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UPPERS TO ZERO IN POLYNOMIAL RINGS OVER GRADED DOMAINS AND UMt-DOMAINS

Haleh Hamdi and Parviz Sahandi

Abstract. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, H be the set of nonzero homogeneous elements of R, and $\star$ be a semistar operation on R. The purpose of this paper is to study the properties of quasi-Prüfer and UMt-domains of graded integral domains. For this reason we study the graded analogue of $\star$-quasi-Prüfer domains called gr-$\star$-quasi-Prüfer domains. We study several ring-theoretic properties of gr-$\star$-quasi-Prüfer domains. As an application we give new characterizations of UMt-domains. In particular it is shown that R is a gr-t-quasi-Prüfer domain if and only if R is a UMt-domain if and only if R_P is a quasi-Prüfer domain for each homogeneous maximal t-ideal P of R. We also show that R is a UMt-domain if and only if H is a t-splitting set in $R[X]$ if and only if each prime t-ideal Q in $R[X]$ such that $Q \cap H = \emptyset$ is a maximal t-ideal.

1. Introduction

Gilmer characterized Prüfer domains as integrally closed domains such that each prime ideal of the polynomial ring contained in an extended prime is extended [20, Theorem 19.15]. The later condition is called a *quasi-Prüfer domain*, see [6] and [16, Chapter 6]. Thus an integral domain D is a Prüfer domain if and only if D is integrally closed and quasi-Prüfer. As a t-operation analogue it is well-known that D is a Prüfer v-multiplication domain (PvMD) if and only if D is an integrally closed UMt-domain [23, Proposition 3.2]. Recall that D is called a *UMt-domain* [23], if every upper to zero in $D[X]$ is a maximal t-ideal and has been studied by several authors (see [8], [10], [12], [14] and [31]). In [9], Chang and Fontana unified quasi-Prüfer and UMt-domains by introducing the notion of $\star$-quasi-Prüfer domain, where $\star$ is a semistar operation on a domain. In this paper, we study quasi-Prüfer and UMt-domain properties of graded integral domains. (Relevant definitions are reviewed in the sequel.)

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded (commutative) integral domain graded by an arbitrary torsionless grading monoid Γ. In [1], Anderson-Anderson defined the

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graded analogue of some classical domains in *Multiplicative Ideal Theory* like a graded-PvMD, graded GCD-domain and graded GGCD-domain. It is known that R is a graded-PvMD (resp., graded GCD-domain, graded GGCD-domain) if and only if R is a PvMD, (resp., GCD-domain, GGCD-domain) [1, Theorem 6.4, Corollary 6.7 and Proposition 6.6]. In [5], Anderson and Chang had begun an investigation on graded integral domains including graded integral domains with a unit of nonzero degree. They defined R to be a *graded-Prüfer* domain if each nonzero finitely generated homogeneous ideal of R is invertible, and gave an example of a graded-Prüfer domain which is not Prüfer [5, Example 3.6]. Then the author in [32] gave some characterizations of graded-Prüfer domains.

For $a \in R = \bigoplus_{\alpha\in\Gamma} R_\alpha$, denote by $C(a)$ the ideal of R generated by homogeneous components of a. In [5] and [32], the authors used $C(a)$ to investigate properties of graded integral domains. Since there was not the role of an indeterminate, in most of the results, the base ring was required to have a unit of nonzero degree or to satisfy some other related condition (see [5, Section 1]). Because of this consideration, the author in [33], introduced a homogeneous content ideal for polynomial rings over graded domains to make use of the role of an indeterminate. For a polynomial $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, define the *homogeneous content ideal of* f by $\mathcal{A}_f := \mathcal{A}_f^R := \sum_{i=0}^{n} C(a_i)$. Using $\mathcal{A}_f$ we no longer need to assume that the base ring has a unit of nonzero degree.

The main purpose of this paper is to study the quasi-Prüfer and UMt-domain properties of graded integral domains. For this reason in Section 2 we introduced the graded analogue of $\star$-quasi-Prüfer domains called gr-$\star$-quasi-Prüfer domains and make use of the homogeneous content ideal $\mathcal{A}_f$. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is called a *gr-$\star$-quasi-Prüfer domain* in case, if Q is a prime ideal in $R[X]$ and $Q \subseteq P[X]$ for some homogeneous quasi-$\star$-prime ideal P of R, then $Q = (Q \cap R)[X]$. When $\star = d$ the identity operation on R, then we call the gr-d-quasi-Prüfer domain a *gr-quasi-Prüfer domain*. It is shown that R is a gr-$\star_f$-quasi-Prüfer domain if and only if each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $\mathcal{A}_g^\star = R^\star$, if and only if for each upper to zero Q in $R[X]$, $\mathcal{A}_Q^{\star_f} = R^\star$. It is also shown that R is a gr-$\star_f$-quasi-Prüfer domain if and only if $\mathrm{NA}(R, \star)$ is a quasi-Prüfer domain if and only if every prime ideal of $\mathrm{NA}(R, \star)$ is extended from a homogeneous prime ideal of R. As an application, in Section 3, we give several new characterizations of UMt-domains. In particular, we show that R is a UMt-domain if and only if R_P is a quasi-Prüfer domain for each *homogeneous* prime t-ideal P of R if and only if R is a gr-t-quasi-Prüfer domain (see Theorem 3.2). Also we show that R is a UMt-domain if and only if H (the multiplicative set of nonzero homogeneous elements of R) is a t-splitting set in $R[X]$ if and only if each prime t-ideal Q in $R[X]$ such that $Q \cap H = \emptyset$ is a maximal t-ideal (see Theorem 3.6). We also connect gr-$\star$-quasi-Prüfer domains to UMt-domains. More precisely, if $\star$ is a (semi)star operation on R, it is shown that R is a gr-$\star_f$-quasi-Prüfer domain if and only if R is a UMt-domain and

$\widetilde{\star}$ and w coincide on nonzero homogeneous ideals of R (see Theorem 3.9). In particular R is a gr-quasi-Prüfer domain if and only if R is a UMt-domain and d and w coincide on nonzero homogeneous ideals of R. Hence if R is a one dimensional graded domain, then R is a gr-quasi-Prüfer domain if and only if R is a quasi-Prüfer domain. Finally, we give an example of a gr-quasi-Prüfer domain that is not a quasi-Prüfer domain (see Example 3.14).

To facilitate the reading of the paper, we review some basic facts on semistar operations on (graded) integral domains. Let Γ be a nonzero torsionless grading monoid, that is, Γ is a commutative cancellative monoid (written additively), and $\langle\Gamma\rangle = \{a - b \mid a, b \in \Gamma\}$ be the quotient group of Γ; so $\langle\Gamma\rangle$ is a torsionfree abelian group. It is known that a cancellative monoid is torsionless if and only if it can be given a total order compatible with the monoid operation [28, page 123]. Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a Γ-graded integral domain. That is, $\deg(x) = \alpha$ for each $0 \neq x \in R_\alpha$ and $\deg(0) = 0$, and thus each nonzero $f \in R$ can be written uniquely as $f = x_{\alpha_1} + \cdots + x_{\alpha_n}$ with $\deg(x_{\alpha_i}) = \alpha_i$ and $\alpha_1 < \cdots < \alpha_n$. A nonzero $x \in R_\alpha$ for all $\alpha \in \Gamma$ is said to be *homogeneous*, and so if $H = \bigcup_{\alpha\in\Gamma}(R_\alpha \setminus \{0\})$, then H is the saturated multiplicative set of nonzero homogeneous elements of R. Then $R_H = \bigoplus_{\alpha\in\langle\Gamma\rangle}(R_H)_\alpha$, called the *homogeneous quotient field of* R, is a $\langle\Gamma\rangle$-graded integral domain whose nonzero homogeneous elements are units. An integral ideal I of R is said to be homogeneous if $I = \bigoplus_{\alpha\in\Gamma}(I \cap R_\alpha)$. A fractional ideal I of R is *homogeneous* if sI is an integral homogeneous ideal of R for some $s \in H$ (thus $I \subseteq R_H$). An overring T of R, with $R \subseteq T \subseteq R_H$ will be called a *homogeneous overring* if $T = \bigoplus_{\alpha\in\langle\Gamma\rangle}(T \cap (R_H)_\alpha)$. Thus T is a ($\langle\Gamma\rangle$-)graded integral domain with $T_\alpha = T \cap (R_H)_\alpha$ for all $\alpha \in \langle\Gamma\rangle$. For more on graded integral domains and their divisibility properties, see [2], [28].

