||b||^2(1 - q^2||b||^2)}{\bar{\beta}^2}(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})a^{\bar{m}i}$$
$$+ \frac{2(1 - q^2||b||^2)}{\bar{\beta}}[\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l b^{\bar{r}} + (\dot{\partial}_{\bar{r}}\Omega_{\bar{m}})b^{\bar{r}}]a^{\bar{m}i}$$
$$+ q^2\{\dot{\partial}_{\bar{r}}[\dot{\partial}_{\bar{s}}(\bar{A}x_{\bar{m}})]\}b^{\bar{r}}b^{\bar{s}}a^{\bar{m}i} - \frac{n-1}{n}\{\dot{\partial}_{\bar{r}}[\dot{\partial}_{\bar{s}}(\bar{A}\frac{1}{|\beta|^2})]\}b^{\bar{r}}b^{\bar{s}}\eta^i = 0.$$

Now, the contractions with b_i and l_i respectively, in the last relation, give:

$$\frac{2(1 - q^2||b||^2)}{\bar{\beta}}X - \frac{2(1 - q^2||b||^2)}{\bar{\beta}}Y = -\frac{(n-1)\alpha^2}{n}\{\dot{\partial}_{\bar{r}}[\dot{\partial}_{\bar{s}}(\bar{A}\frac{1}{|\beta|^2})]\}b^{\bar{r}}b^{\bar{s}}; \tag{3.16}$$

$$-\frac{2||b||^2(1 - q^2||b||^2)}{\bar{\beta}^2}X + \frac{4||b||^2(1 - q^2||b||^2)}{\bar{\beta}^2}Y \tag{3.17}$$

$$= \frac{(n-1)\beta}{n}\{\dot{\partial}_{\bar{r}}[\dot{\partial}_{\bar{s}}(\bar{A}\frac{1}{|\beta|^2})]\}b^{\bar{r}}b^{\bar{s}}$$
$$- \frac{1}{\bar{\beta}}(1-q^2||b||^2)\{[\dot{\partial}_{\bar{r}}(\dot{\partial}_{\bar{m}}\bar{A})]b^{\bar{m}}b^{\bar{r}} - \frac{2\beta||b||^2}{\alpha^2\bar{\beta}}\bar{A}\},$$

where $X := (\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}}$; $Y := -(\dot{\partial}_{\bar{m}}\bar{A})b^{\bar{m}} + \frac{\beta}{\alpha^2}(1+q^2||b||^2)\bar{A}$.

Multiplying the relation (3.16) with $\frac{||b||^2}{\beta}$ and then summing it with the relation (3.17), we obtain $\{\dot{\partial}_{\bar{r}}[\dot{\partial}_{\bar{s}}(\bar{A}\frac{1}{|\beta|^2})]\}b^{\bar{r}}b^{\bar{s}} = 0$, since $1 - q^2||b||^2 \neq 0$. This implies $[\dot{\partial}_{\bar{r}}(\dot{\partial}_{\bar{m}}\bar{A})]b^{\bar{m}}b^{\bar{r}} = \frac{2\beta||b||^2}{|\beta|^2}[(\dot{\partial}_{\bar{m}}\bar{A})b^{\bar{m}} - \frac{\beta||b||^2}{|\beta|^2}\bar{A}]$ and then, $X = Y$, which completes our claim. □

Corollary 3.1. *Let (M,F) be a connected non-purely Hermitian complex Kropina space. If (M,F) is generalized Berwald, α is Kähler and $\Omega_{\bar{m}} = 0$, then (M,F) is a complex Berwald space and $A = 0$.*

Proof. If α is Kähler, then $\Gamma_{l\bar{r}\bar{m}} = 0$ and $\overset{a}{\theta}{}^{*i} = 0$. Our assumption $\Omega_{\bar{m}} = 0$ along with conditions (3.13) and (3.14), then lead to $\theta^{*i} = 0$, i.e., F is weakly Kähler. Hence $K^i = 0$, which implies $A = 0$, that is the space is Berwald. □

Whether or not the space is Douglas, by assuming it is generalized Berwald, it verifies the second relation (3.14). Bearing this in mind, we can prove:

Proposition 3.1. *Let (M,F) be a connected non-purely Hermitian complex Kropina space. If (M,F) is generalized Berwald, then $K^i = \overset{a}{K}{}^i$ if and only if $A = 0$ and $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$.*

Proof. If $K^i = \overset{a}{K}{}^i$, due to (3.14), it follows that

$$q^2(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})a^{\bar{m}i} - \frac{n-1}{n|\beta|^2}\bar{A}\eta^i = 0,$$

which contracted with l_i gives $\bar{A} = 0$, because $n > 1$. Substituting $A = 0$ in the last relation, we obtain $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$.

