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NOTES ON WEAKLY CYCLIC Z-SYMMETRIC MANIFOLDS

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Abstract. In this paper, we study some geometric structures of a weakly cyclic Z-symmetric manifold (briefly, $[WCZS]_n$). More precisely, we prove that a conformally flat $[WCZS]_n$ satisfying certain conditions is special conformally flat and hence the manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface if the manifold is simply connected. Also we show that there exists a $[WCZS]_4$ with one parameter family of its associated 1-forms.

1. Introduction

As a natural generalization of the notion of a space of constant curvature, the notion of a symmetric manifold was introduced by Cartan [5] who obtained a classification of such a manifold. During the last six decades the notion of a symmetric manifold was weakened by many authors in several ways to a different extent such as a conformally symmetric manifold by Chaki and Gupta [8]; a recurrent manifold by Walker [18]; a conformally recurrent manifold by Adati and Miyazawa [1]; a pseudo symmetric manifold by Chaki [6]; a weakly symmetric manifold by Binh and Tamassy [3]; a pseudo Ricci symmetric manifold by Chaki [7]; a weakly Ricci symmetric manifold by Binh and Tamassy [4]; a generalized pseudo Ricci symmetric manifold by Chaki and Koley [9]. As an extending notion of a weakly Ricci symmetric manifold, Jana and Shaikh [13] introduced the notion of a weakly cyclic Ricci symmetric manifold and studied its several geometrical properties with some nontrivial examples. A Riemannian manifold (M^n, g) $(n > 2)$ is said to be weakly cyclic Ricci symmetric if its Ricci tensor r of type (0,2) satisfies the following relation:

$$\begin{aligned}&(\nabla_U r)(V,W) + (\nabla_V r)(W,U) + (\nabla_W r)(U,V)\\&= A(U)r(V,W) + B(V)r(W,U) + C(W)r(U,V),\end{aligned}$$

where A, B, C are 1-forms and ∇ denotes the covariant differentiation with respect to the metric tensor g. A (0,2) symmetric tensor Z is called a generalized

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Z-tensor if it satisfies the relation

$$Z(U,V) = r(U,V) + \phi g(U,V), \tag{1.1}$$

where ϕ is an arbitrary function.

The tensor Z was introduced in [15] and used in [16] and [17]. The classical Z-tensor is obtained with the choice $\phi = -\frac{1}{n}s$, where s is the scalar curvature. Hereafter we refer to the generalized Z-tensor simply as the Z-tensor. In particular, if the Z-tensor of a Riemannian manifold vanishes, then the manifold is Einstein. In [14], Mantica and Molinari introduced a weakly Z-symmetric manifold which is a generalization of the notion of a weakly Ricci symmetric manifold, and studied its several geometric properties. A Riemannian manifold (M^n, g) $(n > 2)$ is said to be weakly Z-symmetric if the Z-tensor fulfills the following relation:

$$(\nabla_U Z)(V,W) = A(U)Z(V,W) + B(V)Z(W,U) + C(W)Z(U,V), \tag{1.2}$$

where A, B, C are the associated 1-forms.

Also recently, De, Mantica and Suh [12] introduced the notion of a weakly cyclic Z-symmetric manifold which is a generalization of the notion of a weakly Z-symmetric manifold, and investigated its various properties.

More precisely, a Riemannian manifold $(M^n, g)(n > 2)$ is said to be weakly cyclic Z-symmetric if its Z-tensor satisfies the condition

$$\begin{aligned} &(\nabla_U Z)(V,W) + (\nabla_V Z)(W,U) + (\nabla_W Z)(U,V) \\ &= A(U)Z(V,W) + B(V)Z(W,U) + C(W)Z(U,V). \end{aligned} \tag{1.3}$$

An n-dimensional manifold of this kind is denoted by $[WCZS]_n$. It is worth to note that $[WCZS]_4$ space-times were investigated in [11]. The purpose of this paper is to study a conformally flat $[WCZS]_n$ and provide a proper example such as a $[WCZS]_4$ with one parameter family of its associated 1-forms.

2. Main results

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be quasi Einstein if there exists a nonzero 1-form T associated with a unit vector field such that its Ricci tensor satisfies the condition

$$r(X,Y) = ag(X,Y) + bT(X)T(Y),$$

where a, b are smooth functions.