Let D be an integral domain with quotient field K. Let $\overline{\mathcal{F}}(D)$ denote the set of all nonzero D-submodules of K, $\mathcal{F}(D)$ be the set of all nonzero fractional ideals of D, and $f(D)$ be the set of all nonzero finitely generated fractional ideals of D. Obviously, $f(D) \subseteq \mathcal{F}(D) \subseteq \overline{\mathcal{F}}(D)$. As in [29], a *semistar operation on* D is a map $\star : \overline{\mathcal{F}}(D) \to \overline{\mathcal{F}}(D)$, $E \mapsto E^\star$, such that, for all $0 \neq x \in K$, and for all $E, F \in \overline{\mathcal{F}}(D)$, the following properties hold: $(\star_1)$ $(xE)^\star = xE^\star$; $(\star_2)$: $E \subseteq F$ implies that $E^\star \subseteq F^\star$; $(\star_3)$ $E \subseteq E^\star$; and $(\star_4)$ $E^{\star\star} := (E^\star)^\star = E^\star$.

A semistar operation $\star$ is called a *(semi)star operation on* D, if $D^\star = D$. Let $\star$ be a semistar operation on D. For every $E \in \overline{\mathcal{F}}(D)$, put $E^{\star_f} := \bigcup F^\star$, where the union is taken over all $F \in f(D)$ with $F \subseteq E$. It is easy to see that $\star_f$ is a semistar operation on D. If $\star = \star_f$, then $\star$ is said to be a semistar operation *of finite type*. We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* of D, if $I^\star \cap D = I$; a *quasi-$\star$-prime* (ideal of D), if I is a prime quasi-$\star$-ideal of D; and a *quasi-$\star$-maximal* (ideal of D), if I is maximal in the set of all proper quasi-$\star$-ideals of D. Each quasi-$\star$-maximal ideal is a prime ideal. It was shown in [15, Lemma 4.20] that if $D^\star \neq K$, then each proper quasi-$\star_f$-ideal of D is contained in a quasi-$\star_f$-maximal ideal of D. We denote by $\mathrm{QMax}^\star(D)$ (resp., QSpec$^\star(D)$) the set of all quasi-$\star$-maximal ideals (resp., quasi-$\star$-prime ideals) of D.

If $\star_1$ and $\star_2$ are semistar operations on D, one says that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\mathcal{F}}(D)$ (cf. [29, page 6]).

Let $\star$ be a semistar operation on D, T an overring of D, and $\iota : D \hookrightarrow T$ the corresponding inclusion map. In a canonical way, one can define an associated semistar operation $\star_\iota$ on T, by $E \mapsto E^{\star_\iota} := E^\star$ for each $E \in \overline{\mathcal{F}}(T)(\subseteq \overline{\mathcal{F}}(D))$.

Given a semistar operation $\star$ on D, it is possible to construct a semistar operation $\widetilde{\star}$, which is defined as follows, for each $E \in \overline{\mathcal{F}}(D)$, $E^{\widetilde{\star}} := \bigcap_{P \in \mathrm{QMax}^{\star_f}(D)} ED_P$.

The most widely studied (semi)star operations on D have been the identity d_D, v_D, $t_D := (v_D)_f$, and $w_D := \widetilde{v_D}$ operations, where $A^{v_D} := (A^{-1})^{-1}$, with $A^{-1} := (D : A) := \{x \in K \mid xA \subseteq D\}$. We usually use these operations without subscripts. If $\star$ is a (semi)star operation on D, then $d \leq \star \leq v$.

Let $\star$ be a semistar operation on D. Recall from [17] that, D is called a *Prüfer $\star$-multiplication domain* (for short, a P$\star$MD) if each nonzero finitely generated ideal of D is *$\star_f$-invertible*; i.e., if $(II^{-1})^{\star_f} = D^\star$ for all $I \in f(D)$. When $\star = v$, we recover the classical notion of PvMD; when $\star = d_D$, the identity (semi)star operation, we recover the notion of Prüfer domain.

Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. We say that $\star$ is *homogeneous preserving* if $\star$ sends homogeneous fractional ideals to homogeneous ones. It is known that d, t, and v are homogeneous preserving [2, Proposition 2.5], $\widetilde{\star}$ is homogeneous preserving [32, Proposition 2.3], and that if $\star$ is homogeneous preserving, then so is $\star_f$ [32, Lemma 2.4]. Denote by h-QSpec$^\star(R)$ the homogeneous elements of QSpec$^\star(R)$ and let h-QMax$^\star(R)$ denote the set of ideals of R which are maximal in the set of all proper homogeneous quasi-$\star$-ideals of R (if $\star$ is a (semi)star operation we denote these sets by h-Spec$^\star(R)$ and h-Max$^\star(R)$ respectively). It is shown that if $R^\star \subsetneq R_H$ and $\star = \star_f$ homogeneous preserving, then h-QMax$^{\star_f}(R)(\subseteq h$-QSpec$^\star(R))$ is nonempty, each proper homogeneous quasi-$\star_f$-ideal is contained in a homogeneous maximal quasi-$\star_f$-ideal [32, Lemma 2.1], and h-QMax$^{\star_f}(R) = h$-QMax$^{\widetilde{\star}}(R)$ [32, Proposition 2.5].

2. Graded $\star$-quasi-Prüfer domains

Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain with quotient field K, H be the set of nonzero homogeneous elements of R, X be an indeterminate over K, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. The following is the key definition in this paper.

Definition 2.1. The graded integral domain R is called a *gr-$\star$-quasi-Prüfer domain* in case, if Q is a prime ideal in $R[X]$ and $Q \subseteq P[X]$ for some $P \in h$-QSpec$^\star(R)$, then $Q = (Q \cap R)[X]$. When $\star = d$ the identity operation on R, then we call the gr-d-quasi-Prüfer domain a *gr-quasi-Prüfer domain.*

It can be seen that if R has trivial grading $\Gamma = \{0\}$, then a gr-$\star$-quasi-Prüfer domain is the same as a $\star$-quasi-Prüfer domain [9].

It is clear from the definition that if R is a $\star$-quasi-Prüfer domain, then it is a gr-$\star$-quasi-Prüfer domain. Assume that $\star_1 \leq \star_2$ are two semistar operations on R. It is easy to see that if R is a gr-$\star_1$-quasi-Prüfer domain, then R is a gr-$\star_2$-quasi-Prüfer domain, since $h\text{-QSpec}^{\star_2}(R) \subseteq h\text{-QSpec}^{\star_1}(R)$.

Assume that L is a fractional ideal of $R[X]$ such that $L \subseteq R_H[X]$, and set $\mathcal{A}_L := \sum_{f\in L} \mathcal{A}_f$. It is easy to see that $L \subseteq \mathcal{A}_L[X]$. By an *upper to zero in* $R[X]$, we mean a nonzero prime ideal Q of $R[X]$ such that $Q \cap R = 0$.

Proposition 2.2. *Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then the following statements are equivalent*:

(1) *R is a gr-$\star$-quasi-Prüfer domain.*
(2) *Let Q be an upper to zero in $R[X]$, then $\mathcal{A}_Q \nsubseteq P$ for each $P \in h$-QSpec$^\star(R)$.*
(3) *Let Q be an upper to zero in $R[X]$, then $Q \nsubseteq P[X]$ for each $P \in h$-QSpec$^\star(R)$.*
(4) *R_P is a quasi-Prüfer domain for each $P \in h$-QSpec$^\star(R)$.*
(5) *$R_{H\setminus P}$ is a gr-quasi-Prüfer domain for each $P \in h$-QSpec$^\star(R)$.*

Proof. (1) $\Rightarrow$ (3). Follows from the definition.

(3) $\Rightarrow$ (2). If Q is an upper to zero in $R[X]$, then by assumption $Q \nsubseteq P[X]$ for all $P \in h$-QSpec$^\star(R)$. Hence $\mathcal{A}_Q \nsubseteq P$ for each $P \in h$-QSpec$^\star(R)$, since $Q \subseteq \mathcal{A}_Q[X]$.

(2) $\Rightarrow$ (1). Assume that Q is a prime ideal in $R[X]$ such that $(Q \cap R)[X] \subsetneq Q \subseteq P[X]$ for some $P \in h$-QSpec$^\star(R)$. Then we can find an upper to zero Q_1 in $R[X]$ such that $Q_1 \subseteq Q$ by [11, Theorem A]. Thus $\mathcal{A}_{Q_1} \subseteq \mathcal{A}_Q \subseteq P$ for some $P \in h$-QSpec$^\star(R)$, and this contradicts the hypothesis.

(1) $\Rightarrow$ (4). Let $P \in h$-QSpec$^\star(R)$. If Q is a prime ideal of $R_P[X]$ with $c_{R_P}(Q) \subsetneq R_P$, then $c_{R_P}(Q) \subseteq PR_P$, and hence $Q \subseteq PR_P[X]$ (where $c_D(f)$ is the fractional ideal of an integral domain D generated by the coefficients of $f \in D[X]$ and $c_D(Q) = \sum_{f\in Q} c_D(f)$ for Q an ideal of $D[X]$). So $Q \cap R[X] \subseteq P[X]$, and by (1) we have $Q \cap R[X] = (Q \cap R)[X]$. Hence $Q = (Q \cap R_P)[X]$. Thus R_P is a quasi-Prüfer domain by [9, Theorem 1.1].

