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SECOND MAIN THEOREM AND UNIQUENESS PROBLEM OF ZERO-ORDER MEROMORPHIC MAPPINGS FOR HYPERPLANES IN SUBGENERAL POSITION

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Abstract. In this paper, we show the Second Main Theorems for zero-order meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes in subgeneral position without truncated multiplicity by considering the p-Casorati determinant with $p \in \mathbb{C}^m$ instead of its Wronskian determinant. As an application, we give some unicity theorems for meromorphic mapping under the growth condition "order=0". The results obtained include p-shift analogues of the Second Main Theorem of Nevanlinna theory and Picard's theorem.

1. Introduction

In 2006, R. Halburd-R. Korhonen [7] considered the Second Main Theorem for complex difference operator with finite order in complex plane. Later, in [8] and [18], difference analogues of the Second Main Theorem for holomorphic curves in $\mathbb{P}^n(\mathbb{C})$ were obtained independently, and in [2] and [12], difference analogues of the Second Main Theorem for meromorphic functions on $\mathbb{C}^m$ were obtained. In particular, Nevanlinna theory for the p-difference operator can be found in [1, 11, 15–17, 19].

Recently, T. B. Cao-R. Korhonen [3] obtained a new natural difference analogue of H. Cartan's Theorem for meromorphic mapping $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$. In which, the counting function $N(r, \nu^0_{W(f)})$ of the Wronskian determinant of f is replaced by the counting function $N(r, \nu^0_{C^c(f)})$ of the Casorati determinant of f (it was called the finite difference Wronskian determinant in [18]) and in addition, the hyper-order of f is strictly less than one.

Our first aim in this paper is to prove a new natural p-difference analogue Second Main Theorem for zero-order meromorphic mapping by considering p-Casorati determinant. For our purpose, we now recall some notations.

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Let $p \in \mathbb{C}^m$, denote by $\mathcal{M}_m$ the set of all meromorphic functions on $\mathbb{C}^m$, denote by ϕ_p the set of all meromorphic functions of $\mathcal{M}_m$ satisfying $f(z) = f(pz)$ and denote by ϕ_p^0 the set of all meromorphic functions of ϕ_p having their zero-orders. Obviously, $\phi_p^0 \subset \phi_p \subset \mathcal{M}_m$.

Definition 1. Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_n)$. Then the map f is said to be linearly nondegenerate over field ϕ_p^0 if the entire functions $f_0, \ldots, f_n$ are linearly independent over field ϕ_p^0.

For $c = (c_1, \ldots, c_m)$ and $p = (p_1, \ldots, p_m)$ with $p_i \neq 0$ $(1 \leq i \leq m)$ and $z = (z_1, \ldots, z_m)$, we write $c + z = (c_1 + z_1, \ldots, c_m + z_m)$ and $pz = (p_1 z_1, \ldots, p_m z_m)$. Denote

$$f(z) \equiv f := \bar{f}^{[0]}, f(z+c) \equiv \bar{f} := \bar{f}^{[1]}, f(z+2c) \equiv \bar{\bar{f}} := \bar{f}^{[2]}, \ldots, f(z+kc) \equiv \bar{f}^{[k]}$$

and

$$f(z) \equiv f := \hat{f}^{[0]}, f(pz) \equiv \hat{f} := \hat{f}^{[1]}, f(p^2 z) \equiv \hat{\hat{f}} := \hat{f}^{[2]}, \ldots, f(p^k z) \equiv \hat{f}^{[k]}.$$

Let

$$D^{(j)} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1(j)} \cdots \left(\frac{\partial}{\partial z_m}\right)^{\alpha_m(j)}$$

be a partial differentiation operator of order at most $j = \sum_{k=1}^m \alpha_k(j)$. Similarly as the Wronskian determinant

$$W(f) = W(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ D^{(1)} f_0 & D^{(1)} f_1 & \cdots & D^{(1)} f_n \\ \vdots & \vdots & \vdots & \vdots \\ D^{(n)} f_0 & D^{(n)} f_1 & \cdots & D^{(n)} f_n \end{vmatrix},$$

the Casorati determinant is defined by

$$C^c(f) = C^c(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \bar{f}_0 & \bar{f}_1 & \cdots & \bar{f}_n \\ \vdots & \vdots & \vdots & \vdots \\ \bar{f}_0^{[n]} & \bar{f}_1^{[n]} & \cdots & \bar{f}_n^{[n]} \end{vmatrix}$$

and the p-Casorati determinant is defined by

$$C_p(f) = C_p(f_0, \ldots, f_n) = \begin{vmatrix} f_0 & f_1 & \cdots & f_n \\ \hat{f}_0 & \hat{f}_1 & \cdots & \hat{f}_n \\ \vdots & \vdots & \vdots & \vdots \\ \hat{f}_0^{[n]} & \hat{f}_1^{[n]} & \cdots & \hat{f}_n^{[n]} \end{vmatrix}.$$

Definition 2. Let $\{H_j\}_{j=1}^q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$. Let $N \geq n$ and $q \geq N+1$. The family $\{H_j\}_{j=1}^q$ is said to be in N-*subgeneral position* in $\mathbb{P}^n(\mathbb{C})$

if for every subset $R \subset \{1, \ldots, q\}$ with the cardinality $|R| = N + 1$, then

$$\bigcap_{j \in R} H_j = \emptyset.$$

If they are in n-subgeneral position, we simply say that they are in general position.

Consider f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(C)$ with reduced representation $f = (f_0 : \cdots : f_n)$ and a hyperplane $H : a_0\omega_0 + \cdots + a_n\omega_n = 0$. We define

$$(f, H) = H(f) := a_0 f_0 + \cdots + a_n f_n,$$

which is a holomorphic function on $\mathbb{C}^m$.

Using above notations, we have the p-difference analogue of H. Cartan's Theorem [4] as follows.

Theorem 1. *Let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all $j \in \{1, \ldots, m\}$ and let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping over the field ϕ_p^0. Let H_j $(1 \leq j \leq q)$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in N-subgeneral position. Assume that f has the zero-order. Then we have*

$$(q - 2N + n - 1)\, T(r, f) \leq \sum_{j=1}^{q} N(r, \nu^0_{H_j(f)}) - \frac{N}{n} N(r, \nu^0_{C_p(f)}) + o\,(T(r, f))$$

for all r on a set of logarithmic density 1.

Here, by ν^0_φ we denote the zero-divisor of holomorphic function φ from $\mathbb{C}^m$ into $\mathbb{C}$.

Definition 3. Let $k \in \mathbb{N}$, $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all $j \in \{1, \ldots, m\}$ and $a \in \mathbb{C}$. An a-point z_0 of meromorphic function $h(z)$ is said that to be k-successive with separated p respect to the rescaling $\tau_p(z) = pz$, if the k functions $h(p^l z)$, $(l = 1, \ldots, k)$ take the value a at $z = z_0$ with multiplicity not less than that of $h(z)$ there. All the other a-points of $h(z)$ are called k-aperiodic of pace p respect to the rescaling $\tau_p(z) = pz$.

Consider H be a hyperplane. By $\hat{N}^{[k,p]}(r, H(f))$, we denote the counting function of k-aperiodic zeros of the function $H(f)$ of pace p respect to the rescaling $\tau_p(z) = pz$.

Note that $\hat{N}^{[k,p]}(r, H(f)) \equiv 0$ when all zeros of $H(f)$ with taking their multiplicities into account are located periodically with period p respect to the rescaling $\tau_p(z) = pz$. This is also the case when the hyperplane H is forward invariant by f with respect to the rescaling $\tau_p(z) = pz$, i.e., $\tau_p(f^{-1}(H)) \subset f^{-1}(H)$ and $f^{-1}(H)$ is considered to be multi-sets in which each point is repeated according to its multiplicity. Then we have the result as follows.

Theorem 2. *Let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all $j \in \{1, \ldots, m\}$ and let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate meromorphic mapping* *over the field ϕ_p^0. Let H_j $(1 \leq j \leq q)$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in N-subgeneral position. Assume that f has the zero-order. Then we have*

$$(q-2N+n-1)\,T(r,f) \leq \sum_{j=1}^{q} \hat{N}^{[n,p]}(r, H_j(f)) + o\,(T(r,f))$$

for all r on a set of logarithmic density 1.

The uniqueness problem for meromorphic mappings was first investigated by R. Nevanlinna. In 1975, H. Fujimoto [5] generalized Nevanlinnas five-value theorem to the case of higher dimension by showing that if two linearly non-degenerate meromorphic mappings $f, g : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ have the same inverse images counted with multiplicities for $q \geq 3n+2$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$, then $f \equiv g$.