At first, we can state the following Proposition which we need for the proofs of main results in this section, and for the sake of completeness, we have provided the proof of this one which was already appeared in [12].

Proposition 2.1 ([12]). *Let (M^n, g) be a $[WCZS]_n$ with $B - C \neq 0$ in* (1.3). *Then the manifold is quasi Einstein.*

Proof. Interchanging V, W in (1.3) and then subtracting the relation obtained thus from (1.3), we have

$$0 = (B - C)(V)Z(W, U) - (B - C)(W)Z(U, V).$$

Let us define $D = B - C(\neq 0)$. Then the last relation reduces to

$$D(V)Z(W, U) = D(W)Z(U, V). \tag{2.4}$$

Contracting (2.4) with respect to U and W, we get

$$D(V)(s + n\phi) = Z(D^\sharp, V), \tag{2.5}$$

where $D^\sharp$ is a vector field associated with the 1-form D, i.e., $g(D^\sharp, V) = D(V)$.

On the other hand, if we replace V by $D^\sharp$ in (2.4), we have

$$D(D^\sharp)Z(W, U) = D(W)Z(U, D^\sharp),$$

which yields from (2.5)

$$Z(U, W) = \frac{(s + n\phi)}{D(D^\sharp)} D(U)D(W) = (s + n\phi)T(U)T(W), \tag{2.6}$$

where $T(U) = \frac{1}{\sqrt{D(D^\sharp)}} D(U)$.

Using (1.1) and (2.6), we obtain

$$r(U, W) = (-\phi)g(U, W) + (s + n\phi)T(U)T(W), \tag{2.7}$$

showing that the manifold is quasi Einstein. This completes the proof. $\square$

A vector field V is said to be a conformally killing vector field on a Riemannian manifold (M^n, g) $(n > 2)$ if it satisfies the relation

$$\pounds_V g = fg,$$

where f and $\pounds$ denote a smooth function and Lie differentiation, respectively. In particular, if $f = 0$, then the vector field V is said to be a Killing vector field. We now prove the following theorem.

Theorem 2.2. *Let (M^n, g) be a compact orientable $[WCZS]_n$ with $B - C \neq 0$ in* (1.3)*. If the associated vector field $T^\sharp$ of 1-form T in* (2.6) *is conformally Killing and the scalar curvature s of (M^n, g) satisfies the condition $s \leq (1-n)\phi$, then the conformally Killing vector field $T^\sharp$ is parallel.*

Proof. It is known from [2,19] that for a vector field V in a compact orientable Riemannian manifold, the following inequality holds

$$\int_M [r(V, V) - ||\nabla V||^2 - \frac{n-2}{n}(divV)^2] dM \leq 0 \tag{2.8}$$

and equality holds if and only if V is a conformally Killing vector field. Here div denotes the divergence. If the associated vector field $T^\sharp$ of 1-form T in (2.6) is conformally Killing, then from (2.7) and (2.8) it follows that

$$\int_M [(s + (n-1)\phi) - ||\nabla T^\sharp||^2 - \frac{n-2}{n}(divT^\sharp)^2] dM = 0,$$

which yields from $s \leq (1-n)\phi$ that $\nabla T^\sharp = 0$ because the integral of non-positive terms is zero only when each term is zero. This completes the proof. $\square$

As a consequence we immediately obtain:

Corollary 2.3. *Let (M^n, g) be a compact orientable $[WCZS]_n$ with $B-C \neq 0$ in (1.3). If the associated vector field $T^\sharp$ of 1-form T in (2.6) is Killing and the scalar curvature s of (M^n, g) satisfies the condition $s \leq (1-n)\phi$, then the Killing vector field $T^\sharp$ is parallel.*

The conformal curvature tensor W of type (0,4) of a Riemannian manifold $(M^n, g)(n > 3)$ is defined by

$$\begin{aligned} W(X,Y,Z,V) = R(X,Y,Z,V) &- \frac{1}{n-2}[r(Y,Z)g(X,V) - r(X,Z)g(Y,V) \\ &+ g(Y,Z)r(X,V) - g(X,Z)r(Y,V)] \\ &+ \frac{s}{(n-1)(n-2)}[g(Y,Z)g(X,V) - g(X,Z)g(Y,V)]. \end{aligned} \tag{2.9}$$