(4) $\Rightarrow$ (1) is the same as the proof of part (iv)$\Rightarrow$(i) of [9, Lemma 2.1].

(1) $\Rightarrow$ (5). Let $P \in h$-QSpec$^\star(R)$. Assume that Q is a prime ideal of $R_{H\setminus P}[X]$ and $Q \subseteq \mathfrak{q}R_{H\setminus P}[X]$ for some $\mathfrak{q}R_{H\setminus P} \in h$-Spec$(R_{H\setminus P})$. Hence $Q \cap R[X] \subseteq \mathfrak{q}R_{H\setminus P}[X] \cap R[X] \subseteq P[X]$. Thus $Q \cap R[X] = (Q \cap R)[X]$ by (1). Therefore $Q = (Q \cap R_{H\setminus P})[X]$ and $R_{H\setminus P}$ is a gr-quasi-Prüfer domain.

(5) $\Rightarrow$ (1). Assume that Q is a prime ideal in $R[X]$ and $Q \subseteq P[X]$ for some $P \in h$-QSpec$^\star(R)$. Thus $Q_{H\setminus P} \subseteq PR_{H\setminus P}[X]$. Hence by (5) one has $Q_{H\setminus P} = (Q_{H\setminus P} \cap R_{H\setminus P})[X]$. Consequently $Q = (Q \cap R)[X]$ and R is a gr-$\star$-quasi-Prüfer domain. □

Recall from [32], that R is called a *graded Prüfer $\star$-multiplication domain* (*GP$\star$MD*) if every nonzero finitely generated homogeneous ideal of R is a $\star_f$-invertible. When $\star = v$ we have the notion of a graded-PvMD(=PvMD) [1]. Also when $\star = d$, a GPdMD is a graded-Prüfer domain [5].

Corollary 2.3. *Every GP$\star$MD is a gr-$\star$-quasi-Prüfer domain.*

Proof. Assume that R is a GP$\star$MD. Then for $P \in h\text{-QSpec}^{\widetilde{\star}}(R)$, we have R_P is a valuation domain by [32, Theorem 4.4], and hence R_P is a quasi-Prüfer domain. So that by Proposition 2.2, R is a gr-$\widetilde{\star}$-quasi-Prüfer domain. Since $\widetilde{\star} \leq \star$ we have R is a gr-$\star$-quasi-Prüfer domain. □

Corollary 2.4. *Let $\star$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Then R is a gr-$\star_f$-quasi-Prüfer domain if and only if R is a gr-$\widetilde{\star}$-quasi-Prüfer domain.*

Proof. Use Proposition 2.2, together with the equality $h\text{-QMax}^{\star_f}(R) = h\text{-QMax}^{\widetilde{\star}}(R)$ of [32, Proposition 2.5]. □

Note that the t-operation is a homogeneous preserving star operation and that $w = \widetilde{t}$. Thus in particular R is a gr-t-quasi-Prüfer domain if and only if R is a gr-w-quasi-Prüfer domain. It is shown [9, Corollary 2.4] that D is a t-quasi-Prüfer domain if and only if D is a UMt-domain. In Theorem 3.2 we will show that R is a gr-t-quasi-Prüfer domain if and only if R is a UMt-domain. Recently Chang defined another notion of graded UMt-domains for graded integral domains R such that R_H is UFD [7]. A graded integral domain R with R_H a UFD is called a *graded UMt-domain* if every upper to zero in R is a maximal t-ideal, in the sense that a prime ideal U in R is called an *upper to zero in* R, if there exists a prime element $f \in R_H$ such that $U = fR_H \cap R$. It is shown in [7, Theorem 3.5] that, if in addition R has a unit of nonzero degree, then R is a UMt-domain if and only if R is a graded UMt-domain.

Assume that $\star$ is a homogeneous preserving semistar operation on R such that $R^\star \subsetneq R_H$. Then using [32, Lemma 2.1], one has $h\text{-QMax}^{\star_f}(R) \neq \emptyset$ and if I is a homogeneous ideal of R, then $I^{\star_f} = R^\star$ if and only if $I \nsubseteq P$ for all $P \in h\text{-QMax}^{\star_f}(R)$.

Lemma 2.5. *Let $\star$ be a homogeneous preserving semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Then the following statements are equivalent*:

1. *R is a gr-$\star_f$-quasi-Prüfer domain.*
2. *Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $\mathcal{A}_g^\star = R^\star$.*
3. *If Q is an upper to zero in $R[X]$, then $\mathcal{A}_Q^{\star_f} = R^\star$.*

Proof. (1) $\Leftrightarrow$ (3). Follows from Proposition 2.2, because the property $\mathcal{A}_Q \nsubseteq P$ for all $P \in h\text{-QMax}^{\star_f}(R)$ is equivalent to $\mathcal{A}_Q^{\star_f} = R^\star$.

(3) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) is the same as the proof of part $(2_{\star_f}) \Rightarrow (1_{\star_f})$ of [9, Lemma 2.3]. □

We say that $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ is a *graded valuation domain* (*gr-valuation domain*) if either $u \in R$ or $u^{-1} \in R$ for every nonzero homogeneous $u \in R_H$. It is known that a gr-valuation domain R has a unique homogeneous maximal ideal M, and in this case, R_M is a valuation domain [32, Lemma 4.3]. It is clear that R is a gr-valuation domain if and only if R is a graded-Prüfer domain with a unique homogeneous maximal ideal. In particular a gr-valuation domain is integrally closed.

The following proposition is the graded version of the celebrated result of Krull [20, Theorem 19.8]. The integral closure of R is denoted by $\bar{R}$.

Proposition 2.6 (cf. [25, Theorem 2.10])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the integral closure of R in K is the intersection of the family $\{V_\lambda\}_{\lambda\in\Lambda}$ of gr-valuation overrings of R. In particular, $\bar{R}$ is a homogeneous overring of R.*

Let R be a graded integral domain and $\star$ a semistar operation on R. By Proposition 2.6, $\bar{R}$ is a homogeneous overring of R. Note that $\bar{R}$ may not be a fractional ideal of R. However the same proof of [32, Proposition 2.3] shows that $\widetilde{\star}$ sends nonzero homogeneous R-submodules of R_H to homogeneous ones. Therefore $\widetilde{R} := (\bar{R})^{\widetilde{\star}}$ is a homogeneous overring of R.

For a fractional ideal I of R let I_h denote the fractional ideal of R generated by the set of homogeneous elements of R in I.

Lemma 2.7. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and T a homogeneous overring of R. If J is an ideal of T, then $(J \cap R)_h = J_h \cap R$.*

Proof. The inclusion $(J\cap R)_h \subseteq J_h \cap R$ is clear. Let $x = \sum x_i \in J_h \cap R$ where x_i are homogeneous components of x. Then $x_i \in J_h \cap R \subseteq J \cap R$. Therefore $x = \sum x_i \in (J \cap R)_h$. □

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and T be a homogeneous overring of R. Let $\star$ and $\star'$ be semistar operations on R and T, respectively. Recall from [32] that T is called a *homogeneously* $(\star,\star')$*-linked overring of* R if

$$F^\star = R^\star \Rightarrow (FT)^{\star'} = T^{\star'}$$

for each nonzero homogeneous finitely generated ideal F of R.

Let $N(\star) := \{f \in R[X] \mid f \neq 0 \text{ and } \mathcal{A}_f^\star = R^\star\}$ and set $\mathrm{NA}(R,\star) := R[X]_{N(\star)}$ and $\mathrm{NA}(R) := \mathrm{NA}(R,d)$. Then it is shown in [33], that $\mathrm{NA}(R,\star)$ is compatible with the graded structure of the base ring R and that if R has the trivial grading, then $\mathrm{NA}(R,\star) = \mathrm{Na}(R,\star)$ the usual $\star$-Nagata ring [18]. It is known that $N(\star) = N(\star_f) = N(\widetilde{\star}) = R[X] \setminus \bigcup\{P[X] \mid P \in h\text{-}\operatorname{QMax}^{\widetilde{\star}}(R)\}$ and $\operatorname{Max}(\mathrm{NA}(R,\star)) = \{P\,\mathrm{NA}(R,\star) \mid P \in h\text{-}\operatorname{QMax}^{\widetilde{\star}}(R)\}$ [33, Proposition 2.3].

Proposition 2.8 ([33, Theorem 3.6])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a semistar operation on R such that $R^\star \subsetneq R_H$. Then, the following statements are equivalent*:

1. *R is a GP$\star$MD.*
2. *Every ideal of* $\mathrm{NA}(R,\star)$ *is extended from a homogeneous ideal of R.*
3. $\mathrm{NA}(R,\star)$ *is a Prüfer domain.*

In particular if R is a GP$\star$MD, then $R^{\widetilde{\star}}$ is integrally closed.

We are now ready to state and prove the main result of this section which gives some characterizations of gr-$\star_f$-quasi-Prüfer domains.