By considering the uniqueness problem for holomorphic curves $f(z)$ and $f(z+c)$ also for holomorphic curves $f(z)$ and $f(pz)$ intersecting hyperplanes in general position, R. Halburd, R. Korhonen, K. Tohge [8, Theorem 1.1 and Theorem 6.1] obtained a difference analogue of Picard's theorem. Recently, T. B. Cao, R. Korhonen [3] generalized the this result [8, Theorem 1.1] for the case of meromorphic mappings $f(z)$ and $f(z+c)$ intersecting hyperplanes in subgeneral position.

Our final aim in this paper is to extend the result in [8, Theorem 6.1] to meromorphic mappings $f(z)$ and $f(pz)$ of $\mathbb{C}^m$ into $\mathbb{P}^n(C)$ intersecting hyperplanes in N-subgeneral position. Our result is a difference analogue of Picard's theorem. Namely, we will prove the following theorem.

Theorem 3. *Let f be a zero-order meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0, 1$ for all $j \in \{1, \ldots, m\}$. Assume that f is forward invariant over q hyperplanes in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$ respect to the rescaling $\tau_p(z) = pz$. Then the image of f is contained in a projective linear subspace over ϕ_p^0 of dimension $\leq \left[\frac{N}{q-N}\right]$. Special, if $q \geq 2N+1$, then $f(z) = f(pz)$.*

Note that when $|p_i| \neq 1$ for all $i \in \{1, \ldots, m\}$, then $f(z) = f(pz)$ implies that f must be a constant mapping. Immediately, we have the following corollary.

Corollary 4. *Let f be a zero-order meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$ and let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ satisfying $|p_j| \neq 0, 1$ for all $j \in \{1, \ldots, m\}$. Assume that f is forward invariant over q hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ respect to the rescaling $\tau_p(z) = pz$. If $q \geq 2n+1$, then f is constant.*

2. Preliminaries and auxiliary lemmas

2.1. We set $||z|| = \left(|z_1|^2 + \cdots + |z_n|^2\right)^{1/2}$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and define

$$B_m(r) := \{z \in \mathbb{C}^m : ||z|| < r\}, \quad S_m(r) := \{z \in \mathbb{C}^m : ||z|| = r\}\ (0 < r < \infty).$$

Define

$$\sigma_m(z) := \big(dd^c\|z\|^2\big)^{m-1} \qquad \text{and}$$

$$\eta_m(z) := d^c\log\|z\|^2 \wedge \big(dd^c\log\|z\|^2\big)^{m-1} \text{on} \quad \mathbb{C}^m \setminus \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbb{C}^m$. For a set $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\mathcal{D}^\alpha F = \frac{\partial^{|\alpha|}F}{\partial^{\alpha_1}z_1\cdots\partial^{\alpha_m}z_m}$. We define the map $\nu_F : \Omega \to \mathbb{Z}$ by

$$\nu_F(z) := \max\,\{n : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < n\}\ (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbb{C}^m$ a map $\nu : \Omega \to \mathbb{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighbourhood $U \subset \Omega$ of a such that $\nu(z) = \nu_F(z) - \nu_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m-2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m-2$. For a divisor ν on Ω we set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or empty.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbb{C}^n$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighbourhood $U \subset \Omega$ such that $\varphi = \frac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m-2$, and we define the divisors ν^0_φ, ν^∞_φ by $\nu^0_\varphi := \nu_F$, $\nu^\infty_\varphi := \nu_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor ν on $\mathbb{C}^m$, we define the counting functions of ν by

$$n(t) = \begin{cases} \int\limits_{|\nu|\,\cap B(t)} \nu(z)\sigma_{m-1} & \text{if } m \geq 2 \\ \sum\limits_{|z|\leq t} \nu(z) & \text{if } m = 1 \end{cases}$$

and

$$N(r,\nu) = \int_1^r \frac{n(t)}{t^{2m-1}}dt \quad (1 < r < \infty).$$

Let $\varphi : \mathbb{C}^m \longrightarrow \mathbb{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi).$$

2.4. Let $f : \mathbb{C}^m \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = \big(f_0(z) : \cdots : f_n(z)\big)$ outside the analytic set $I(f) = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(\sum_{j=0}^n |f_j|^2\big)^{1/2}$. The characteristic function of f is defined by

$$\begin{aligned} T(r,f) &= \int_{r_0}^r \frac{dt}{2^{m-1}} \int\limits_{B_m(r)} dd^c\log\|f\|^2 \wedge \sigma_m(z) \\ &= \int\limits_{S_m(r)} \log\|f\|\eta_m - \int\limits_{S_m(r_0)} \log\|f\|\eta_m(z). \end{aligned}$$

Note that $T(r, f)$ is independent of the choice of the representation of f. The order and hyperorder of f are respectively defined by

$$\sigma(f) := \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r} \text{ and } \zeta(f) := \limsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r},$$

where $\log^+ x := \max\{\log x, 0\}$ for any $x > 0$.

2.5. Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^n(C)$ with reduced representation $f = (f_0 : \cdots : f_n)$ and a hyperplane $H : a_0\omega_0 + \cdots + a_n\omega_n = 0$ satisfies

$$(f, H) = a_0 f_0 + \cdots + a_n f_n \not\equiv 0.$$

The proximity function is defined as

$$m_{f,H}(r) := \int_{S_m(r)} \log^+ \frac{||f|| \cdot ||H||}{|(f,H)|} \eta_m(z) + \int_{S_m(1)} \log^+ \frac{||f|| \cdot ||H||}{|(f,H)|} \eta_m(z).$$

We have the First Main Theorem of Nevanlinna theory

$$m_{f,H}(r) + N(r, \nu^0_{H(f)}) = T(r, f) + O(1),$$

where $O(1)$ is a constant independent of r.

2.6. Let φ be a nonzero meromorphic function on $\mathbb{C}^m$, which is occationally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S_m(r)} \log^+ |\varphi| \eta_m.$$

Lemma 5 ([1, Lemmas 5.1, 5.2, and 5.3])**.** *Let f be a non-constant zero-order meromorphic function of $\mathbb{C}$ into $\mathbb{C}$ and let $p \in \mathbb{C} \setminus \{0\}$. Then*

$$m\left(r, \frac{f(pz)}{f(z)}\right) < \frac{4D_1 + 2D_2}{2^n} T(r, f(z))$$

on a set of logarithmic density 1 for all $n \in \mathbb{N}$, where D_1, D_2 are positive constants.

Lemma 6 ([9, Lemma 4])**.** *If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that order*

$$\sigma(T) = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = 0,$$

then the set

$$E := \{r \in \mathbb{R}^+ : T(C_1 r) \geq C_2 T(r)\}$$

has logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

Lemma 7 ([1, Lemma 5.4])**.** *Let $T : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function and $U : \mathbb{R}^+ \to \mathbb{R}^+$. If there exits a decreasing sequence $\{c_n\}_{n\in\mathbb{N}}$ such that $c_n \to 0$ as $n \to \infty$ and for all $n \in \mathbb{N}$, the set*

$$F_n = \{r \geq 1 : U(r) < c_n T(r)\}$$

has logarithmic density 1*, then* $U(r) = o(T(r))$ *on a set of logarithmic density* 1*.*

Lemma 8. *Let* T *be a function as in Lemma 6 and let* $p \in \mathbb{R}^+$*. Then we have*

$$T(pr) = T(r) + o(T(r))$$

on a set of logarithmic density 1*.*

Proof. **Case 1:** $p \leq 1$**.** Since $T(r)$ is an increasing function, we have $T(pr) \leq T(r)$ for all $r > 0$. Obviously, the conclusion holds.
Case 2: $p > 1$**.** By Lemma 6, for each $n \in \mathbb{N}$, we have

$$E_n := \left\{ r \geq 1 : T(pr) < \left(1 + \frac{1}{n}\right) T(r) \right\}$$

has logarithmic density 1. Put $U(r) = T(pr) - T(r)$, we deduce that

$$0 < U(r) < \frac{1}{n} T(r)$$

on a set of logarithmic density 1. It follows from Lemma 7 that $U(r) = o(T(r))$ on a set of logarithmic density 1. Therefore, we get

$$T(pr) = T(r) + o(T(r)) \tag{2.1}$$

on a set of logarithmic density 1. Therefore, the proof of the Lemma 8 is finished. □

For each $\omega \in \overline{B}_{m-1}(r)$, we define a function $p_r(\omega) = \sqrt{r^2 - |\omega|^2}$. We need the following lemma from W. Stoll.