Using again (3.14), the converse implication is obvious. □

The sufficiency from Theorem 3.2 is checked only in a particular case. Indeed, considering a non-purely Hermitian complex Kropina space which is generalized Berwald and $A = 0$ and $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$, the conditions (3.15) are held. Some necessary and sufficient circumstances for complex Douglas property of a non-purely Hermitian complex Kropina space are given in the following.

Theorem 3.3. *Let (M,F) be a connected non-purely Hermitian complex Kropina space. Then (M,F) is a complex Douglas space with $K^i = \overset{a}{K}{}^i$ if* *and only if* $G^i = \overset{a}{G^i}$ *and* $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. *Moreover, given any of them,* $\theta^{*i} = \overset{a}{\theta^{*i}} - \frac{\bar{\beta}}{|\beta|^2}\Gamma_{l\bar{r}\bar{m}}\eta^l b^{\bar{m}}\bar{\eta}^r\eta^i$.

Proof. The direct implication results from Theorem 3.1 and Corollary 3.1.

Conversely, since $G^i = \overset{a}{G^i}$ the space (M,F) is generalized Berwald, and owing to (3.5), $A = 0$. Now, due to Corollary 3.1, $K^i = \overset{a}{K^i}$. Since $\overset{a}{K^i}$ are homogeneous polynomials of first degree in η and in $\bar{\eta}$, then K^i are also are homogeneous polynomials of first degree in η and in $\bar{\eta}$. Taking into account Theorem 2.1, we obtain that (M,F) is complex Douglas.

Moreover, $A = 0$ and $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$ substituted in (3.13), give $\theta^{*i} = \overset{a}{\theta^{*i}} - \frac{\bar{\beta}}{|\beta|^2}\Gamma_{l\bar{r}\bar{m}}\eta^l b^{\bar{m}}\bar{\eta}^r\eta^i$. □

Corollary 3.2. *Let* (M,F) *be a connected non-purely Hermitian complex Kropina space.* (M,F) *is a complex Douglas space with* $K^i = \overset{a}{K^i}$ *and* $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}} = 0$ *if and only if* $G^i = \overset{a}{G^i}$ *and* $\theta^{*i} = \overset{a}{\theta^{*i}}$.

Proof. According to Theorem 3.3, the direct implication is obvious. Conversely, under our assumption, the space is generalized Berwald with $A = 0$ and owing to (3.12), it results

$$\frac{\bar{\beta}}{|\beta|^2}\Gamma_{l\bar{r}\bar{m}}\eta^l b^{\bar{m}}\bar{\eta}^r\eta^i = q^2(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})(a^{\bar{m}i} + \frac{\bar{\beta}}{|\beta|^2}b^{\bar{m}}\eta^i). \tag{3.18}$$

The contractions with l_i and b_i in the above formula (3.18) yield the homogeneous linear system

$$\begin{cases} q^2X - Z = 0 \\ q^2X - \frac{1}{2}Z = 0 = 0 \end{cases} \tag{3.19}$$

with the unknowns $X := (\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}}$ and $Z := \Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^r b^{\bar{m}}$. Since its determinant is nonzero, $(\Delta = \frac{q^2}{2} \neq 0)$, then the system (3.19) admits only the null solution, i.e.,

$$(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}} = 0; \tag{3.20}$$
$$\Gamma_{l\bar{r}\bar{m}}\eta^l\bar{\eta}^r b^{\bar{m}} = 0.$$