Theorem 2.9. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain, and $\star$ be a homogeneous preserving semistar operation on R such that $R^\star \subsetneq R_H$. Then the following statements are equivalent*:

1. *R is a gr-$\star_f$-quasi-Prüfer domain.*
2. *Set $\widetilde{R} = (\bar{R})^{\widetilde{\star}}$ and let $\widetilde{\iota} : R \hookrightarrow \widetilde{R}$ be the canonical embedding, then $\widetilde{R}$ is a GP$(\widetilde{\star})_{\widetilde{\iota}}$MD.*
3. *Each homogeneous overring T of R is a gr-$(\star_f)_\iota$-quasi-Prüfer domain, where $\iota : R \hookrightarrow T$ is the canonical embedding.*
4. *Each homogeneously $(\star,\star')$-linked overring T of R is a gr-$\star'_f$-quasi-Prüfer domain.*
5. *Every prime ideal of* $\mathrm{NA}(R,\star)$ *is extended from a homogeneous prime ideal of R.*
6. $\mathrm{NA}(R,\star_f)$ *is a quasi-Prüfer domain.*
7. *The integral closure of* $\mathrm{NA}(R,\star_f)$ *is a Prüfer domain.*
8. *R_P is a quasi-Prüfer domain, for each $P \in h\text{-}\mathrm{QMax}^{\star_f}(R)$ (or, for each $P \in h\text{-}\mathrm{QSpec}^{\star_f}(R)$).*

Proof. (6) $\Rightarrow$ (8). Let $P \in h\text{-}\mathrm{QMax}^{\star_f}(R)$ (or $P \in h\text{-}\mathrm{QSpec}^{\star_f}(R)$). Then $P\,\mathrm{NA}(R,\star)$ is a maximal ideal of $\mathrm{NA}(R,\star)$ [33, Proposition 2.3]. Since $R_P(X) = R_P[X]_{PR_P[X]} = \mathrm{NA}(R,\star)_{P\,\mathrm{NA}(R,\star)}$ and $\mathrm{NA}(R,\star)$ is a quasi-Prüfer domain, then $R_P(X)$ is a quasi-Prüfer domain by [9, Theorem 1.1 (1) $\Leftrightarrow$ (11)]. Then [9, Theorem 1.1 (1) $\Leftrightarrow$ (9)] implies that R_P is a quasi-Prüfer domain.

(8) $\Rightarrow$ (6). Let $Q \in \mathrm{Max}(\mathrm{NA}(R,\star))$. Then there exists a $P \in h\text{-}\mathrm{QMax}^{\widetilde{\star}}(R)$ such that $Q = P\,\mathrm{NA}(R,\star)$ and $\mathrm{NA}(R,\star)_Q = R_P(X)$ [33, Proposition 2.3]. Thus using [9, Theorem 1.1], one has $\mathrm{NA}(R,\star_f)$ is a quasi-Prüfer domain.

(1) $\Leftrightarrow$ (8) is Proposition 2.2.

(6) $\Rightarrow$ (2). Assume that $\mathrm{NA}(R,\star_f)(= \mathrm{NA}(R,\widetilde{\star}))$ is a quasi-Prüfer domain and thus the integral closure $\overline{\mathrm{NA}(R,\widetilde{\star})}$ is a Prüfer domain by [9, Theorem 1.1]. Note that $\overline{\mathrm{NA}(R,\widetilde{\star})} = \bar{R}[X]_{N(\widetilde{\star})}$, where $N(\widetilde{\star}) = N(\star_f) = \{g \in R[X] \mid \mathcal{A}_g^{\widetilde{\star}} = R^{\widetilde{\star}}\}$. Set $* := (\widetilde{\star})_{\widetilde{\iota}}$. Clearly $*$ is a (semi)star operation of finite type on $\widetilde{R}$. Moreover $\mathrm{NA}(\widetilde{R},*) = \widetilde{R}[X]_{\widetilde{N}}$ where $\widetilde{N} = \{h \in \widetilde{R}[X] \mid (\mathcal{A}_h^{\widetilde{R}})^* = \widetilde{R}\}$. Then $\widetilde{N}$ is a multiplicatively closed subset of $\widetilde{R}[X]$ and it is easy to see that $N(\widetilde{\star}) \subseteq \widetilde{N}$ (indeed if $f \in N(\widetilde{\star})$, then $\mathcal{A}_f^{\widetilde{\star}} = R^{\widetilde{\star}}$ and so $(\mathcal{A}_f^{\widetilde{R}})^* = (\mathcal{A}_f\widetilde{R})^{\widetilde{\star}} = \widetilde{R}$). Hence

$\overline{\mathrm{NA}(R,\widetilde{\star})} \subseteq \mathrm{NA}(\widetilde{R},*)$ and so $\mathrm{NA}(\widetilde{R},*)$ is a Prüfer domain by [20, Theorem 26.1]. Therefore $\widetilde{R}$ is a GP$*$MD by Proposition 2.8.

(2) $\Rightarrow$ (7). With the notation used in part (6) $\Rightarrow$ (2), since $\widetilde{R}$ is a GP$*$MD, we have $\mathrm{NA}(\widetilde{R},*)$ is a Prüfer domain by Proposition 2.8. The conclusion will trivially follows if we show that $\overline{\mathrm{NA}(R,\widetilde{\star})} = \mathrm{NA}(\widetilde{R},*)$, i.e., $\bar{R}[X]_{N(\widetilde{\star})} = \widetilde{R}[X]_{\widetilde{N}}$.

Note that $N(\widetilde{\star}) = R[X]\setminus\bigcup\{P[X] \mid P \in h\text{-QMax}^{\widetilde{\star}}(R)\}$, $\widetilde{N} = \widetilde{R}[X]\setminus\bigcup\{Q[X] \mid Q \in h\text{-Max}^*(\widetilde{R})\}$ and $\bar{R}[X]_{N(\widetilde{\star})} \subseteq \widetilde{R}[X]_{\widetilde{N}}$. By [9, Lemma 2.15(b)], the natural embedding $\widetilde{\iota}\colon R \hookrightarrow \widetilde{R}$ verifies $\widetilde{\star}$-INC and $\widetilde{\star}$-GU.

Let Q be a prime ideal of $\widetilde{R}$. We show that $Q \in h\text{-Max}^*(\widetilde{R})$ if and only if $Q \cap R \in h\text{-QMax}^{\widetilde{\star}}(R)$. First we show that if $P = Q \cap R$ is a quasi-$\widetilde{\star}$-prime ideal of R, then Q is a $*$-prime ideal of $\widetilde{R}$. Since $Q \in \overline{\mathcal{F}}(\widetilde{R}) \subseteq \overline{\mathcal{F}}(R)$, we have $Q^* = Q^{(\widetilde{\star})_{\widetilde{\iota}}} = Q^{\widetilde{\star}} = \bigcap_{P\in \mathrm{QSpec}^{\widetilde{\star}}(R)} QR_P$. Assume that $P = Q \cap R$ is a quasi-$\widetilde{\star}$-prime ideal of R and $x \in Q^*$. Then $x \in QR_P$. So there exist $a \in Q$ and $b \in R \setminus P$ such that $x = a/b$. Therefore $xb = a \in Q$ implies that $x \in Q$, since Q is a prime ideal of $\widetilde{R}$ and $b \in \widetilde{R} \setminus Q$ and $x \in Q^* \subseteq (\widetilde{R})^* = \widetilde{R}$. Therefore $Q^* \subseteq Q$ and so Q is a $*$-prime ideal of $\widetilde{R}$. Now assume that $P := Q \cap R \in h$-QMax$^{\widetilde{\star}}(R)$. Note that $P = P_h = (Q \cap R)_h = Q_h \cap R$ by Lemma 2.7, and Q_h is a homogeneous $*$-prime ideal by [32, Page 186]. If $Q_h \subsetneq Q$, then by $\widetilde{\star}$-INC we have $Q_h \cap R \subsetneq Q \cap R$, that is $P \subsetneq P$ which is a contradiction. Therefore Q is a homogeneous $*$-prime ideal of $\widetilde{R}$. Let $M \in h\text{-Max}^*(\widetilde{R})$ such that $Q \subsetneq M$. By $\widetilde{\star}$-INC we have $P = Q \cap R \subsetneq M \cap R$. Therefore $M \cap R \subseteq (M \cap R)^{\widetilde{\star}} \cap R = (M^{\widetilde{\star}} \cap R^{\widetilde{\star}}) \cap R \subseteq (M^{\widetilde{\star}} \cap \widetilde{R}) \cap R = (M^* \cap \widetilde{R}) \cap R = M \cap R$, which is a contradiction since $P \in h\text{-QMax}^{\widetilde{\star}}(R)$.