Lemma 9 ([10])**.** *Let* $r > 0$ *and let* h *be a function on* $S_m(r)$ *such that* $h\eta_m$ *is integrable over* $S_m(r)$*. Then*

$$\int\limits_{S_m(r)} h(z)\eta_m(z) = \frac{1}{r^{2m-2}} \int\limits_{\overline{B}_{m-1}(r)} \sigma_{m-1}(\omega) \int\limits_{S_1(P_r(\omega))} h(\omega, \zeta)\eta_1(\zeta).$$

Consider a non-constant meromorphic function f on $\mathbb{C}^m$, take $\omega \in \mathbb{C}^{m-1}$ and define $f_\omega(z) := f(\omega, z)$ on $\mathbb{C}$. We will prove the following lemma.

Lemma 10. *Let* f *be a meromorphic function on* $\mathbb{C}^m$ *of zero-order such that* $f(0) \neq 0, \infty$ *and let* $\tilde{p}_j := (1, \ldots, p_j, \ldots, 1)$ *with* $p_j \neq 0$*. Then*

$$m\left(r, \frac{f(\tilde{p}_j z)}{f(z)}\right) = \int\limits_{S_m(r)} \log^+ \left| \frac{f(\tilde{p}_j z)}{f(z)} \right| \eta_m(z) = o(T(r, f(z)))$$

on a set of logarithmic density 1*.*

Proof. By applying Lemma 9 for $h(z) = \log^+ \left| \frac{f(\tilde{p}_j z)}{f(z)} \right|$, we have

$$m\left(r, \frac{f(\tilde{p}_j z)}{f(z)}\right) = \int\limits_{S_m(r)} \log^+ \left| \frac{f(\tilde{p}_j z)}{f(z)} \right| \eta_m(z)$$

$$= \frac{1}{r^{2m-2}} \int\limits_{\overline{B}_{m-1}(r)} \sigma_{m-1}(\omega) \int\limits_{S_1(P_r(\omega))} \log^+ \left| \frac{f_\omega(p_j z_j)}{f_\omega(z_j)} \right| \eta_1(\zeta)$$

$$= \frac{1}{r^{2m-2}} \int\limits_{\overline{B}_{m-1}(r)} m\left(P_r(\omega), \frac{f_\omega(p_j z_j)}{f_\omega(z_j)} \right) \sigma_{m-1}(\omega).$$

By Lemma 5, there exist two positive constants D_1 and D_2 which are independent of $P_r(\omega)$ such that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} & m\left(r, \frac{f(\tilde{p}_j z)}{f(z)}\right) \\ & < \frac{1}{r^{2m-2}} \int\limits_{\overline{B}_{m-1}(r)} \frac{4D_1 + 2D_2}{2^n} T\left(P_r(\omega), f_\omega(z_j)\right) \sigma_{m-1}(\omega) \\ & = \frac{4D_1 + 2D_2}{2^n} \cdot \frac{1}{r^{2m-2}} \int\limits_{\overline{B}_{m-1}(r)} \sigma_{m-1}(\omega) \int\limits_{S_1(P_r(\omega))} \log \|f_\omega(z_j)\| \, \eta_1(z_j) + O(1) \\ & = \frac{4D_1 + 2D_2}{2^n} \int\limits_{S_m(r)} \log \|f(\omega, z_j)\| \, \eta_m(z) + O(1) \\ & = \frac{4D_1 + 2D_2}{2^n} T(r, f(z)) + O(1) \end{aligned}$$

on a set of logarithmic density 1 for all $n \in \mathbb{N}$. By applying the Lemma 7, we get

$$m\left(r, \frac{f(\tilde{p}_j z)}{f(z)}\right) = o(T(r, f(z)))$$

on a set of logarithmic density 1. We finish the proof of Lemma 10. □

The lemma on the Logarithmic Derivative [4–6, 14] plays an important role in Nevanlinna theory. Here, it is replaced by the following lemma.

Lemma 11. *Let f be a non-constant zero-order meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}$ and $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all j. Then*

$$m\left(r, \frac{f(pz)}{f(z)}\right) = o(T(r, f(z)))$$

on a set of logarithmic density 1.

Proof. Since f is a meromorphic function on $\mathbb{C}^m$ of zero-order, according to Lemma 10, it follows that

$$m\left(r, \frac{f(pz)}{f(z)}\right) = \int\limits_{S_m(r)} \log^+ \left| \frac{f(pz)}{f(z)} \right| \eta_m(z)$$

$$= \int\limits_{S_m(r)} \log^+ \prod_{k=1}^{n} \left| \frac{f(\prod\limits_{j=0}^{k} \tilde{p}_j z)}{f(\prod\limits_{j=0}^{k-1} \tilde{p}_j z)} \right| \eta_m(z)$$

$$\leq \sum_{k=1}^{n} \int\limits_{S_m(r)} \log^+ \left| \frac{f(\prod\limits_{j=0}^{k} \tilde{p}_j z)}{f(\prod\limits_{j=0}^{k-1} \tilde{p}_j z)} \right| \eta_m(z) = o(T(r,f))$$

on a set of logarithmic density 1. The proof of Lemma 11 is finished. □

Lemma 12. *Let f be a meromorphic function on $\mathbb{C}^m$ of zero-order such that $f(0) \neq 0, \infty$ and let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all j. Then we have*

$$T(r, f(pz)) = T(r, f(z)) + o(T(r, f(z)))$$

on a set of logarithmic density 1.

Proof. By the First Main Theorem, we have

$$T\left(r, \frac{f(pz)}{f(z)}\right) = m\left(r, \frac{f(pz)}{f(z)}\right) + N\left(r, \frac{f(pz)}{f(z)}\right) + O(1).$$

Therefore, by Lemma 11, we get

$$T(r, f(pz)) - T(r, f(z)) = N(r, f(pz)) - N(r, f(z)) + o(T(r, f(z))) \tag{2.2}$$

on a set of logarithmic density 1. Also by the First Main Theorem, we deduce that

$$\overline{\lim_{r\to\infty}} \frac{\log(N(r,f))}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r} = \sigma(f) = 0.$$

This, by Lemma 8, we have

$$N(|p|r, f) = N(r, f) + o(N(r, f) \leq N(r, f) + o(T(r, f)) \tag{2.3}$$

on a set of logarithmic density 1. Together (2.2) with (2.3), we get

$$T(r, f(pz)) \leq T(r, f(z)) + o(T(r, f(z)))$$

on a set of logarithmic density 1. We have the assertion of Lemma 12. □

The similar results to Lemmas 10, 11, and 12 can be found in [1,11,16,17,19].

It is known that holomorphic functions $f_0, \ldots, f_n$ on $\mathbb{C}^m$ are linearly dependent over $\mathbb{C}$ if and only if their Wronskian determinants $W(f_0, \ldots, f_n)$ vanish identically [6, 13, 14]. Similarly, holomorphic functions $f_0, \ldots, f_n$ on $\mathbb{C}^m$ are linearly dependent over $\mathcal{P}_c^\lambda$ if and only if their Casorati determinants $C^c(f_0, \ldots, f_n)$ vanish identically [3], where $\mathcal{P}_c^\lambda$ is the field of c-periodic meromorphic functions having hyper-order of λ.

Here, we introduce a similar result for the case of p-Casorati determinant by the same method as in [8]. Namely, we have the following.

Lemma 13. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with reduce presentation $f = (f_0 : \cdots : f_n)$ and let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0$ for all j. Assume that $\sigma(f) = 0$. Then p-Casorati determinant $C_p(f_0, \ldots, f_n) \equiv 0$ if and only if the functions $f_0, \ldots, f_n$ are linear dependent over the field ϕ_p^0.*

Proof. Suppose first that $f_0, \ldots, f_n$ are linear dependent over the field ϕ_p^0. Then there exist $\varphi_0, \ldots, \varphi_n \in \phi_p^0$ such that $\varphi_0 f_0 + \cdots + \varphi_n f_n = 0$ and so

$$\left\{ \begin{array}{l} \varphi_0 f_0 + \cdots + \varphi_n f_n = 0 \\ \varphi_0 \hat{f}_0 + \cdots + \varphi_n \hat{f}_n = 0 \\ \vdots \\ \varphi_0 \hat{f}_0^{[n]} + \cdots + \varphi_n \hat{f}_n^{[n]} = 0. \end{array} \right. \tag{2.4}$$

Since (2.4) has a nontrivial solution, we get p-Casorati determinant

$$C_p(f_0, \ldots, f_n) \equiv 0.$$

We apply induction on n to prove the converse assertion.