By derivations with respect to η and $\bar{\eta}$, the second relation (3.20) implies $\Gamma_{l\bar{r}\bar{m}}b^{\bar{m}} = 0$, which substituted in the relation (3.18) gives $(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})a^{\bar{m}i} = 0$ and so, $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. These along with (3.14) lead to $K^i = \overset{a}{K^i}$. Thus, all conditions are fulfilled and the space is complex Douglas. □

Theorem 3.4. *Let (M, F) be a 2-dimensional connected non-purely Hermitian complex Kropina space. If (M, F) is a complex Douglas space with $K^i = \overset{a}{K}{}^i$ and $\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = 0$, then it is complex Berwald.*

Proof. The condition $\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = 0$ can be rewritten as

$$\Gamma_{l\bar{r}\bar{1}} b^{\bar{1}} + \Gamma_{l\bar{r}\bar{2}} b^{\bar{2}} = 0, \text{ with } l, r = 1, 2. \tag{3.21}$$

Since $\Gamma_{l\bar{m}\bar{m}} = 0$ and $\Gamma_{l\bar{1}\bar{2}} = -\Gamma_{l\bar{2}\bar{1}}$, with $l, m = 1, 2$, (3.21) is reduced to $\Gamma_{l\bar{1}\bar{2}} b^{\bar{1}} = 0$ and $\Gamma_{l\bar{1}\bar{2}} b^{\bar{2}} = 0$. These give $\Gamma_{l\bar{1}\bar{2}} = 0$, because at least one of coefficients $b^{\bar{m}}$ is nonzero. This means that the metric α is Kähler, and so, by Corollary 3.2, $\theta^{*i} = \overset{a}{\theta}{}^{*i} = 0$. Thus, the space (M, F) is complex Berwald. $\square$

Theorem 3.4 attests that a complex Douglas-Kropina space of dimension two with $K^i = \overset{a}{K}{}^i$ and $\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = 0$ is a complex Berwald space. However, there exist complex Douglas-Kropina spaces with $K^i = \overset{a}{K}{}^i$ and $\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = 0$ that are not complex Berwald for dimension $n \geq 3$. We show this fact by constructing some explicit examples.

Example 3.1. On $M = \mathbb{C}^3$ we set the purely Hermitian metric

$$\alpha^2 = e^{z^1+\bar{z}^1} \left|\eta^1\right|^2 + e^{z^2+\bar{z}^2} \left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3} \left|\eta^3\right|^2 \tag{3.22}$$

and we choose the $(1, 0)$-differential form β as

$$\beta = e^{z^2} \eta^2 \neq 0. \tag{3.23}$$

Then, $|\beta|^2 = e^{z^2+\bar{z}^2} \left|\eta^2\right|^2$ and so, $b_i = b^i = 0$, $i = 1, 3$, $b_2 = e^{z^2}$, $b^2 = e^{-z^2}$ and $||b|| = 1$. Also, we have $\Gamma_{l\bar{r}\bar{m}} = 0$, except for the coefficients $\Gamma_{3\bar{1}\bar{3}} = -\Gamma_{3\bar{3}\bar{1}} = e^{z^1+\bar{z}^1+z^3+\bar{z}^3} \neq 0$. Thus, the metric (3.22) is not Kähler.

With these tools we construct a complex Kropina metric

$$F = \frac{e^{z^1+\bar{z}^1} \left|\eta^1\right|^2 + e^{z^2+\bar{z}^2} \left|\eta^2\right|^2 + e^{z^1+\bar{z}^1+z^3+\bar{z}^3} \left|\eta^3\right|^2}{\sqrt{e^{z^2+\bar{z}^2}} \left|\eta^2\right|}, \tag{3.24}$$

which is non-purely Hermitian, with $\det(g_{i\bar{j}}) = 4q^{2n} \det\left(a_{i\bar{j}}\right) > 0$, $i, j = 1, 2, 3$.