Conversely, assume that $Q \in h\text{-Max}^*(\widetilde{R})$ and that $P := Q \cap R \subsetneq P'$ for some $P' \in h\text{-QMax}^{\widetilde{\star}}(R)$. By $\widetilde{\star}$-GU, there exists a $*$-prime ideal Q' of $\widetilde{R}$ such that $Q' \cap R = P'$ and $Q \subsetneq Q'$. Note that using Lemma 2.7 and $\widetilde{\star}$-INC, we can assume that Q' is a homogeneous prime ideal, and this is a contradiction.

From the fact that $Q \in h\text{-Max}^*(\widetilde{R})$ if and only if $Q \cap D \in h\text{-QMax}^{\widetilde{\star}}(R)$, it can be seen that the ideals of $\bar{R}[X]$ that are maximal with respect to the property of being disjoint from $N(\widetilde{\star})$ are the ideals $\{(Q \cap \bar{R})[X] \mid Q \in h\text{-Max}^*(\widetilde{R})\}$. From this, [20, Proposition 4.8 and Theorem 4.10] and [30, Proposition 1.5], it follows easily that $\bar{R}[X]_{N(\widetilde{\star})} = \widetilde{R}[X]_{\widetilde{N}}$.

(7) $\Rightarrow$ (6) is true by [9, Theorem 1.1].

(1) $\Rightarrow$ (4). Assume that T is a homogeneously $(\star,\star')$-linked overring of R. Thus $\mathrm{NA}(T,\star'_f)$ is an overring of $\mathrm{NA}(R,\star_f)$ by [33, Lemma 2.8]. Since we already proved that (1) is equivalent to (7), we have $\mathrm{NA}(R,\star_f)$ has Prüfer integral closure. Hence $\mathrm{NA}(T,\star'_f)$ also has Prüfer integral closure. Therefore T is a gr-$\star'_f$-quasi-Prüfer domain.

(4) $\Rightarrow$ (3). Assume that T is a homogeneous overring of R. Then it can be seen that T is homogeneously $(\star_f, (\star_f)_\iota)$-linked overring of R. Hence T is a gr-$(\star_f)_\iota$-quasi-Prüfer domain.

(3) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (5). Let $\mathfrak{Q} = Q\,\mathrm{NA}(R,\star) = QR[X]_{N(\star)}$ be a prime ideal of $\mathrm{NA}(R,\star)$ for some prime ideal Q of $R[X]$ such that $Q \cap N(\star) = \emptyset$. In part (2) $\Rightarrow$ (7), we showed that $\overline{\mathrm{NA}(R,\star)} = \mathrm{NA}(\widetilde{R},\ast)$. So there exists a prime ideal $\mathcal{L}$ of $\mathrm{NA}(\widetilde{R},\ast)$ such that $\mathcal{L} \cap \mathrm{NA}(R,\star) = QR[X]_{N(\star)}$. Note that we proved (1) $\Leftrightarrow$ (2), hence $\widetilde{R}$ is a GP$\ast$MD. Thus by Proposition 2.8, there exists a homogeneous prime ideal L of $\widetilde{R}$ such that $\mathcal{L} = L\,\mathrm{NA}(\widetilde{R},\ast)$. Whence $L\,\mathrm{NA}(\widetilde{R},\ast) \cap \mathrm{NA}(R,\star) = QR[X]_{N(\star)}$ and intersecting with $R[X]$ one obtains that $Q = (L \cap R)[X]$. Note that $L \cap R$ is a homogeneous prime ideal of R such that $\mathfrak{Q} = (L \cap R)\,\mathrm{NA}(R,\star)$.

(5) $\Rightarrow$ (1). Suppose that R is not a gr-$\star_f$-quasi-Prüfer domain. Then by Lemma 2.5, there is an upper to zero Q in $R[X]$ such that $Q \cap N(\star) = \emptyset$. Hence $Q\,\mathrm{NA}(R,\star) = QR[X]_{N(\star)}$ is a proper prime ideal of $\mathrm{NA}(R,\star)$. Note that $Q\,\mathrm{NA}(R,\star) \neq P\,\mathrm{NA}(R,\star)$ for all nonzero homogeneous prime ideals P of R, since Q is an upper to zero in $R[X]$. This fact contradicts the assumption (5). □

The following corollary is immediate from Theorem 2.9, Proposition 2.2 and Lemma 2.5.

Corollary 2.10. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent:*

(1) *R is a gr-quasi-Prüfer domain.*
(2) *$\bar{R}$ is a graded-Prüfer domain.*
(3) *Each homogeneous overring T of R is a gr-quasi-Prüfer domain.*
(4) *Every prime ideal of $\mathrm{NA}(R)$ is extended from a homogeneous prime ideal of R.*
(5) *$\mathrm{NA}(R)$ is a quasi-Prüfer domain.*
(6) *The integral closure of $\mathrm{NA}(R)$ is a Prüfer domain.*
(7) *R_P is a quasi-Prüfer domain, for each $P \in h\text{-}\mathrm{Max}(R)$ (or, for each $P \in h\text{-}\mathrm{Spec}(R)$).*
(8) *$R_{H\setminus P}$ is a gr-quasi-Prüfer domain, for each $P \in h\text{-}\mathrm{Max}(R)$ (or, for each $P \in h\text{-}\mathrm{Spec}(R)$).*
(9) *Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $\mathcal{A}_g = R$.*
(10) *If Q is an upper to zero in $R[X]$, then $\mathcal{A}_Q = R$.*

Remark 2.11. Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Note that by [32, Proposition 2.3], $R^{\widetilde{\star}}$ is a homogeneous overring of R and let $\iota : R \hookrightarrow R^{\widetilde{\star}}$. Then exactly by the same way as the proof of [18, Corollary 3.5], one can show that $h\text{-}\mathrm{QMax}^{(\widetilde{\star})_\iota}(R^{\widetilde{\star}}) = \{QR_Q \cap R^{\widetilde{\star}} | Q \in h\text{-}\mathrm{QMax}^{\widetilde{\star}}(R)\}$, and hence $\mathrm{NA}(R,\star) = \mathrm{NA}(R^{\widetilde{\star}}, (\widetilde{\star})_\iota)$.

Proposition 2.12. *Let $\star$ be a semistar operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ such that $R^\star \subsetneq R_H$. Then the following statements are equivalent:*

(1) *R is a GP$\star$MD.*
(2) *R is a gr-$\star_f$-quasi-Prüfer domain and R_Q is integrally closed for all $Q \in h$-$\mathrm{QMax}^{\widetilde{\star}}(R)$.*
(3) *R is a gr-$\star_f$-quasi-Prüfer domain and $R^{\widetilde{\star}}$ is integrally closed.*

Proof. (1) $\Rightarrow$ (3) holds by Corollary 2.3 and Proposition 2.8.

(3) $\Rightarrow$ (1). Note that $\mathrm{NA}(R,\star) = \mathrm{NA}(R^{\widetilde{\star}}, (\widetilde{\star})_\iota)$ by Remark 2.11. On the other hand $\mathrm{NA}(R^{\widetilde{\star}}, (\widetilde{\star})_\iota)$ is integrally closed since $R^{\widetilde{\star}}$ is integrally closed and $\mathrm{NA}(R,\star)$ is a quasi-Prüfer domain by Theorem 2.9. Thus $\mathrm{NA}(R,\star)$ is a Prüfer domain and hence R is a GP$\star$MD by Proposition 2.8.

(1) $\Leftrightarrow$ (2) holds by [13, Proposition 3.8] and Corollary 2.3. □

3. Graded integral UMt-domains

Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and H be the set of nonzero homogeneous elements of R. In this section we give several new characterizations of UMt-domains. In particular we show that R is a UMt-domain if and only if R is a gr-t-quasi-Prüfer domain. We also connect the gr-$\star_f$-quasi-Prüfer domains to UMt-domains for (semi)star operation $\star$ on R.

Let D be an integral domain. Then D is a trivially graded domain with $\Gamma = \{0\}$, and each nonzero element of D is homogeneous, i.e., $H = D \setminus \{0\}$. Hence in this case, a prime ideal Q of $D[X]$ is an upper to zero if and only if $Q \cap H = \emptyset$. Also note that each upper to zero in $D[X]$ is a prime t-ideal. The following proposition is a useful graded version of the well-known result of Houston and Zafrullah [23, Theorem 1.4] (see also [19, Theorem 3.3]). Recall from [22, Proposition 4.3] that $(I[X])^t = I^t[X]$ for each fractional ideal I of R.

Proposition 3.1. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain and Q be a prime t-ideal in $R[X]$ such that $Q \cap H = \emptyset$. Consider the following statements.*

(1) *$(\mathcal{A}_Q)^t = R$.*
(2) *Q is a maximal t-ideal.*
(3) *Q is t-invertible.*

Then (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) and, if Q is an upper to zero, then (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2). Suppose that Q is not a maximal t-ideal, and let M be a maximal t-ideal of $R[X]$ which contains Q. Since the containment is proper, we have that $M \cap R \neq 0$. Then by [23, Proposition 1.1], $M = (M \cap R)[X]$ and $M \cap R$ is a t-ideal of R. Since $Q \subseteq M$, $\mathcal{A}_Q$ is contained in the t-ideal $M \cap R$, so that $(\mathcal{A}_Q)^t \neq R$.