In the case when $n = 1$, suppose that $C_p(f_0, f_1) \equiv 0$. We consider the system of equations

$$\left\{ \begin{array}{l} \varphi_0 f_0 + \varphi_1 f_1 = 0 \\ \varphi_0 \hat{f}_0 + \varphi_1 \hat{f}_1 = 0. \end{array} \right. \tag{2.5}$$

Since $C_p(f_0, f_1) \equiv 0$, it is easy to see that $\varphi_0 = \frac{f_1}{f_0}$, $\varphi_1 = -1$ is a solution of (2.5). Moreover, by assumption $\sigma(f) = 0$, we have $\sigma(\tilde{f}) = 0$ where $\tilde{f} := (f_0 : f_1)$. Then the order of φ_0 satisfies $\sigma(\varphi_0) = \sigma\left(\frac{f_1}{f_0}\right) \leq \sigma(\tilde{f}) \leq \sigma(f) = 0$. Obviously, $\varphi_1 = -1 \in \phi_p^0$ and $\varphi_0 = \frac{f_1}{f_0} = \frac{\hat{f}_1}{\hat{f}_0}$. Therefore, we also have $\varphi_0 \in \phi_p^0$. This implies that f_0, f_1 are linearly dependent over ϕ_p^0.

Suppose now that $C_p(f_0, \ldots, f_j) \equiv 0$ implies that $f_0, \ldots, f_j$ are linearly dependent over ϕ_p^0 for all $j \in \{1, \ldots, k-1\}$, where $k \leq n$ and assume that $C_p(f_0, \ldots, f_k) \equiv 0$. Then the linear system

$$\left\{ \begin{array}{l} \varphi_0 f_0 + \cdots + \varphi_{k-1} f_{k-1} = f_k \\ \varphi_0 \hat{f}_0 + \cdots + \varphi_{k-1} \hat{f}_{k-1} = \hat{f}_k \\ \vdots \\ \varphi_0 \hat{f}_0^{[k-1]} + \cdots + \varphi_{k-1} \hat{f}_{k-1}^{[k-1]} = \hat{f}_k^{[k-1]} \\ \varphi_0 \hat{f}_0^{[k]} + \cdots + \varphi_{k-1} \hat{f}_{k-1}^{[k]} = \hat{f}_k^{[k]}, \end{array} \right. \tag{2.6}$$

where we have made the choice $\varphi_k = -1$. If $C_p(f_0, \ldots, f_{k-1}) \equiv 0$, then $f_0, \ldots, f_{k-1}$ are linearly dependent over ϕ_p^0 by the induction assumption. Thus also $f_0, \ldots, f_{k-1}, f_k$ are linearly dependent over ϕ_p^0. If $C_p(f_0, \ldots, f_{k-1}) \not\equiv 0$, then by Cramer's rule for each $i = 0, \ldots, k-1$, we have

$$\varphi_i = \frac{C_p(f_0, \ldots, f_{i-1}, f_k, f_{i+1}, \ldots, f_{k-1})}{C_p(f_0, \ldots, f_{k-1})},$$

where f_k occurs in the i^{th} entry of p-Casorati determinent in the numerator instead of f_i. By writing

$$\varphi_i = \frac{f_i \hat{f}_i \cdots \hat{f}_i^{[k-1]} \cdot C_p\left(\frac{f_0}{f_i}, \ldots, \frac{f_{i-1}}{f_i}, \frac{f_k}{f_i}, \frac{f_{i+1}}{f_i}, \ldots, \frac{f_{k-1}}{f_i}\right)}{f_k \hat{f}_k \cdots \hat{f}_k^{[k-1]} \cdot C_p\left(\frac{f_0}{f_k}, \ldots, \frac{f_{k-1}}{f_k}\right)},$$

it can be seen that

$$T(r, \varphi_i) = O\left(\sum_{j=0}^{k} \sum_{l=0}^{k-1} \left(T\left(r, \frac{\hat{f}_j^{[l]}}{\hat{f}_i^{[l]}}\right) + T\left(r, \frac{\hat{f}_j^{[l]}}{\hat{f}_k^{[l]}}\right)\right)\right)$$

for all $i = 0, \ldots, k-1$. Now by Lemma 12, we have $T\left(r, \hat{f}\right) = T(r, f) + o(T(r, f))$ for all meromorphic mappings $f(z)$ with $\sigma(f) = 0$, and it follows that $\sigma(\varphi_i) = 0$ for all $i = 0, \ldots, k-1$.

We still need to prove that φ_i satisfies $\varphi_i(pz) = \varphi_i(z)$ for all $i = 0, \ldots, k-1$. By applying the operator $\hat{\Delta}_p$ to k equations in the system (2.6), where $\hat{\Delta}_p f = \hat{f} - f$, it follows that

(2.7)
$$\begin{cases} \left(\varphi_0 \hat{\Delta}_p f_0 + \cdots + \varphi_{k-1} \hat{\Delta}_p f_{k-1}\right) + \left(\hat{f}_0 \hat{\Delta}_p \varphi_0 + \cdots + \hat{f}_{k-1} \hat{\Delta}_p \varphi_{k-1}\right) = \hat{\Delta}_p f_k \\ \left(\varphi_0 \hat{\Delta}_p \hat{f}_0 + \cdots + \varphi_{k-1} \hat{\Delta}_p \hat{f}_{k-1}\right) + \left(\hat{\hat{f}}_0 \hat{\Delta}_p \varphi_0 + \cdots + \hat{\hat{f}}_{k-1} \hat{\Delta}_p \varphi_{k-1}\right) = \hat{\Delta}_p \hat{f}_k \\ \vdots \\ \left(\varphi_0 \hat{\Delta}_p \hat{f}_0^{[k-1]} + \cdots + \varphi_{k-1} \hat{\Delta}_p \hat{f}_{k-1}^{[k-1]}\right) + \left(\hat{f}_0^{[k]} \hat{\Delta}_p \varphi_0 + \cdots + \hat{f}_{k-1}^{[k]} \hat{\Delta}_p \varphi_{k-1}\right) = \hat{\Delta}_p \hat{f}_k^{[k-1]}. \end{cases}$$

On the other hand also from (2.6), we have

$$(2.8) \qquad \begin{cases} \varphi_0 \hat{\Delta}_p f_0 + \cdots + \varphi_{k-1} \hat{\Delta}_p f_{k-1} = \hat{\Delta}_p f_k \\ \varphi_0 \hat{\Delta}_p \hat{f}_0 + \cdots + \varphi_{k-1} \hat{\Delta}_p \hat{f}_{k-1} = \hat{\Delta}_p \hat{f}_k \\ \vdots \\ \varphi_0 \hat{\Delta}_p \hat{f}_0^{[k-1]} + \cdots + \varphi_{k-1} \hat{\Delta}_p \hat{f}_{k-1}^{[k-1]} = \hat{\Delta}_p \hat{f}_k^{[k-1]}. \end{cases}$$

Together (2.7) with (2.8), we get

$$\begin{cases} \hat{f}_0 \hat{\Delta}_p \varphi_0 + \cdots + \hat{f}_{k-1} \hat{\Delta}_p \varphi_{k-1} = 0 \\ \hat{\hat{f}}_0 \hat{\Delta}_p \varphi_0 + \cdots + \hat{\hat{f}}_{k-1} \hat{\Delta}_p \varphi_{k-1} = 0 \\ \vdots \\ \hat{f}_0^{[k]} \hat{\Delta}_p \varphi_0 + \cdots + \hat{f}_{k-1}^{[k]} \hat{\Delta}_p \varphi_{k-1} = 0, \end{cases}$$

which has only trivial solution. Therefore, $\hat{\Delta}_p \varphi_0 \equiv \cdots \equiv \hat{\Delta}_p \varphi_{k-1} \equiv 0$. It follows that $\varphi_i(pz) = \varphi_i(z)$ for all $i = 0, \ldots, k-1$. We finish the proof of Lemma 13. $\square$

3. The proof of Theorem 1

We recall the lemma due to Nochka (see [5,6,13,14]) as follows.

Lemma 14. *Let $H_1, \dots, H_q$ $(q > 2N - n + 1)$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in N-subgeneral position. Then there exist a function $\omega : \{1, \dots, q\} \to (0, 1]$ called a Nochka weight and a real number $\tilde{\omega} \geq 1$ called a Nochka constant satisfying the following conditions*:

(i) *If $j \in \{1, \dots, q\}$, then $0 < \omega(j)\tilde{\omega} \leq 1$.*

(ii) $q - 2N + n - 1 = \tilde{\omega}(\sum_{j=1}^{q} \omega(j) - n - 1)$.