Some computations give that the metric (3.24) is generalized Berwald with $A = 0$, i.e.,

$$A = (\bar{\beta} l_{\bar{r}} \frac{\partial \bar{b}^r}{\partial z^j} + \beta \frac{\partial b_{\bar{r}}}{\partial z^j} \bar{\eta}^r) \eta^j = (\bar{\beta} l_{\bar{2}} \frac{\partial \bar{b}^2}{\partial z^j} + \beta \frac{\partial b_{\bar{2}}}{\partial z^j} \bar{\eta}^2) \eta^j = 0.$$

Moreover, we have

$$\Gamma_{l\bar{r}\bar{m}} b^{\bar{m}} = \Gamma_{l\bar{r}\bar{2}} b^{\bar{2}} = 0 \; ; \; \Gamma_{l\bar{r}\bar{m}} b^l = \Gamma_{2\bar{r}\bar{m}} b^2 = 0,$$
$$\Omega_{\bar{r}} = 0 \; , \; r = 1, 2, 3,$$

which substituted into (3.13) and (3.14) give

$$\overset{a}{\theta}{}^{*i} = \theta^{*i} \text{ and } \overset{a}{K}{}^i = K^i, \; i = 1, 2, 3.$$

Thus, by Corollary 3.2, it results that (3.24) is a complex Douglas metric.

Note that the above example can be generalized to a class of complex Douglas metrics, taking on $M = \mathbb{C}^n$,

$$\alpha^2 = \sum_{\substack{k=1 \\ k \neq 3}}^{n} e^{z^k + \bar{z}^k} \left|\eta^k\right|^2 + e^{z^1 + \bar{z}^1 + z^3 + \bar{z}^3} \left|\eta^3\right|^2 .$$

For β we can choose one of the following possibilities $\beta = e^{z^k} \eta^k$, where $k = \overline{1, n}$, excepting $k = 1$ and 3.

Theorem 3.5. *Let (M, F) be a connected non-purely Hermitian complex Berwald-Kropina space. Then, α is Kähler if and only if $A = 0$. Moreover, given any of them, $\Omega_{\bar{m}} b^{\bar{m}} = 0$.*

Proof. Under complex Berwald assumption, it results $\theta^{*i} = K^i = 0$ which contracted with l_i give

$$(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}} b^l \bar{\eta}^r + \Omega_{\bar{m}}) b^{\bar{m}} = \frac{1}{q^2} \Gamma_{l\bar{r}\bar{m}} \eta^l \bar{\eta}^r b^{\bar{m}}; \tag{3.25}$$

$$\bar{A} = \frac{|\beta|}{n-1} \Gamma_{l\bar{r}\bar{m}} a^{\bar{m}l} \bar{\eta}^r .$$

The last relations establish the equivalence between the Kähler property of α and $A = 0$, and then the first relation (3.25) leads to $\Omega_{\bar{m}} b^{\bar{m}} = 0$. □

4. Projectiveness of a complex Kropina metric

Our aim is to determine the necessary and sufficient conditions under which the complex Kropina metric, $F = \frac{\alpha^2}{|\beta|}$, $|\beta| \neq 0$, is projectively related to the Hermitian metric α, and then to find other characterizations for complex Douglas-Kropina metric.

In order to apply Corollary 2.1, some computations are entailed. We easily obtain that

$$(\overset{a}{\delta}_k F)\eta^k = \alpha^2 (\overset{a}{\delta}_k \frac{1}{|\beta|})\eta^k = -\frac{q^2}{2|\beta|} A, \tag{4.1}$$

because $\overset{a}{\delta}_k = \frac{\partial}{\partial z^k} - \overset{a}{N^l_k} \dot{\partial}_l$ and $(\overset{a}{\delta}_k \alpha^2)\eta^k = 0$, and

$$\overset{a}{\theta}{}^{*i} (\dot{\partial}_i F) = \frac{q^2 \bar{\beta}}{2|\beta|} \Gamma_{l\bar{r}m} \eta^l \bar{\eta}^r b^{\bar{m}}. \tag{4.2}$$

Theorem 4.1. *Let (M, F) be a connected complex Kropina space. Then, α and F are projectively related if and only if $\tilde{F}$ is generalized Berwald with $A = 0$ and $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}} b^l \bar{\eta}^r$. Moreover, given any of them, the projective change is $G^i = \overset{a}{G^i}$ and $B^i = -P\eta^i$, for any $i = \overline{1, n}$, where $P = \frac{\bar{\beta}}{2|\beta|^2} \Gamma_{l\bar{r}m} \eta^l \bar{\eta}^r b^{\bar{m}}$.*