(2) $\Rightarrow$ (1). Since $Q \cap H = \emptyset$ and $\mathcal{A}_Q$ is homogeneous, one has $Q \subsetneq \mathcal{A}_Q[X]$. Then $(\mathcal{A}_Q)^t = R$ using [22, Proposition 4.3].

(3) $\Rightarrow$ (2) is true by [23, Proposition 1.3].

Now assume that Q is an upper to zero in $R[X]$. Then (2) $\Rightarrow$ (3) is true by [23, Theorem 1.4]. □

In the following result which is the first main result of this section, we show that R is a gr-t-quasi-Prüfer domain if and only if R is a UMt-domain if and only if R_P is a quasi-Prüfer domain, for each homogeneous prime (or maximal) t-ideal P of R, which are new characterizations of UMt-domains.

Theorem 3.2. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a gr-t-quasi-Prüfer domain.*
(2) *Let Q be an upper to zero in $R[X]$, then $\mathcal{A}_Q \nsubseteq P$ for each $P \in h$-$\text{Spec}^t(R)$.*
(3) *Let Q be an upper to zero in $R[X]$, then $Q \nsubseteq P[X]$ for each $P \in h$-$\text{Spec}^t(R)$.*
(4) *R_P is a quasi-Prüfer domain for each $P \in h$-$\text{Spec}^t(R)$.*
(5) *$R_{H\setminus P}$ is a gr-quasi-Prüfer domain for each $P \in h$-$\text{Spec}^t(R)$.*
(6) *Each upper to zero in $R[X]$ contains a nonzero polynomial $g \in R[X]$ with $(\mathcal{A}_g)^t = R$.*
(7) *If Q is an upper to zero in $R[X]$, then $(\mathcal{A}_Q)^t = R$.*
(8) *Each upper to zero in $R[X]$ is a t-invertible.*
(9) *Each upper to zero in $R[X]$ is a maximal t-ideal.*
(10) *R is a UMt-domain.*

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) ⇔ (5) follows from Proposition 2.2.
(1) ⇔ (6) ⇔ (7) follows from Lemma 2.5.
(7) ⇔ (8) ⇔ (9) follows from Proposition 3.1.
(9) ⇔ (10) is the definition of UMt-domains [23]. □

Note that a consequence of Proposition 2.12 is that R is a graded-PvMD if and only if R is an integrally closed gr-t-quasi-Prüfer domain. Thus Theorem 3.2 implies the following corollary.

Corollary 3.3 ([1, Theorem 6.4])**.** *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then R is a graded-PvMD if and only if R is a PvMD.*

Proposition 3.4. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a UMt-domain.*
(2) *Every prime ideal of $\text{NA}(R, v)$ is extended from a homogeneous prime ideal of R.*
(3) *$\text{NA}(R, v)$ is a quasi-Prüfer domain.*
(4) *Each homogeneously (t_R, d_T)-linked overring T of R is a gr-quasi-Prüfer domain.*

Proof. The equivalences (1) ⇔ (2) ⇔ (3) are immediate from Theorems 2.9 and 3.2, and (1) ⇔ (4) follows from Theorem 2.9. □

Let D be an integral domain. A multiplicative subset S of D is called a *t-splitting set* if each $0 \neq d \in D$ can be written as $dD = (AB)^t$, where A and

B are integral ideals of D such that $A^t \cap sD = sA^t$ (equivalently, $(A, s)^t = D$) for all $s \in S$ and $B^t \cap S \neq \emptyset$. The notion of t-splitting sets was introduced in [3], where it is shown that S is a t-splitting set of D if and only if $dD_S \cap D$ is t-invertible for all $0 \neq d \in D$. It is known that D is a UMt-domain if and only if $D \setminus \{0\}$ is a t-splitting set of $D[X]$ [8, Corollary 2.9]. In the following theorem we generalized this result to the graded case among other things, which is the second main result in this section. Before that we need a lemma and for an ideal I of R set $C(I) := \sum_{a \in I} C(a)$.

Lemma 3.5. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain and I be an ideal of R. Then $\mathcal{A}_{I[X]} = C(I)$.*

Proof. Assume that $a \in I$. Then $aX \in I[X]$ and so $C(a) = \mathcal{A}_{aX} \subseteq \mathcal{A}_{I[X]}$. Hence $C(I) \subseteq \mathcal{A}_{I[X]}$. Conversely let $f = \sum_{i=0}^{n} a_i X^i \in I[X]$. Then $\mathcal{A}_f = \sum_{i=0}^{n} C(a_i) \subseteq C(I)$. Hence $\mathcal{A}_{I[X]} \subseteq C(I)$. □

Theorem 3.6. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

(1) *R is a UMt-domain.*
(2) *If Q is a prime t-ideal in $R[X]$ such that $Q \cap H = \emptyset$, then $(\mathcal{A}_Q)^t = R$.*
(3) *Each prime t-ideal Q in $R[X]$ such that $Q \cap H = \emptyset$, is a maximal t-ideal.*
(4) *H is a t-splitting set in $R[X]$.*

Proof. (1) $\Rightarrow$ (2). Assume that Q is a prime t-ideal of $R[X]$ such that $Q \cap H = \emptyset$. If Q is an upper to zero, then $(\mathcal{A}_Q)^t = R$ by Theorem 3.2. Otherwise $P := Q \cap R \neq 0$ and $P \cap H = \emptyset$. If $P[X] \subsetneq Q$, pick $q \in Q \setminus P[X]$, and let Q_1 be an upper to zero in $R[X]$ such that $q \in Q_1 \subseteq Q$ by [11, Theorem A]. Thus using Theorem 3.2, one has $R = (\mathcal{A}_{Q_1})^t \subseteq (\mathcal{A}_Q)^t \subseteq R$, and then $(\mathcal{A}_Q)^t = R$. Now assume that $Q = P[X]$ and that $(\mathcal{A}_Q)^t = (C(P))^t \subsetneq R$ (Lemma 3.5). Then there exists a homogeneous maximal t-ideal M of R such that $(C(P))^t \subseteq M$. So that $Q = P[X] \subseteq M[X]$ and hence $Q \cap N(v) = \emptyset$. Thus $Q\,\mathrm{NA}(R, v)$ is a proper prime ideal of $\mathrm{NA}(R, v)$ and there exists a homogeneous prime ideal P_0 of R such that $Q\,\mathrm{NA}(R, v) = P_0\,\mathrm{NA}(R, v)$ by Proposition 3.4. By intersecting this last equality with R, we have $P = P_0$, the desired contradiction, since $P \cap H = \emptyset$.

(2) $\Leftrightarrow$ (3) holds by Proposition 3.1.

(3) $\Rightarrow$ (1). Assume that Q is an upper to zero in $R[X]$. Then $Q \cap H = \emptyset$ and Q is a prime t-ideal. Hence Q is a maximal t-ideal by (3), which implies that R is a UMt-domain by Theorem 3.2.

(4) $\Rightarrow$ (2). Suppose that H is a t-splitting set in $R[X]$, and let Q be a prime t-ideal of $R[X]$ with $Q \cap H = \emptyset$. For any $0 \neq f \in Q$, let $fR[X] = (AB)^t$, where A and B are integral ideals of $R[X]$ such that $A^t \cap sR[X] = sA^t$ for all $s \in H$ and $B^t \cap H \neq \emptyset$. Since $Q \cap H = \emptyset$, $B \nsubseteq Q$; so $A \subseteq Q$. Thus if s is a nonzero homogeneous element in $\mathcal{A}_Q$, then $(A, s)^t \subseteq (\mathcal{A}_Q[X])^t = (\mathcal{A}_Q)^t[X]$ [22, Proposition 4.3]. Therefore $R = \mathcal{A}_{R[X]} = \mathcal{A}_{(A,s)^t} \subseteq \mathcal{A}_{(\mathcal{A}_Q)^t[X]} = (\mathcal{A}_Q)^t \subseteq R$ by Lemma 3.5, and then $(\mathcal{A}_Q)^t = R$.