A Riemannian manifold $(M^n, g)(n > 3)$ is said to be conformally flat if its conformal curvature tensor W vanishes. Also a Riemannian manifold $(M^n, g)(n > 3)$ is called a conformally flat $[WCZS]_n$ if the manifold is a conformally flat and weakly cyclic Z-symmetric manifold. Concerning a conformally flat $[WCZS]_n$, we have:

Theorem 2.4. *Let (M^n, g) be a conformally flat $[WCZS]_n$ with $B - C \neq 0$ in (1.3). If the Z-tensor of (M^n, g) has trace $Z \neq 0$ and $\phi =$ constant, then the associated 1-form T in (2.7) is closed.*

Proof. Differentiating (2.9) covariantly and then contracting the relation obtained thus, we have

$$\begin{aligned} (divW)(X,Y,Z) = (divR)(X,Y,Z) &- \frac{1}{n-2}[(\nabla_X r)(Y,Z) - (\nabla_Y r)(X,Z) \\ &+ \frac{1}{2}ds(X)g(Y,Z) - \frac{1}{2}ds(Y)g(X,Z)] \\ &+ \frac{1}{(n-1)(n-2)}[ds(X)g(Y,Z) - ds(Y)g(X,Z)]. \end{aligned} \tag{2.10}$$

Also it is well known that the relation

$$(divR)(X,Y,Z) = (\nabla_X r)(Y,Z) - (\nabla_Y r)(X,Z)$$

holds. By virtue of the last relation and (2.10), we get

$$\begin{aligned} (divW)(X,Y,Z) = \frac{n-3}{n-2}&[(\nabla_X r)(Y,Z) - (\nabla_Y r)(X,Z) \\ &- \frac{1}{2(n-1)}(ds(X)g(Y,Z) - ds(Y)g(X,Z))], \end{aligned} \tag{2.11}$$

which yields from conformal flatness that

$$(\nabla_X r)(Y,Z) - (\nabla_Y r)(X,Z) = \frac{1}{2(n-1)}(ds(X)g(Y,Z) - ds(Y)g(X,Z))]. \tag{2.12}$$

From (2.7) and (2.12), it follows that

$$\begin{aligned}
&- d\phi(X)g(Y,Z) + d(s+n\phi)(X)T(Y)T(Z)\\
&+ (s+n\phi)(\nabla_X T)(Y)T(Z) + (s+n\phi)T(Y)(\nabla_X T)(Z)\\
&+ d\phi(Y)g(X,Z) - d(s+n\phi)(Y)T(X)T(Z)\\
&- (s+n\phi)(\nabla_Y T)(X)T(Z) - (s+n\phi)T(X)(\nabla_Y T)(Z)\\
&= \frac{1}{2(n-1)}(ds(X)g(Y,Z) - ds(Y)g(X,Z)).
\end{aligned} \tag{2.13}$$

Considering $\phi =$ constant, we obtain

$$\begin{aligned}
&ds(X)T(Y)T(Z) + (s+n\phi)(\nabla_X T)(Y)T(Z)\\
&+ (s+n\phi)T(Y)(\nabla_X T)(Z)\\
&- ds(Y)T(X)T(Z) - (s+n\phi)(\nabla_Y T)(X)T(Z)\\
&- (s+n\phi)T(X)(\nabla_Y T)(Z)\\
&= \frac{1}{2(n-1)}(ds(X)g(Y,Z) - ds(Y)g(X,Z)).
\end{aligned} \tag{2.14}$$

Setting $Z = T^\sharp$ in (2.14), we have

$$\begin{aligned}
&ds(X)T(Y) + (s+n\phi)(\nabla_X T)(Y)\\
&- ds(Y)T(X) - (s+n\phi)(\nabla_Y T)(X)\\
&= \frac{1}{2(n-1)}(ds(X)T(Y) - ds(Y)T(X)).
\end{aligned} \tag{2.15}$$