Proof. Since α is purely Hermitian, it is generalized Berwald. If α and F are projectively related, then by Corollary 2.1, F is also generalized Berwald. So that, by Theorem 3.1 and (3.13), the conditions (2.4) are reduced to $B^i = -P\eta^i$, with $P = \frac{\bar{\beta}}{2|\beta|^2}\Gamma_{l\bar{r}m}\eta^l\bar{\eta}^r b^{\bar{m}}$ and so,

$$(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})(a^{\bar{m}i} + \frac{\bar{\beta}}{|\beta|^2}b^{\bar{m}}\eta^i) = 0.$$

Contracting the last relation with l_i, it results $(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}} = 0$, and then $\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r$. Now, differentiating the previous relation with respect to η^s, and then contracting the result with b^s, due to (3.11), we obtain $A = 0$ and $G^i = \overset{a}{G^i}$.

Conversely, if F is generalized Berwald with $A = 0$, then by

$$\Omega_{\bar{m}} = -\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}}b^l\bar{\eta}^r,$$

and the relations (3.13) and (4.2), all conditions from (2.4) are fulfilled and α and F are projectively related. □

Theorems 2.2, 3.3, 3.5 and 4.1 lead to the next results.

Theorem 4.2. *Let (M, F) be a connected non purely Hermitian complex Kropina space. Then,*

i) *(M, F) is a complex Douglas space with $K^i = \overset{a}{K^i}$ if and only if α and F are projectively related.*

ii) *(M, F) is a complex Berwald space with $A = 0$ if and only if α is Kähler and α and F are projectively related.*

Note that complex the Kropina metric constructed in Example 3.1 is projectively related with the metric (3.22).

Example 4.1. Let $\Delta = \{(z, w) \in \mathbb{C}^2,\ |w| < |z| < 1\}$ be the Hartogs triangle with the Kähler-purely Hermitian metric

$$a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(\log \frac{1}{(1 - |z|^2)(|z|^2 - |w|^2)});\ \alpha^2(z, w; \eta, \theta) = a_{i\bar{j}}\eta^i\bar{\eta}^j, \tag{4.3}$$

where z, w, η, θ are the local coordinates z^1, z^2, η^1, η^2, respectively, and $|z^i|^2 := z^i\bar{z}^i$, $z^i \in \{z, w\}$, $\eta^i \in \{\eta, \theta\}$, and we choose

$$b_1 = \frac{w}{|z|^2 - |w|^2};\ b_2 = -\frac{z}{|z|^2 - |w|^2}. \tag{4.4}$$

With these tools we construct the complex Kropina metric $F = \frac{\alpha^2}{|\beta|}$, where $\alpha(z, w, \eta, \theta) := \sqrt{a_{i\bar{j}}(z, w)\eta^i\bar{\eta}^j}$ and $\beta(z, \eta) = b_i(z, w)\eta^i = \frac{w\eta - z\theta}{|z|^2 - |w|^2} \neq 0$. Since $A = B = 0$ and α is Kähler, $F = \frac{\alpha^2}{|\beta|}$ is a complex Berwald metric and α and F are projectively related.

Example 4.2. Now, if we choose

$$b_1 = \frac{w}{\sqrt{|z|^2 - |w|^2}};\ b_2 = -\frac{z}{\sqrt{|z|^2 - |w|^2}} \tag{4.5}$$

and the purely Hermitian metric $\alpha(z, w, \eta, \theta) := \sqrt{a_{i\bar{j}}(z, w)\eta^i \bar{\eta}^j}$ with

$$a_{1\bar{1}} = \frac{1}{1 - |z|^2} + b_1 b_{\bar{1}};\ a_{1\bar{2}} = b_1 b_{\bar{2}};\ a_{w\overline{w}} = b_2 b_{\bar{2}}; \tag{4.6}$$

$$a^{\bar{1}1} = 1 - |z|^2;\ a^{\bar{2}1} = \frac{\overline{w}z\left(1 - |z|^2\right)}{|z|^2};\ a^{\bar{2}2} = 1 - |w|^2,$$