(2) ⇒ (4). Let $0 \neq g \in R[X]$, and let $J = gR[X]_H \cap R[X] = gR_H[X] \cap R[X]$. By [3, Corollary 2.3], to show that H is a t-splitting set, it suffices to show that J is t-invertible. We first show that $(\mathcal{A}_J)^t = R$. Assume $(\mathcal{A}_J)^t \subsetneq R$, and let P be a maximal t-ideal of R containing $(\mathcal{A}_J)^t$. Note that we have $J \subseteq \mathcal{A}_J[X] \subseteq (\mathcal{A}_J)^t[X] \subseteq P[X]$. Let Q be a prime ideal of $R[X]$ minimal over J such that $Q \subseteq P[X]$. Then Q is a t-ideal (since J is a t-ideal of $R[X]$ by [26, Lemma 3.17]), and since P is homogeneous by [4, Lemma 1.2], we have $(\mathcal{A}_Q)^t \subseteq (\mathcal{A}_{P[X]})^t = C(P)^t = P$ using Lemma 3.5. Assume that $Q \cap H = \emptyset$. Then by the hypothesis $R = (\mathcal{A}_Q)^t \subseteq P$ which is a contradiction. Hence we have $Q \cap H \neq \emptyset$. Let $0 \neq x \in Q \cap H$. Then there are a $y \notin Q$ and a nonnegative integer n such that $yx^n \in J$ [24, Theorem 2.1], whence $y \in gR[X]_H \cap R[X] = J \subseteq Q$. This contradiction shows that $(\mathcal{A}_J)^t = R$.

Let $f_1, \dots, f_n \in J$ such that $(\mathcal{A}_{f_1} + \cdots + \mathcal{A}_{f_n})^t = R$, and let $I = (g, f_1, \dots, f_n)^t$ (so $IR_H[X] = gR_H[X] = JR_H[X]$ using [22, Proposition 4.3]). Let M be a maximal t-ideal of $R[X]$. If $M \cap H = \emptyset$, then M_H is a prime ideal of $R_H[X]$, and thus $IR[X]_M = (IR_H[X])_{M_H} = (gR_H[X])_{M_H} = (JR_H[X])_{M_H} = JR[X]_M$. If $M \cap H \neq \emptyset$, then $P := M \cap R \neq 0$, and $M = (M \cap R)[X] = P[X]$ by [23, Proposition 1.1]. Note that P is a homogeneous maximal t-ideal of R ([23, Proposition 1.1] and [4, Lemma 1.2]). If $I \subseteq M$, then $R = (\mathcal{A}_I)^t \subseteq (\mathcal{A}_M)^t \subseteq R$. But by Lemma 3.5, we have $(\mathcal{A}_M)^t = (\mathcal{A}_{P[X]})^t = (C(P))^t = P$ which is a contradiction. Therefore we have $I \nsubseteq M$. By the same reasoning $J \nsubseteq M$. Hence $IR[X]_M = R[X]_M = JR[X]_M$. Thus $J = I$ by [26, Proiposition 2.8], and since I is t-locally principal, $I = J$ is t-invertible by [26, Corollary 2.7]. □

Corollary 3.7 ([8, Corollary 2.9])**.** *Let D be an integral domain. Then D is a UMt-domain if and only if $D \setminus \{0\}$ is a t-splitting set of $D[X]$.*

A saturated multiplicative subset S of D is called a *splitting set* if for each $0 \neq d \in D$, $d = sa$ for some $s \in S$ and $a \in D$ with $aD \cap s'D = as'D$ for all $s' \in S$. The concept of splitting sets was introduced by Gilmer and Parker [21], where they proved that if S is a splitting set generated by prime elements, then D is a UFD if D_S is a UFD. Note that a t-splitting set of a GCD-domain is a splitting set. The following corollary gives a new characterization of (graded) GCD-domains. Recall from [1] that a graded integral domain $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ is a *graded GCD-domain* if each pair of nonzero homogeneous elements of R has a GCD.

Corollary 3.8. *Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent*:

1. *H is a splitting set in $R[X]$.*
2. *R is a graded GCD-domain.*
3. *R is a GCD-domain.*

Proof. (1) ⇒ (2). Assume that H is a splitting set in $R[X]$. Then for each $0 \neq f \in R[X]$, $f = ag$ where $a \in H$ and $g \in R[X]$ with $(g, s)_v = R[X]$ for all

$s \in H$. This means that $(\mathcal{A}_g)_v = R$ and so $(\mathcal{A}_f)_v = (\mathcal{A}_{ag})_v = a(\mathcal{A}_g)_v = aR$. Hence R is a graded GCD-domain.

(2) $\Rightarrow$ (3) holds by [1, Corollary 6.7].

(3) $\Rightarrow$ (1). Since R is a GCD-domain, H is a t-splitting set in $R[X]$ by Theorem 3.6, and hence is a splitting set in $R[X]$, because $R[X]$ is a GCD-domain. $\square$

The following theorem connects the gr-$\star_f$-quasi-Prüfer domains to UMt-domains for (semi)star operation $\star$ on R.

Theorem 3.9. *Assume that $\star$ is a (semi)star operation on R. Then the following statements are equivalent*:

1. *R is a gr-$\star_f$-quasi-Prüfer domain.*
2. *Each homogeneously $(\star_f, t)$-linked overring of R is a UMt-domain and each element of h-$\mathrm{Max}^{\star_f}(R)$ is a t_R-ideal.*
3. *R is a UMt-domain and each element of h-$\mathrm{Max}^{\widetilde{\star}}(R)$ is a t_R-ideal.*
4. *R is a UMt-domain and, $\widetilde{\star}$ and w_R coincide on nonzero homogeneous ideals.*

Proof. (1) $\Rightarrow$ (3). Since $\star_f \leq t_R$ and R is a gr-$\star_f$-quasi-Prüfer domain, then R is a gr-t_R-quasi-Prüfer domain and thus is a UMt-domain by Theorem 3.2. Let P be an element of h-$\mathrm{Max}^{\widetilde{\star}}(R)$. By Theorem 2.9, R_P is a quasi-Prüfer domain and by [9, Corollary 1.3], PR_P is a t-ideal of R_P. Thus $P = PR_P \cap R$ is a t_R-ideal by [26, Lemma 3.17].

(3) $\Rightarrow$ (4). The second part of (3) implies that h-$\mathrm{Max}^{\widetilde{\star}}(R) = h$-$\mathrm{Max}^{t_R}(R)$. Thus $\widetilde{\star}$ and w_R coincide on homogeneous ideals using [32, Proposition 2.6].

(4) $\Rightarrow$ (3). If $\widetilde{\star}$ and w_R coincide on homogeneous ideals, then h-$\mathrm{Max}^{\widetilde{\star}}(R) = h$-$\mathrm{Max}^{w_R}(R) = h$-$\mathrm{Max}^{t_R}(R)$ by [32, Proposition 2.5]. So that each element of h-$\mathrm{Max}^{\widetilde{\star}}(R)$ is a t_R-ideal.

(3) $\Rightarrow$ (1). Since each element of h-$\mathrm{Max}^{\widetilde{\star}}(R)$ is a t_R-ideal, one has h-$\mathrm{Max}^{\widetilde{\star}}(R) = h$-$\mathrm{Max}^{t_R}(R)$. Thus $N(\star) = N(t_R)$ and hence $\mathrm{NA}(R, \star) = \mathrm{NA}(R, t_R)$. Now Theorem 2.9, completes the proof.

(1) $\Rightarrow$ (2). Assume that T is a homogeneously $(\star_f, t_T)$-linked overring of R. Thus $\mathrm{NA}(T, t_T)$ is an overring of $\mathrm{NA}(R, \star_f)$ by [33, Lemma 2.8]. Using Theorem 2.9, we have $\mathrm{NA}(R, \star_f)$ has Prüfer integral closure. Hence $\mathrm{NA}(T, t_T)$ has Prüfer integral closure. Therefore T is a UMt-domain by Proposition 3.4. Moreover each element of h-$\mathrm{Max}^{\star_f}(R)$ is a t_R-ideal by (1) $\Rightarrow$ (3).

(2) $\Rightarrow$ (3) is trivial. $\square$

A homogeneous overring T of R is called a *homogeneously t-linked overring* of R if, it is homogeneously (t_R, t_T)-linked overring of R

Corollary 3.10. *A graded integral domain R is a UMt-domain if and only if each homogeneously t-linked overring of R is a UMt-domain.*

The following corollary shows that a graded integral domain R is a gr-quasi-Prüfer domain if and only if it is a UMt-domain and d_R and w_R coincide on homogeneous ideals.

Corollary 3.11. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$ be a graded integral domain. Then the following statements are equivalent:*

1. *R is a gr-quasi-Prüfer domain.*
2. *Each homogeneous overring T of R is a UMt-domain and each element of h-$\mathrm{Max}(R)$ is a t-ideal.*
3. *R is a UMt-domain and each element of h-$\mathrm{Max}(R)$ is a t-ideal.*
4. *R is a UMt-domain and, d_R and w_R coincide on nonzero homogeneous ideals.*

From Corollary 3.11, and the fact that height one primes are t-ideals we can show that if R is a one dimensional graded integral domain, then R is a gr-quasi-Prüfer domain if and only if R is a quasi-Prüfer domain. But it is not the case in general, see Example 3.14(2).