On the other hand, setting $Y = Z = e_i$ in (2.13) and taking summation over $i = 1, \ldots, n$, we have

$$\begin{aligned}
&- d\phi(X)n + d(s+n\phi)(X) + d\phi(X) - d(s+n\phi)(T^\sharp)T(X)\\
&- (s+n\phi)(\nabla_{T^\sharp} T)(X) - (s+n\phi)T(X)(\sum_{i=1}^{n}(\nabla_{e_i} T)(e_i))\\
&= \frac{1}{2(n-1)}(ds(X)n - ds(X)).
\end{aligned} \tag{2.16}$$

Again putting $Y = Z = T^\sharp$ in (2.13), we obtain

$$\begin{aligned}
&- d\phi(X) + d(s+n\phi)(X) + d\phi(T^\sharp)T(X) - d(s+n\phi)(T^\sharp)T(X)\\
&- (s+n\phi)(\nabla_{T^\sharp} T)(X)\\
&= \frac{1}{2(n-1)}(ds(X) - ds(T^\sharp)T(X)).
\end{aligned} \tag{2.17}$$

By virtue of the common term $(s+n\phi)(\nabla_{T^\sharp}T)(X)$ in (2.16) and (2.17), we have from (2.16) and (2.17)

$$\begin{aligned}&(2-n)d\phi(X)-d\phi(T^\sharp)T(X)-(s+n\phi)T(X)(\sum_{i=1}^n(\nabla_{e_i}T)(e_i))\\ &=\frac{n-2}{2(n-1)}ds(X)+\frac{1}{2(n-1)}ds(T^\sharp)T(X).\end{aligned}\tag{2.18}$$

Putting $X=T^\sharp$ in (2.18), we get

$$(1-n)d\phi(T^\sharp)-(s+n\phi)(\sum_{i=1}^n(\nabla_{e_i}T)(e_i))=\frac{1}{2}ds(T^\sharp),$$

which leads to

$$-(s+n\phi)T(X)(\sum_{i=1}^n(\nabla_{e_i}T)(e_i))=(n-1)d\phi(T^\sharp)T(X)+\frac{1}{2}ds(T^\sharp)T(X).\tag{2.19}$$

Considering (2.18) and (2.19), we obtain

$$d\phi(X)-d\phi(T^\sharp)T(X)=\frac{-1}{2(n-1)}ds(X)+\frac{1}{2(n-1)}ds(T^\sharp)T(X).\tag{2.20}$$

It follows from $\phi=$ constant and (2.20) that

$$ds(X)=ds(T^\sharp)T(X).\tag{2.21}$$

Taking account of (2.15) and (2.21), we have

$$(s+n\phi)((\nabla_XT)(Y)-(\nabla_YT)(X))=0,$$

which yields from $s+n\phi\neq 0$ that

$$(\nabla_XT)(Y)-(\nabla_YT)(X)=0,$$

showing that T is closed. This completes the proof. □

Concerning a conformally flat $[WCZS]_n$ with constant scalar curvature, we have the following results:

Lemma 2.5. *Let (M^n,g) be a conformally flat $[WCZS]_n$ with $s=$ constant and $B-C\neq 0$ in (1.3). If the Z-tensor of (M^n,g) has trace $Z\neq 0$ and $\phi=$ constant, then for the vector field $T^\sharp$ associated with 1-form T in (2.7), the integral curve of $T^\sharp$ is geodesic.*

Proof. Taking account of (2.17) and $s=$ constant, we get

$$-d\phi(X)+nd\phi(X)+d\phi(T^\sharp)T(X)-nd\phi(T^\sharp)T(X)-(s+n\phi)(\nabla_{T^\sharp}T)(X)=0,\tag{2.22}$$

which yields from $\phi=$ constant and trace $Z\neq 0$ that

$$(\nabla_{T^\sharp}T)(X)=0$$

or equivalently

$$\nabla_{T^\sharp}T^\sharp=0,$$

showing that the integral curve of $T^\sharp$ is geodesic. This completes the proof. $\square$

As a consequence we have:

Theorem 2.6. *Let (M^n, g) be a conformally flat $[WCZS]_n$ with $s =$ constant and $B - C \neq 0$ in (1.3). If the Z-tensor of (M^n, g) has trace $Z \neq 0$ and $\phi =$ constant, then the vector field $T^\sharp$ associated with 1-form T in (2.7) is parallel.*

Proof. By virtue of trace $Z \neq 0$, $s =$ constant and $\phi =$ constant, we obtain from (2.13)