on Hartogs triangle Δ, then $b^1 = 0$; $b^2 = -\frac{\sqrt{|z|^2 - |w|^2}}{z}$. These tools induce a complex Kropina metric which is only generalized Berwald with $A = 0$. It is not a complex Douglas metric because $(\frac{1}{2}\beta\Gamma_{l\bar{r}\bar{m}} b^l \bar{\eta}^r + \Omega_{\bar{m}})b^{\bar{m}} = (\frac{1}{2}\beta\Gamma_{2\bar{1}\bar{2}} b^2 \bar{\eta} + \Omega_{\bar{2}})b^{\bar{2}} \neq 0$, and so, α and F are not projectively related.

Our next goal is to find when a complex Kropina metric $F := \frac{\alpha^2}{|\beta|}|$, $|\beta| \neq 0$, on a domain D in $\mathbb{C}^n$ is projectively related to the complex Euclidean metric, denoted by ε, on D.

For this reason, we impose some assumptions. On one hand, we assume that F is a complex Berwald metric with $A = 0$. Thus, due to Theorem 4.2 ii) it results α and F are projectively related, α is Kähler and $G^i = \overset{a}{G^i}$. On the other hand, we assume that α is projectively related to the Euclidean metric ε. Therefore, Theorem 3.7 from [5] implies that $\overset{a}{G^i} = \frac{1}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k \eta^i$. Under these statements, some computations lead us to

$$\begin{aligned}
\frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k \eta^i &= \frac{2\alpha}{\tilde{F}|\beta|}\frac{\partial \alpha}{\partial z^k}\eta^k \eta^i - \frac{q^2}{2\tilde{F}|\beta|}\frac{\partial |\beta|^2}{\partial z^k}\eta^k \eta^i \\
&= \frac{2}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k \eta^i - \frac{q^2}{2\alpha^2}\left((\overset{a}{\delta_k}|\beta|)\eta^k + 2\bar{\beta}\overset{a}{G^l} b_l\right)\eta^i \\
&= 2\overset{a}{G^i} - \frac{1}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k \eta^i = \overset{a}{G^i},
\end{aligned}$$

because $A = (\overset{a}{\delta_k}|\beta|)\eta^k = 0$. Thus, $G^i = \frac{1}{\tilde{F}}\frac{\partial \tilde{F}}{\partial z^k}\eta^k \eta^i$, for any $i = \overline{1, n}$, which together with the Berwald assumption for F, give that F is projectively related to the complex Euclidean metric ε.

Conversely, by [5, Theorem 3.7] it results that ε and F are projectively related if and only if the complex Kropina metric F is weakly Kähler and $G^i = \frac{1}{F}\frac{\partial F}{\partial z^k}\eta^k \eta^i$, for any $i = \overline{1, n}$. These induce the Berwald property for F. In order to apply Theorem 4.2 ii), we assume that $A = 0$ and then, it results that F and α are projectively related, α is Kähler and $G^i = \overset{a}{G^i}$.

Also, we thus have

$$\overset{a}{G^i} = \frac{2}{\alpha}\frac{\partial \alpha}{\partial z^k}\eta^k \eta^i - \frac{q^2 \bar{\beta}}{\alpha^2}\overset{a}{G^l} b_l \eta^i. \tag{4.7}$$

The contraction with b_i of (4.7) gives $\overset{a}{G^i} b_i = \frac{\beta}{\alpha}\frac{\partial\alpha}{\partial z^k}\eta^k$, which substituted into (4.7) yields $\overset{a}{G^i} = \frac{1}{\alpha}\frac{\partial\alpha}{\partial z^k}\eta^k\eta^i$, i.e., α is projectively related to the Euclidean metric ε.

Thus, the following theorem is proved:

Theorem 4.3. *Let $F := \frac{\alpha^2}{|\beta|}$, $|\beta| \neq 0$, be a complex Kropina metric with $A = 0$, on a domain D in $\mathbb{C}^n$ and ε the complex Euclidean metric on D. Then, ε and F are projectively related if and only if α is projectively related to the Euclidean metric ε and F is a complex Berwald metric.*

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