Lemma 3.12. *Let $\star$ be a (semi)star operation on a graded integral domain $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Then the following statements are equivalent:*

1. *R is a GP$\star$MD.*
2. *R is a PvMD and $\widetilde{\star}$ and t coincide on homogeneous ideals.*
3. *R is a PvMD and $\star_f$ and t coincide on homogeneous ideals.*

Proof. (1) $\Rightarrow$ (2). Since R is a GP$\star$MD, and $\star \leq v$, one has R is a GPvMD by [32], and hence R is a PvMD by [1, Theorem 6.4] (or Corollary 3.3). Also h-$\mathrm{Spec}^t(R) \subseteq h$-$\mathrm{Spec}^{\widetilde{\star}}(R)$. On the other hand if $P \in h$-$\mathrm{Spec}^{\widetilde{\star}}(R)$, then R_P is a valuation domain by [32, Theorem 4.4], and so P is a t-ideal of R by [27, Proposition 4.1]. This means that h-$\mathrm{Spec}^t(R) = h$-$\mathrm{Spec}^{\widetilde{\star}}(R)$. So that $\widetilde{\star}$ and $w = \widetilde{t}$ coincide on homogeneous ideals by [32, Proposition 2.6]. Note that in a PvMD, $t = w$.

(2) $\Rightarrow$ (3) is true since $\widetilde{\star} \leq \star_f \leq t$.

(3) $\Rightarrow$ (1) is clear. ☐

Corollary 3.13. *Let $R = \bigoplus_{\alpha\in\Gamma} R_\alpha$. Set $\widetilde{R} = (\bar{R})^{w_R}$ and let $\widetilde{\iota}\colon R \hookrightarrow \widetilde{R}$ be the canonical embedding. Then the following statements are equivalent:*

1. *R is a gr-t_R-quasi-Prüfer (or a UMt-)domain.*
2. *$\widetilde{R}$ is a GP$(w_R)_{\widetilde{\iota}}$MD.*
3. *$\widetilde{R}$ is a P$(w_R)_{\widetilde{\iota}}$MD.*
4. *$\widetilde{R}$ is a P$v_{\widetilde{R}}$MD and $(w_R)_{\widetilde{\iota}}$ and $w_{\widetilde{R}}(= t_{\widetilde{R}})$ coincide on homogeneous ideals.*
5. *$\widetilde{R}$ is a P$v_{\widetilde{R}}$MD and $(w_R)_{\widetilde{\iota}} = w_{\widetilde{R}} = t_{\widetilde{R}}$.*

Proof. (1) $\Leftrightarrow$ (2) is true by Theorem 2.9, (2) $\Leftrightarrow$ (4) holds by Lemma 3.12, and (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) holds by [9, Corollary 2.18]. ☐

In the following we give an example of a gr-quasi-Prüfer domain which is not a quasi-Prüfer domain.

Example 3.14. (1) Let $R = \bigoplus_{\alpha \in \Gamma} R_\alpha$. If d_R and w_R coincide on homogeneous ideals, then we do not have necessarily $d_R = w_R$. Let D be an integral domain, X be an indeterminate over D, and $R := D[X, X^{-1}]$. It is shown in [5, Example 3.6], that R is a graded-Prüfer domain if and only if D is a Prüfer domain and R is a Prüfer domain if and only if D is a field. Assume further that D is a non-field Prüfer domain. Then R is a graded-Prüfer domain. Thus by Lemma 3.12, R is a PvMD and d_R and w_R coincide on homogeneous ideals. If $d_R = w_R$, then R must be a Prüfer domain, and so D is a field, a contradiction.

(2) Assume that D is a non-Prüfer quasi-Prüfer domain (e.g. $D = K[Y^2, Y^3]$ for a field K and Y an indeterminate over K) and set $R := D[X, X^{-1}]$. Then $\bar{R} := \bar{D}[X, X^{-1}]$ is a graded-Prüfer domain and so R is a gr-quasi-Prüfer domain by Corollary 2.10. Now if R is a quasi-Prüfer domain, we have $\bar{D}$ is a field which implies that D is a field which is a contradiction.

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References

[1] D. D. Anderson and D. F. Anderson, *Divisorial ideals and invertible ideals in a graded integral domain*, J. Algebra **76** (1982), no. 2, 549–569.

[2] ———, *Divisibility properties of graded domains*, Canad. J. Math. **34** (1982), no. 1, 196–215.

[3] D. D. Anderson, D. F. Anderson and M. Zafrullah, *The ring $D+XD_S[X]$ and t-splitting sets*, Commutative Algebra Arab. J. Sci. Eng. Sect. C, Theme Issues **26** (2001), no. 1, 3–16.

[4] D. F. Anderson and G. W. Chang, *Homogeneous splitting sets of a graded integral domain*, J. Algebra **288** (2005), no. 2, 527–544.

[5] ———, *Graded integral domains and Nagata rings*, J. Algebra **387** (2013), 169–184.

[6] A. Ayache, P. Cahen, and O. Echi, *Anneaux quasi-Prüfériens et P-anneaux*, Boll. Un. Mat. Ital. B**10** (1996), no. 1, 1–24.

[7] G. W. Chang, *Graded integral domains and Prüfer-like domains*, Preprint, 2016.

[8] G. W. Chang, T. Dumitrescu, and M. Zafrullah, *t-splitting sets in integral domains*, J. Pure Appl. Algebra **187** (2004), no. 1-3, 71–86.

[9] G. W. Chang and M. Fontana, *Uppers to zero in polynomial rings and Prüfer-like domains*, Commun. Algebra **37** (2009), no. 1, 164–192.

[10] G. W. Chang and M. Zafrullah, *The w-integral closure of integral domains*, J. Algebra **295** (2006), no. 1, 195–210.

[11] A. De Souza Doering and Y. Lequain, *Chain of prime ideals in polynomial rings*, J. Algebra **78** (1982), no. 1, 163–180.

[12] D. E. Dobbs, E. G. Houston, T. G. Lucas, M. Roitman, and M. Zafrullah, *On t-linked overrings*, Commun. Algebra **20** (1992), no. 5, 1463–1488.

[13] D. E. Dobbs and P. Sahandi, *On semistar Nagata rings, Prüfer-like domains and semistar going-down domains*, Houston J. Math. **37** (2011), no. 3, 715–731.

[14] M. Fontana, S. Gabelli, and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Commun. Algebra **26** (1998), no. 4, 1017–1039.

[15] M. Fontana and J. A. Huckaba, *Localizing systems and semistar operations*, in: S. Chapman and S. Glaz (Eds.), Non Noetherian Commutative Ring Theory, Kluwer, Dordrecht, 2000, 169–197.
[16] M. Fontana, J. Huckaba, and I. Papick, *Prüfer Domains*, New York, Marcel Dekker, 1997.
[17] M. Fontana, P. Jara, and E. Santos, *Prüfer ⋆-multiplication domains and semistar operations*, J. Algebra Appl. **2** (2003), no. 1, 21–50.
[18] M. Fontana and K. A. Loper, *Nagata rings, Kronecker function rings and related semistar operations*, Commun. Algebra **31** (2003), no. 4, 4775–4805.
[19] S. Gabelli, E. Houston, and T. Lucas, *The t#-property for integral domains*, J. Pure Appl. Algebra **194** (2004), no. 3, 281–298.
[20] R. Gilmer, *Multiplicative Ideal Theory*, New York, Dekker, 1972.
[21] R. Gilmer and T. Parker, *Divisibility properties in semigroup rings*, Michigan Math. J. **21** (1974), 65–86.
[22] J. Hedstrom and E. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), no. 1, 37–44.
[23] E. Houston and M. Zafrullah, *On t-invertibility II*, Commun. Algebra **17** (1989), no. 8, 1955–1969.
[24] J. Huckaba, *Commutative Rings with Zero Divisors*, Dekker, New York, 1988.
[25] J. L. Johnson, *Integral closure and generalized transform in graded domains*, Pacific J. Math. **107** (1983), no. 1, 173–178.
[26] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra **123** (1989), 151–170.
[27] J. L. Mott and M. Zafrullah, *On Prüfer v-multiplication domains*, Manuscripta Math. **35** (1981), no. 1-2, 1–26.
[28] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge Univ. Press, Cambridge, 1968.
[29] A. Okabe and R. Matsuda, *Semistar-operations on integral domains*, Math. J. Toyama Univ. **17** (1994), 1–21.
[30] G. Picozza, *Star operations on overrings and semistar operations*, Commun. Algebra **33** (2005), no. 6, 2051–2073.
[31] P. Sahandi, *On quasi-Prüfer and UMt domains*, Commun. Algebra **42** (2014), 299–305.
[32] ———, *Characterizations of graded Prüfer ⋆-multiplication domains*, Korean J. Math. **22** (2014), 181–206.
[33] ———, *Characterizations of graded Prüfer ⋆-multiplication domains. II*, Bull. Iranian Math. Soc. to appear.

Haleh Hamdi
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz 51666-14766, Iran
Email address: h.hamdimoghadam@tabrizu.ac.ir

Parviz Sahandi
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz 51666-14766, Iran
Email address: sahandi@ipm.ir