(2.23)
$$(\nabla_X T)(Y)T(Z) + T(Y)(\nabla_X T)(Z) - (\nabla_Y T)(X)T(Z) - T(X)(\nabla_Y T)(Z) = 0.$$

Putting $Y = T^\sharp$ in (2.23), we get

$$(\nabla_X T)(Z) - (\nabla_{T^\sharp} T)(X)T(Z) - T(X)(\nabla_{T^\sharp} T)(Z) = 0.$$

From Lemma 2.5, it follows that the last relation reduces to

$$(\nabla_X T)(Z) = 0$$

or equivalently

$$\nabla_X T^\sharp = 0,$$

showing that the vector field $T^\sharp$ is parallel. This completes the proof. $\square$

According to Chen and Yano [10], if a (0,2) tensor H defined by

$$H(X,Y) = -\frac{1}{n-2}r(X,Y) + \frac{s}{2(n-1)(n-2)}g(X,Y) \tag{2.24}$$

is expressible in the form

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha),$$

where $\alpha(> 0)$ and β are scalar functions, then the conformally flat manifold with the above mentioned H is said to be special conformally flat. Now we are in a position to state the following:

Theorem 2.7. *Let (M^n, g) be a conformally flat $[WCZS]_n$ with $B - C \neq 0$ in (1.3). If the manifold has a nonconstant scalar curvature s and a constant ϕ in (1.1) satisfying the condition $s < 2(1-n)\phi$, then the manifold is special conformally flat.*

Proof. By virtue of (2.7) and (2.24), we get

$$H(X,Y) = \left(\frac{s + 2(n-1)\phi}{2(n-1)(n-2)}\right)g(X,Y) - \left(\frac{s + n\phi}{n-2}\right)T(X)T(Y). \tag{2.25}$$

According to $s + 2(n-1)\phi < 0$, we can define a scalar function α (> 0) such as

$$\alpha^2 = -\left(\frac{s + 2(n-1)\phi}{(n-1)(n-2)}\right) > 0. \tag{2.26}$$

Then we have from (2.26) and $\phi =$ constant

$$2\alpha(X\alpha) = \frac{-ds(X)}{(n-1)(n-2)},$$

which yields from (2.21)

$$2\alpha(X\alpha) = \frac{-ds(T^\sharp)T(X)}{(n-1)(n-2)}. \tag{2.27}$$

Hence from (2.27) and nonconstant s, it follows that

$$T(X)T(Y) = \frac{4\alpha^2(X\alpha)(Y\alpha)(n-1)^2(n-2)^2}{(ds(T^\sharp))^2},$$

which yields from (2.26)

$$T(X)T(Y) = \frac{-4(s+2(n-1)\phi)(X\alpha)(Y\alpha)(n-1)(n-2)}{(ds(T^\sharp))^2},$$

which leads to

$$\begin{aligned} -(\frac{s+n\phi}{n-2})T(X)T(Y) &= \frac{4(s+n\phi)(s+2(n-1)\phi)(X\alpha)(Y\alpha)(n-1)}{(ds(T^\sharp))^2} \\ &= \beta(X\alpha)(Y\alpha), \end{aligned} \tag{2.28}$$

where $\beta = \frac{4(s+n\phi)(s+2(n-1)\phi)(n-1)}{(ds(T^\sharp))^2}$.

Therefore taking account of (2.25), (2.26) and (2.28), we obtain

$$H(X,Y) = -\frac{\alpha^2}{2}g(X,Y) + \beta(X\alpha)(Y\alpha),$$

showing that the manifold under consideration is special conformally flat. This completes the proof. □

Also in [10] Chen and Yano showed that every simply connected and special conformally flat manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface. Therefore by virtue of Theorem 2.7, we can state:

Corollary 2.8. *Let (M^n, g) be a simply connected and conformally flat $[WCZS]_n$ with $B - C \neq 0$ in (1.3). If the manifold has a nonconstant scalar curvature s and a constant ϕ in (1.1) satisfying the condition $s < 2(1-n)\phi$, then the manifold is isometrically immersed in an Euclidean E^{n+1} as a hypersurface.*

Now we show that there exists a $[WCZS]_4$ with one parameter family of its associated 1-forms.