(iii) *For $R \subset \{1, \dots, q\}$ with $|R| = N + 1$, then $\sum_{i \in R} \omega(i) \leq n + 1$.*

(iv) $\frac{N}{n} \leq \tilde{\omega} \leq \frac{2N-n+1}{n+1}$.

(v) *Given real numbers $\lambda_1, \dots, \lambda_q$ with $\lambda_j \geq 1$ for $1 \leq j \leq q$ and given any $R \subset \{1, \dots, q\}$ and $|R| = N + 1$, there exists a subset $R^1 \subset R$ such that $|R^1| = \operatorname{rank}\{H_i\}_{i \in R^1} = n + 1$ and*

$$\prod_{j \in R} \lambda_j^{\omega(j)} \leq \prod_{i \in R^1} \lambda_i.$$

Lemma 15. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be an linearly nondegenerate meromorphic mapping over ϕ_p^0, and $H_j, j \in Q = \{1, \dots, q\}$ are hyperplanes, located in N-subgeneral position in $\mathbb{P}^n(\mathbb{C})$. Let $\omega(j)$ be the Nochka weights of $\{H_j\}_{j \in Q}$. Assume that $q > 2N - n + 1$. Then we get*

$$||f||^{\sum_{j \in S} \omega(j)} \cdot \prod_{t_j \in R} ||\hat{f}^{[j]}||^{\omega(t_j)} \cdot \prod_{j=0}^{n} ||\hat{f}^{[j]}||^{-1}$$
$$\leq K \cdot \frac{\Pi_{t_j \in R} |H_j(\hat{f}^{[j]})|^{\omega(t_j)} \cdot \Pi_{j \in S} |H_j(f)|^{\omega(j)}}{|C_p(f)|} \frac{|C_p(H_j(f) : j \in R^0)|}{\Pi_{t_j \in R^0} |(H_{t_j}(\hat{f}^{[j]}))|}$$

for an arbitrarily $z \in \mathbb{C}^m \setminus \Big(\big\{z \in \mathbb{C}^m : \Pi_{t_j \in R} H_j(\hat{f}^{[j]}) \cdot \Pi_{j \in S} H_j(f) = 0\big\} \cup I(f)\Big)$, where $I(f) = \{z \in \mathbb{C}^m : f_0(z) = \cdots = f_n(z) = 0\}$ and K depends on $\{H_j\}_{j \in Q}$, and R^0, R, S are some subsets of Q such that

$$R^0 = \{t_0, t_1, \dots, t_n\} \subset R = \{t_0, t_1, \dots, t_n, t_{n+1}, \dots, t_N\} \subset Q \setminus S.$$

Proof. Since the hyperplanes $\{H_j\}_{j=1}^{q}$ are in N-subgeneral position of $\mathbb{P}^n(\mathbb{C})$, we have $\cap_{j \in R} H_j = \emptyset$ for any $R \subset Q$ with $|R| = N + 1$. This implies that there exists a subset $S \subset Q$ with $|S| = q - N - 1$ such that $\Pi_{j \in S} H_j(\omega) \neq 0$.

For each $j \in S$, we consider function $h_j(\omega) = \frac{|H_j(\omega)|}{||\omega||}$ with $\omega \in \mathbb{P}^n(\mathbb{C})$. It is a positive continuous function on $\mathbb{P}^n(\mathbb{C})$. By the compactness of $\mathbb{P}^n(\mathbb{C})$, there exists a positive constant K_j such that $\frac{1}{K_j} \leq h_j(\omega) \leq K_j$. Therefore, we have

$$\frac{1}{K_j} \leq \frac{|H_j(\hat{f}^{[k_j]})|}{||\hat{f}^{[k_j]}||} \leq K_j \tag{3.9}$$

for each $j \in S$, $k_j \in \mathbb{N}^*$. It is easy to see that for each $j \in Q \setminus S$ and $k_j \in \mathbb{N}^*$, there exists a positive constant K_j such that

$$\frac{|H_j(\hat{f}^{[k_j]})|}{||\hat{f}^{[k_j]}||} \le K_j.$$

Put $R = Q \setminus S$. Then $|R| = N+1$. Choose $R^0 \subset R$ such that $|R^0| = n+1$ and R^0 satisfies Lemma 14(v) with respect to numbers $\frac{||\hat{f}^{[k_j]}||K_j}{|H_j(\hat{f}^{[k_j]})|}$ for arbitrary fixed point $z \in \mathbb{C}^m \setminus \Big(\{z \in \mathbb{C}^m : \Pi_{j\in Q}|H_j(\hat{f}^{[k_j]})| = 0\} \cup I(f)\Big)$ and $k_j \in \mathbb{N}$. We may assume that

$$R = \{t_0, t_1, \ldots, t_n, t_{n+1}, \ldots, t_N\} \text{ and } R^0 = \{t_0, t_1, \ldots, t_n\}.$$

For Q, we can rewrite its elements as follows.

$$Q = \{t_0, t_1, \ldots, t_n, t_{n+1}, \ldots, t_N, t_{N+1}, \ldots, t_{q-1}\}.$$

Then

$$\prod_{t_j \in R} \left(\frac{||\hat{f}^{[j]}||K_{t_j}}{|H_{t_j}(\hat{f}^{[j]})|}\right)^{\omega(t_j)} \le \prod_{t_j \in R^0} \frac{||\hat{f}^{[j]}||K_{t_j}}{|H_{t_j}(\hat{f}^{[j]})|}. \tag{3.10}$$

Since f is linearly nondegenerate over field ϕ_p^0, it follows from Lemma 13 that the Casorati determinant $C_p(f) \not\equiv 0$. By $\operatorname{rank}\{H_{t_j}\}_{j\in R^0} = n+1$, there exists a positive constant K_{R^0} such that $|C_p(f)| = K_{R^0} \cdot |C_p(H_j(f) : j \in R^0)|$. Thus

$$\frac{K_{R^0} \cdot |C_p(H_j(f) : j \in R^0)|}{|C^d(f)|} = 1. \tag{3.11}$$

Since (3.9) and (3.10), for an arbitrarily

$$z \in G := \mathbb{C}^m \setminus \Big(\{z \in \mathbb{C}^m : \Pi_{t_j \in R} H_j(\hat{f}^{[j]}) \cdot \Pi_{j \in S} H_j(f) = 0\} \cup I(f)\Big),$$

we have

(3.12)

$$\begin{aligned}
&\prod_{j\in S}\left(\frac{1}{K_j^2}\right)^{\omega(j)} \\
&\le \prod_{j\in S}\left(\frac{|H_j(f)|}{||f||K_j}\right)^{\omega(j)} \\
&\le \prod_{t_j\in R}\left(\frac{||\hat{f}^{[j]}||K_{t_j}}{|H_{t_j}(\hat{f}^{[j]})|}\right)^{\omega(t_j)} \cdot \frac{\prod_{t_j\in R}|H_{t_j}(\hat{f}^{[j]})|^{\omega(t_j)} \prod_{j\in S}|H_j(f)|^{\omega(j)}}{||f||^{\sum_{j\in S}\omega(j)} \cdot \prod_{t_j\in R}||\hat{f}^{[j]}||^{\omega(t_j)} \cdot K_0^{\sum_{j=1}^q \omega(j)}} \\
&\le \prod_{t_j\in R^0}\frac{||\hat{f}^{[j]}||K_{t_j}}{|H_{t_j}(\hat{f}^{[j]})|} \cdot \frac{\prod_{t_j\in R}|H_{t_j}(\hat{f}^{[j]})|^{\omega(t_j)} \prod_{j\in S}|H_j(f)|^{\omega(j)}}{||f||^{\sum_{j\in S}\omega(j)} \cdot \prod_{t_j\in R}||\hat{f}^{[j]}||^{\omega(t_j)} \cdot K_0^{\sum_{j=1}^q \omega(j)}}
\end{aligned}$$

$$= \frac{\prod_{t_j\in R^0} K_{t_j}}{K_0^{\sum_{j=1}^q \omega(j)}} \cdot \frac{\prod_{t_j\in R} |H_{t_j}(\hat{f}^{[j]})|^{\omega(t_j)} \prod_{j\in S} |H_j(f)|^{\omega(j)}}{|H_{t_0}(f)\cdot H_{t_1}(\hat{f})\cdots H_{t_n}(\hat{f}^{[n]})|}$$
$$\times \frac{1}{||f||^{\sum_{j\in S}\omega(j)} \cdot \prod_{t_j\in R} ||\hat{f}^{[j]}||^{\omega(t_j)} \cdot \prod_{t_j\in R^0} ||\hat{f}^{[j]}||^{-1}},$$

where $K_0 := \min\{K_1, \dots, K_q\}$. Together (3.11) with (3.12), for $z \in G$, we have

$$\prod_{j\in S}\left(\frac{1}{K_j^2}\right)^{\omega(j)}$$
$$\leq \frac{\prod_{t_j\in R^0} K_{t_j}\cdot K_{R^0}}{K_0^{\sum_{j=1}^q \omega(j)}} \cdot \frac{1}{||f||^{\sum_{j\in S}\omega(j)} \cdot \prod_{t_j\in R} ||\hat{f}^{[j]}||^{\omega(t_j)} \cdot \prod_{t_j\in R^0} ||\hat{f}^{[j]}||^{-1}}$$
$$\times \frac{\prod_{t_j\in R} |H_{t_j}(\hat{f}^{[j]})|^{\omega(t_j)} \prod_{j\in S} |H_j(f)|^{\omega(j)}}{|C_p(f)|} \cdot \frac{|C_p(H_j(f): j\in R^0)|}{|H_{t_0}(f)\cdot H_{t_1}(\hat{f})\cdots H_{t_n}(\hat{f}^{[n]})|}.$$

It implies that

$$||f||^{\sum_{j\in S}\omega(j)} \cdot \prod_{t_j\in R} ||\hat{f}^{[j]}||^{\omega(t_j)} \cdot \prod_{j=0}^{n} ||\hat{f}^{[j]}||^{-1}$$
$$\leq \frac{\prod_{t_j\in R^0} K_{t_j}\cdot K_{R^0}\cdot \Pi_{j\in S}(K_j)^{2\omega(j)}}{K_0^{\sum_{j=1}^q \omega(j)}} \cdot \frac{\prod_{t_j\in R} |H_{t_j}(\hat{f}^{[j]})|^{\omega(t_j)} \prod_{j\in S} |H_j(f)|^{\omega(j)}}{|C_p(f)|}$$
$$\times \frac{|C_p(H_j(f): j\in R^0)|}{|H_{t_0}(f)\cdot H_{t_1}(\hat{f})\cdots H_{t_n}(\hat{f}^{[n]})|}$$

for an arbitrarily $z \in G$. We obtain Lemma 15 by setting

$$K = \frac{\prod_{t_j\in R^0} K_{t_j}\cdot K_{R^0}\cdot \Pi_{j\in S}(K_j)^{2\omega(j)}}{K_0^{\sum_{j=1}^q \omega(j)}}$$

which is a positive constant depending on $\{H_j\}_{j=1}^q$, R^0, R and S. We finish the proof of Lemma 15. □

Proof of Theorem 1. By Lemma 15, for $r > 1$, we have

$$\sum_{j\in S}\omega(j)\log||f|| + \sum_{t_j\in R}\omega(t_j)\log||\hat{f}^{[j]}|| - \sum_{j=0}^{n}\log||\hat{f}^{[j]}||$$
$$\leq \sum_{t_j\in R}\omega(t_j)\log|H_{t_j}(\hat{f}^{[j]})| + \sum_{j\in S}\omega(j)\log|H_j(f)| - \log|C_p(f)|$$
$$+ \log\frac{|C_p(H_j(f): j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat{f}^{[j]}))|} + O(1).$$

Integrating both sides of this inequality and using Jensen's theorem and by definition of the characteristic function of f, we have

$$
\begin{aligned}
(3.13)\qquad & \sum_{j\in S}\omega(j)T_f(r)+\sum_{t_j\in R}\omega(t_j)T_{\hat{f}^{[j]}}(r)-\sum_{j=0}^{n}T_{\hat{f}^{[j]}}(r)\\
\le\ & \sum_{t_j\in R}\omega(t_j)N(r,\nu^0_{H_{t_j}(\hat{f}^{[j]})})+\sum_{j\in S}\omega(j)N(r,\nu^0_{H_j(f)})-N(r,\nu^0_{C_p(f))})\\
& +\int_{S_m(r)}\log^+\frac{|C_p(H_j(f):j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat{f}^{[j]}))|}\eta_m(z)+O(1)\\
\le\ & \sum_{t_j\in R}\omega(t_j)N(|p|r,\nu^0_{H_{t_j}(f)})+\sum_{j\in S}\omega(j)N(r,\nu^0_{Q_j(f)})-N(r,\nu^0_{C^d(f))})\\
& +\int_{S_m(r)}\log^+\frac{|C_p(H_j(f):j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat{f}^{[j]}))|}\eta_m(z)+O(1).
\end{aligned}
$$

By the First Main Theorem, the order of $N(r,\nu^0_{H_j(f)})$ satisfies

$$\limsup_{r\to\infty}\frac{\log^+N(r,\nu^0_{H_{t_j}(f)})}{\log r}\le\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r}=\sigma(f)=0.$$

So, by Lemma 8, the below inequality holds on a set of logarithmic density 1

$$
\begin{aligned}
N(|p|r,\nu^0_{H_{t_j}(f)})&=N(r,\nu^0_{H_{t_j}(f)})+o\left(N(r,\nu^0_{H_{t_j}(f)})\right)\\
&\le N(r,\nu^0_{H_{t_j}(f)})+o\left(T(r,f)\right).
\end{aligned}
$$

It follows from (3.13) that

$$
\begin{aligned}
(3.14)\qquad & \sum_{j\in S}\omega(j)T_f(r)+\sum_{t_j\in R}\omega(t_j)T_{\hat{f}^{[j]}}(r)-\sum_{j=0}^{n}T_{\hat{f}^{[j]}}(r)\\
\le\ & \sum_{t_j\in R}\omega(t_j)N(r,\nu^0_{H_{t_j}(f)})+\sum_{j\in S}\omega(j)N(r,\nu^0_{H_j(f)})-N(r,\nu^0_{C_p(f))})\\
& +\int_{S_m(r)}\log^+\frac{|C_p(H_j(f):j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat{f}^{[j]}))|}\eta_m(z)+o\left(T(r,f)\right)\\
=\ & \sum_{j\in Q}\omega(j)N(r,\nu^0_{H_j(f)})-N(r,\nu^0_{C_p(f))})\\
& +\int_{S_m(r)}\log^+\frac{|C_p(H_j(f):j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat{f}^{[j]}))|}\eta_m(z)+o\left(T(r,f)\right).
\end{aligned}
$$

We have

$$\frac{C_p(H_j(f):j\in R^0)}{\Pi_{t_j\in R^0}|(Q_{t_j}(\hat f^{[j]}))|}=\frac{\begin{vmatrix} 1 & \frac{H_{t_1}(f)}{H_{t_0}(f)} & \cdots & \frac{H_{t_n}(f)}{H_{t_0}(f)}\\ 1 & \frac{H_{t_1}(\hat f)}{H_{t_0}(\hat f)} & \cdots & \frac{H_{t_n}(\hat f)}{H_{t_0}(\hat f)}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \frac{H_{t_1}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})} & \cdots & \frac{H_{t_n}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})}\end{vmatrix}}{|\frac{H_{t_1}(\hat f)}{H_{t_0}(\hat f)}\cdots\frac{H_{t_n}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})}|}$$

$$=\frac{\begin{vmatrix} 1 & 1 & \cdots & 1\\ 1 & \frac{H_{t_1}(\hat f)}{H_{t_0}(\hat f)}\Big/\frac{H_{t_1}(f)}{H_{t_0}(f)} & \cdots & \frac{H_{t_n}(\hat f)}{H_{t_0}(\hat f)}\Big/\frac{H_{t_n}(f)}{H_{t_0}(f)}\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \frac{H_{t_1}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})}\Big/\frac{H_{t_1}(f)}{H_{t_0}(f)} & \cdots & \frac{H_{t_n}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})}\Big/\frac{H_{t_n}(f)}{H_{t_0}(f)}\end{vmatrix}}{|\frac{H_{t_1}(\hat f)}{H_{t_0}(\hat f)}\Big/\frac{H_{t_1}(f)}{H_{t_0}(f)}\cdots\frac{H_{t_n}(\hat f^{[n]})}{H_{t_0}(\hat f^{[n]})}\Big/\frac{H_{t_n}(f)}{H_{t_0}(f)}|}.$$