Example 1 ([12])**.** Let (R^4_+, g) be a Riemannian manifold given by

$$R^4_+ = \{(x^1, x^2, x^3, x^4) \,|\, x^4 > 0\}$$

and

$$g = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

This kind of metric was appeared in [12]. In the metric described as above, the only nonvanishing components for the Christoffel symbols Γ^k_{ij}, the curvature tensors R_{ijkl} and the Ricci tensors r_{jk} are

$$\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4},$$

$$\Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = \frac{-2}{3}(x^4)^{\frac{1}{3}},$$

$$R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{\frac{2}{3}},$$

$$R_{1441} = R_{2442} = R_{3443} = \frac{-2}{9(x^4)^{\frac{2}{3}}},$$

$$r_{11} = r_{22} = r_{33} = \frac{2}{3(x^4)^{\frac{2}{3}}},$$

$$r_{44} = \frac{-2}{3(x^4)^2}.$$

Now we define a scalar function ϕ in (1.1) as

$$\phi = \frac{1}{(x^4)^2}.$$

Therefore the nonvanishing components of the Z-tensor Z_{ij} and their covariant derivatives $Z_{ij;k}$ are

$$Z_{11} = Z_{22} = Z_{33} = \frac{5}{3(x^4)^{\frac{2}{3}}}, Z_{44} = \frac{1}{3(x^4)^2}, \tag{2.29}$$

$$Z_{11;4} = Z_{22;4} = Z_{33;4} = \frac{-10}{9(x^4)^{\frac{5}{3}}}, Z_{44;4} = \frac{-2}{3(x^4)^3}. \tag{2.30}$$

It is easy to see that the scalar curvature s of (R^4_+, g) is $\frac{4}{3(x^4)^2}$.

Let us define the associated 1-forms A, B, C of (1.3) on (R^4_+, g) as follows: $A_i = \frac{-2}{3(x^4)}$ for $i = 4$ and 0 otherwise; $B_i = \frac{-t}{3(x^4)}$ for $i = 4$ and 0 otherwise; $C_i = \frac{-16+t}{3(x^4)}$ for $i = 4$ and 0 otherwise. Here $t \in R$.

In the manifold (R^4_+, g), (1.3) reduces to the following equations:

$$Z_{11;4} + Z_{14;1} + Z_{41;1} = A_4 Z_{11} + B_1 Z_{14} + C_1 Z_{41}, \tag{2.31}$$

$$Z_{22;4} + Z_{24;2} + Z_{42;2} = A_4 Z_{22} + B_2 Z_{24} + C_2 Z_{42}, \tag{2.32}$$

$$Z_{33;4} + Z_{34;3} + Z_{43;3} = A_4 Z_{33} + B_3 Z_{34} + C_3 Z_{43}, \tag{2.33}$$

$$Z_{44;4} + Z_{44;4} + Z_{44;4} = A_4 Z_{44} + B_4 Z_{44} + C_4 Z_{44} \tag{2.34}$$

because for the other cases, the components of each term of (1.3) vanishes identically. From (2.29), (2.30) and the definition of A, B, C, it follows that the last equations hold. For instance, in case of (2.31),

$$Z_{11;4} + Z_{14;1} + Z_{41;1} = \frac{-10}{9(x^4)^{\frac{5}{3}}}$$

and

$$A_4 Z_{11} + B_1 Z_{14} + C_1 Z_{41} = (\frac{-2}{3x^4})(\frac{5}{3(x^4)^{\frac{2}{3}}}),$$

showing that (2.31) holds. By similar arguments, it can be shown that (2.32) and (2.33) hold. In case of (2.34),

$$Z_{44;4} + Z_{44;4} + Z_{44;4} = 3(\frac{-2}{3(x^4)^3})$$

and

$$A_4 Z_{44} + B_4 Z_{44} + C_4 Z_{44} = (\frac{-2}{3x^4})(\frac{1}{3(x^4)^2}) + (\frac{-t}{3x^4})(\frac{1}{3(x^4)^2}) + (\frac{-16+t}{3x^4})(\frac{1}{3(x^4)^2}),$$

showing that (2.34) holds too. Hence the Riemannian manifold (R^4_+, g) with ϕ,A, B, C mentioned in the above is a $[WCZS]_4$.

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