It is easy to see that $\sigma\left(\frac{H_i(f)}{H_j(f)}\right)\le\sigma(f)=0$ for all i,j. Therefore, by Lemma 11, we have

$$\begin{aligned}\int_{S_m(r)}\log^+\frac{|C_p(H_j(f):j\in R^0)|}{\Pi_{t_j\in R^0}|(H_{t_j}(\hat f^{[j]}))|}\eta_m(z)&\le\sum_{j=1}^n o\left(T\left(r,\frac{H_{t_j}(\hat f^{[j]})}{H_{t_0}(\hat f^{[j]})}\right)\right)\\&=o\left(T(r,f)\right)\end{aligned}$$

on a set of logarithmic density 1. Hence, together this with (3.14), we get

$$\begin{aligned}&\sum_{j\in S}\omega(j)T_f(r)+\sum_{t_j\in R}\omega(t_j)T_{\hat f^{[j]}}(r)-\sum_{j=0}^n T_{\hat f^{[j]}}(r)\\ \le&\sum_{j\in Q}\omega(j)N(r,\nu^0_{H_j(f)})-N(r,\nu^0_{C_p(f))})+o\left(T(r,f)\right)\end{aligned}\tag{3.15}$$

on a set of logarithmic density 1. From (3.15) and Lemma 12, we get

$$\Big(\sum_{j\in Q}\omega(j)-n-1\Big)T(r,f)\le\sum_{j\in Q}\omega(j)N(r,\nu^0_{H_j(f)})-N(r,\nu^0_{C_p(f))})+o\left(T(r,f)\right)\tag{3.16}$$

on a set of logarithmic density 1.

By (i), (ii) and (iv) of Lemma 14, the inequality (3.16) implies that the below inequality holds on a set of logarithmic density 1

$$(q-2N+n-1)T(r,f)\le\sum_{j\in Q}N(r,\nu^0_{H_j(f)})-\frac{N}{n}N(r,\nu^0_{C_p(f))})+o\left(T(r,f)\right).$$

The proof of Theorem 1 is completed. □

4. The proof of Theorem 2

Let z_0 be a n-successive zero with separation p of $H_j(f)$ respect to the rescaling $\tau_p(z) = pz$ for some $j \in \{1, \ldots, q\}$. Since $\{H_j\}_{j=1}^q$ is in N-subgeneral position, there are at most N functions $H_j(f)$ vanishing at z_0. Without loss of generality, we may assume that z_0 is a n-successive with separation p zero of $H_j(f)$ respect to the rescaling $\tau_p(z) = pz$ with all $j \in A$ and z_0 is a n-aperiodic zero with separation p of $H_j(f)$ respect to the rescaling $\tau_p(z) = pz$ with all $j \in B$ and z_0 is not a zero of $H_j(f)$ with all $j \notin A \cup B$, where $|A \cup B| = N$. Take $R \subset \{1, \ldots, q\}$ containing A such that $|R| = N+1$ and $R \cap B = \emptyset$. Choose subset $R^1 \subset R$ with $|R^1| = \operatorname{rank}\{H_j\}_{j \in R^1} = n+1$ such that R^1 satisfies (v) of Lemma 14 with respect to numbers $\{\lambda_j = e^{\nu^0_{H_j(f)}(z_0)}\}_{j=1}^q$. Then we have

$$\prod_{j \in R} e^{\omega(j)\nu^0_{H_j(f)}(z_0)} \leq \prod_{j \in R^1} e^{\nu^0_{H_j(f)}(z_0)}.$$

Therefore,

$$\sum_{j \notin B} \omega(j)\nu^0_{H_j(f)}(z_0) \leq \sum_{j \in A \cap R^1} \nu^0_{H_j(f)}(z_0). \tag{4.17}$$

By rearrangement index if necessary, we may assume that $R^1 = \{t_0, \ldots, t_n\}$ and $A \cap R^1 = \{t_0, \ldots, t_k\}$ with $0 \leq k \leq n$. Since $\operatorname{rank}\{H_{t_j}\}_{j=0}^n = n+1$, there exists a nonzero constant C_{R^1} such that

$$C_p(f) = C_{R^1} \cdot C_p(H_{t_0}(f), \ldots, H_{t_n}(f)).$$

This deduces that $\nu^0_{C_p(f)} = \nu^0_{C_p(H_{t_0}(f), \ldots, H_{t_n}(f))}$. We have

$$\begin{aligned}
&C_p(H_{t_0}(f), \ldots, H_{t_n}(f)) \\
&= H_{t_0}(f) \cdots H_{t_k}(f) \\
&\times \begin{vmatrix}
1 & \cdots & 1 & H_{t_{k+1}}(f) & \cdots & H_{t_n}(f) \\
\frac{H_{t_0}(\hat{f})}{H_{t_0}(f)} & \cdots & \frac{H_{t_k}(\hat{f})}{H_{t_k}(f)} & H_{t_{k+1}}(\hat{f}) & \cdots & H_{t_n}(\hat{f}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{H_{t_0}(\hat{f}^{[n]})}{H_{t_0}(f)} & \cdots & \frac{H_{t_k}(\hat{f}^{[n]})}{H_{t_n}(f)} & H_{t_{k+1}}(\hat{f}^{[n]}) & \cdots & H_{t_n}(\hat{f}^{[n]})
\end{vmatrix}.
\end{aligned}$$

It follows that

$$\nu^0_{C_p(f)}(z_0) \geq \nu^0_{H_{t_0}(f) \cdots H_{t_k}(f)}(z_0) = \sum_{j=0}^k \nu^0_{H_{t_j}(f)}(z_0).$$

Together this inequality with (4.17), we get

$$\nu^0_{C_p(f)}(z_0) \geq \sum_{j \notin B} \omega(j)\nu^0_{H_j(f)}(z_0).$$

This, by going through all points $z_0 \in \mathbb{C}^m$ and by definitions of $\hat{N}^{[n,p]}(r, H_j(f))$ implies that

$$\sum_{j=1}^{q} \omega(j) N(r, \nu^0_{H_j(f)}) - N(r, \nu^0_{C_p(f)}) \leq \sum_{j=1}^{q} \omega(j) \hat{N}^{[n,p]}(r, H_j(f)).$$

This and (3.16) yield

$$(\sum_{j=1}^{q} \omega(j) - n - 1) T(r, f) \quad \leq \sum_{j=1}^{q} \omega(j) \hat{N}^{[n,p]}(r, H_j(f)) + o\left(T(r, f)\right)$$

on a set of logarithmic density 1. By (i), (ii) and (iv) of Lemma 14, the above inequality implies that

$$(q - 2N + n - 1)\, T(r, f) \leq \sum_{j=1}^{q} \hat{N}^{[n,p]}(r, H_j(f)) + o\left(T(r, f)\right)$$

on a set of logarithmic density 1. The proof of Theorem 2 is completed.

5. The proof of Theorem 3

Lemma 16. *Let $f : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping with reduce presentation $f = (f_0 : \cdots : f_n)$ and let $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0, 1$ for all j. Assume that $\sigma(f) = 0$ and all zeros of $f_0, \ldots, f_n$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$. If $\frac{f_i}{f_j} \notin \phi^0_p$ for all $i, j \in \{0, \ldots, n\}$ such that $i \neq j$, then $f_0, \ldots, f_n$ are linearly independent over the field ϕ^0_p.*

Proof. Assume that f is linearly degenerate over ϕ^0_p. Without loss generality we assume that there exist $\varphi_0, \ldots, \varphi_n \in \phi^0_p \backslash \{0\}$ such that $\varphi_0 f_0 + \cdots + \varphi_{n-1} f_{n-1} = \varphi_n f_n$. Since all zeros of $f_0, \ldots, f_n$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$ and since $\varphi_0, \ldots, \varphi_n \in \phi^0_p \backslash \{0\}$, we can choose a meromorphic h such that $h\varphi_0 f_0, \ldots, h\varphi_n f_n$ are holomorphic functions on $\mathbb{C}^m$ without common zeros and such that preimages of all zeros of $h\varphi_0 f_0, \ldots, h\varphi_n f_n$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$. Then we get

$$\overline{\lim_{r \to \infty}} \frac{\log^+ \left(N(r, \nu^0_h) + N(r, \nu^\infty_h)\right)}{\log r} = 0 \tag{5.18}$$

and $h\varphi_0 f_0, \ldots, h\varphi_{n-1} f_{n-1}$ can not have any common zeros.

Put $g_i = h\varphi_i f_i$ for $0 \leq i \leq n$ and $G = (g_0 : \cdots : g_{n-1})$ is a holomorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^{n-1}(\mathbb{C})$. Then by definition of characteristic function, we have

$$T\left(r, G\right) = \int\limits_{S_m(r)} \log||G|| \eta_m(z) + O(1)$$

$$
\begin{aligned}
&\le \int\limits_{S_m(r)} \log|h|\eta_m(z) + \int\limits_{S_m(r)} \log\left(\sum_{j=0}^{n-1}|f_j|^2\right)^{\frac{1}{2}} \eta_m(z)\\
&\quad + \sum_{j=0}^{n-1}\int\limits_{S_m(r)} \log|\varphi_j|\eta_m(z) + O(1)\\
&\le N(r,\nu_h^0) + N(r,\nu_h^\infty) + T_f(r) + \sum_{j=0}^{n-1} T_{\varphi_j}(r) + O(1).
\end{aligned}
$$

This together (5.18) deduce that $\sigma(G) = 0$.

Assume that $G : \mathbb{C}^m \to \mathbb{P}^n(\mathbb{C})$ is linearly nondegenerate over ϕ_p^0. Since Lemma 13, it follows that $C_p(g_0, \dots, g_{n-1}) \not\equiv 0$. Take $n+1$ hyperplanes

$$H_0 : \omega_0 = 0, H_1 : \omega_1 = 0, \dots, H_{n-1} : \omega_{n-1} = 0$$

and

$$H_n : \omega_0 + \cdots + \omega_{n-1} = 0,$$

where $(\omega_0, \dots, \omega_{n-1})$ is homogeneous coordinate system of $\mathbb{P}^{n-1}(\mathbb{C})$. So $(G, H_j) = g_j$ for $0 \le j \le n-1$ and $(G, H_n) = g_0 + \cdots + g_{n-1} = h\varphi_n f_n = g_n$. Obviously, $\{H_j\}_{j=0}^n$ are in general position in $P^{n-1}(\mathbb{C})$. Applying Theorem 2, we have

$$T(r, G) \le \sum_{j=0}^{n} \hat{N}^{[n,p]}(r, H_j(G)) + o(T(r, G))$$

on a set of logarithmic density 1. Since all zeros of $H_j(G) = (G, H_j) = g_j$ $(0 \le j \le n)$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$, $\hat{N}^{[n,p]}(r, H_j(G)) \equiv 0$ and therefore, $T(r, G) \le o(T(r, G))$ on a set of logarithmic density 1. This is a contradiction. It follows that G is linearly dependent over ϕ_p^0. Thus there exist $\psi_0, \dots, \psi_{n-1}$ satisfying

$$\psi_0 f_0 + \cdots + \psi_{n-2} f_{n-2} = \psi_{n-1} f_{n-1}$$

and not all ψ_i are identically zero. By continuing in this fashion it follows after at most $n-2$ time, we have $\frac{f_i}{f_j} \in \phi_p^0$ for some $i \neq j$. This is contradiction. Hence, f is linearly nondegenerate over ϕ_p^0. We finish the proof of Lemma 16. $\square$

Lemma 17. *Let $f = (f_0 : \cdots : f_n)$ be a meromorphic mapping of $\mathbb{C}^m$ to $\mathbb{P}^n(\mathbb{C})$ such that $\sigma(f) = 0$ and let $p = (p_1, \dots, p_m) \in \mathbb{C}^m$ with $p_j \neq 0, 1$ for all j. Assume that all zeros of $f_0, \dots, f_n$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$. Let $S_1 \cup \cdots \cup S_l$ be the partion of $\{0, \dots, n\}$ formed in such a way that i and j are in the same class S_k if only if $\frac{f_i}{f_j} \in \phi_p^0$. If $f_0 + \cdots + f_n = 0$, then $\sum\limits_{j \in S_k} f_j = 0$ for all $k \in \{1, \dots, l\}$.*

Proof. For each $i \in S_k, k \in \{1, \dots, l\}$ we have $f_i = \varphi_{i,j_k} f_{j_k}$ for $\varphi_{i,j_k} \in \phi_p^0$ whenever the $i, j_k \in S_k$. It implies that

$$0 = \sum_{k=0}^{n} f_k = \sum_{k=1}^{l} \sum_{i \in S_k} \varphi_{i,j_k} f_{j_k} = \sum_{k=1}^{l} B_k f_{j_k},$$

where $B_k = \sum\limits_{i \in S_k} \varphi_{i,j_k} \in \phi_p^0$. This deduces that $f_{j_1}, \dots, f_{j_l}$ are linearly dependent over ϕ_p^0 if not all B_k are identically zeros. This contradicts to Lemma 16. Then $B_k \equiv 0$ for all $k \in \{1, \dots, l\}$. Thus $\sum\limits_{i \in S_k} f_i = \sum\limits_{i \in S_k} \varphi_{i,j_k} f_{j_k} = B_k f_{i_k} \equiv 0$ for all $k \in \{1, \dots, l\}$. Lemma 17 is proved. □

Proof of Theorem 3. By assumptions of the theorem, holomorphic functions

$$G_j = H_j(f) = \sum_{i=0}^{n} a_{ji} f_i,$$

satisfying

$$\{\tau_p(G_j^{-1}(0))\} \subset \{G_j^{-1}(0)\},\ j \in \{1, \dots, q\},$$

where $H_j : \sum_{i=0}^{n} a_{ji}\omega_i = 0$, and $\{\cdot\}$ denotes a multiset with counting multiplicities of its elements. We say that $i \sim j$ if $G_i = \alpha G_j$ for some $\alpha \in \phi_p^0 \setminus \{0\}$. Therefore, the set of indexes $\{1, \dots, q\}$ may be split into disjoint equivalence classes S_j,

$$\{1, \dots, q\} = \cup_{j=1}^{l} S_j$$

for some $l \leq q$.

The first, we assume that S_j has as most $q - N - 1$ elements for some $j \in \{1, \dots, l\}$. Put $R = Q \setminus S_j$ then, $|R| \geq N+1$. Let $s_0 \in S_j$ and put $U = R \cup \{s_0\}$. Without loss of generality, we may assume that $U = \{s_0, \dots, s_{N+1}\}$. Then since the $\{H_j\}_{j=1}^{q}$ are in N-subgeneral position, there exist $\alpha_j \in \mathbb{C} \setminus \{0\}$ such that $\sum_{j=0}^{N+1} \alpha_j H_{s_j} = 0$ and therefore, we have $\sum_{j=0}^{N+1} \alpha_j H_{s_j}(f) = \sum_{j=0}^{N+1} \alpha_j G_{s_j} \equiv 0$. By assumptions of the theorem, we can see that all zeros of $\alpha_j G_{s_j}$ are forward invariant with respect to the rescaling $\tau_p(z) = pz$. We have

$$G := (\alpha_0 G_{s_0} : \cdots : \alpha_{N+1} G_{s_{N+1}})$$

is a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^{N+1}(\mathbb{C})$ with its order $\sigma(G) \leq \sigma(f) = 0$. By Lemma 17, we have $\alpha_0 G_{s_0} \equiv 0$. Hence, $H_{s_0}(f) \equiv 0$. This implies that the image $f(\mathbb{C}^m)$ is included in the hyperplane H_{s_0} of $\mathbb{P}^n(\mathbb{C})$. We may consider f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{P}^{n-1}(\mathbb{C})$.

The second, we assume that S_j has as least $q - N$ elements for all $j \in \{1, \dots, l\}$. Then

$$l \leq \frac{q}{q - N}.$$

Since $\{H_j\}_{j=1}^{q}$ is in N-subgeneral position, we can choose a subset $V \subset \{1, \dots, q\}$ with $|V| = n+1$ such that $\{H_j\}_{j \in V}$ is linearly independent. Put $V_j = V \cap S_j$

for each $1 \leq j \leq l$. Then we have $V = \cup_{j=1}^{l} V_j$. Since each V_j gives raise to $|V_j| - 1$ equations over the field ϕ_p^0, it is easy to see that there are at least

$$\sum_{j=1}^{l}(|V_j| - 1) = n + 1 - l \geq n + 1 - \frac{q}{q-N} = n - \frac{N}{q-N}$$

linearly independent relations over the field ϕ_p^0. It follows that the image of f is contained in a projective linear subspace over ϕ_p^0 of dimension $\leq \left[\frac{N}{q-N}\right]$. Obviously, if $q \geq 2N + 1$, then $\left[\frac{N}{q-N}\right] = 0$, and therefore $f(z) = f(pz)$. The Theorem 3 is proved. □

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