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SUBGRADIENT ESTIMATES FOR A NONLINEAR SUBELLIPTIC EQUATION ON COMPLETE PSEUDOHERMITIAN MANIFOLD

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Abstract. Let (M, J, θ) be a complete pseudohermintian $(2n+1)$-manifold. In this paper, we derive the subgradient estimate for positive solutions to a nonlinear subelliptic equation $\triangle_b u + au\log u + bu = 0$ on M, where $a \leq 0$, b are two real constants.

1. Introduction

In [4] and [9], S. Y. Cheng and S. T. Yau derived a well known gradient estimate for positive harmonic functions in a complete noncompact Riemannian manifold.

Proposition 1.1 ([4, 9])**.** *Let M be a complete noncompact Riemannian m-manifold with Ricci curvature bounded from below by $-K(K \geq 0)$. If $u(x)$ is a positive harmonic function on M, then there exists a positive constant $C = C(m)$ such that $|\nabla f|^2 \leq C(\sqrt{K}+\frac{1}{R})$ on the ball $B(R)$ with $f(x) = \ln u(x)$. As a consequence, the Liouville theorem holds for complete noncompact Riemannian m-manifolds of nonnegative Ricci curvature.*

In [3], S.-C. Chang, T. J. Kuo and J. Z. Tie modified the arguments of [4], [9] and [1] and obtained the following result.

Theorem 1.2 ([3])**.** *Let (M, J, θ) be a complete noncompact pseudohermitian $(2n+1)$-manifold with*

$$2Ric(X,X) - (n-2)Tor(X,X) \geq -2k|X|$$

for all $Z \in T_{1,0}$ and $k \geq 0$. Furthermore, we assume that (M, J, θ) satisfies the CR sub-Laplacian comparison property (5). If $u(x)$ is a positive pseudoharmonic function with $[\triangle b, T] = 0$ on M. Then for each $b > 0$, there exists a

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positive constant $C_2 = C_2(K)$ *such that*

$$\frac{|\nabla_b u|^2}{u^2} + b\frac{u_0^2}{u^2} < \frac{(n+5+2bk)^2}{(5+2bk)}(k + \frac{2}{b} + \frac{C_2}{R}) \tag{1}$$

on the ball $B_{x_0}(R)$ *of a large enough radius* R *which depends only on* b, k.

On the other hand, for Riemannian case, L. Ma in [8] studied the local gradient estimate for the positive solution to the following equation:

$$\triangle u + au\log u + bu = 0$$

on a complete noncompact Riemannian manifold, where $a < 0$, b are real constants.

In this paper, we consider the following nonlinear subelliptic equation

$$\triangle_b u + au\log u + bu = 0 \tag{2}$$

in a complete pseudohermintian $(2n+1)$-manifold (M, J, θ). We obtain the following results:

Theorem 1.3 (cf. Theorem 3.1)**.** *Let* (M, J, θ) *be a complete pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric(X, X) - (n-2)Tor(X, X) \geq -2k|X|$$

for all $X \in T_{1,0} \oplus T_{0,1}$ *and* $k > 0$. *Furthermore, we assume that* (M, J, θ) *satisfies the CR sub-Laplacian comparison property* (5). *If* u *is the positive solution of*

$$\triangle_b u + au\log u = 0 \tag{3}$$

with $[\triangle_b, T] = 0$ *on* M, *let* $f(x) = \log u(x)$. *Then we have*

$$\begin{aligned} &\frac{|\nabla_b u|^2}{u^2} + a\frac{n+a\beta+4+\gamma+2\beta k}{n+a\beta}\ln u + \beta\frac{u_0^2}{u^2} \\ &< \frac{(n+a\beta+4+\gamma+2\beta k)^2}{2(4+\gamma+2\beta k)^2}[2k + \frac{4}{\beta} - a\frac{4+\gamma+2\beta k}{n+a\beta} + \frac{C}{R}] \end{aligned} \tag{4}$$

on the ball $B_{x_0}(R)$ *of large enough radius* R *which depends only on* a, β, γ, k, *where* $a \leq 0$, $\beta \geq 0$ *and* $\gamma > 0$ *are constants such that* $n + a\beta > 0$.

Remark 1.4. By replacing u by $e^{\frac{b}{a}}u$, equation (2) reduces to equation (3).

Remark 1.5. When $a = 0$ and $\gamma = 1$, the estimate (4) reduces to the estimate (1).

Remark 1.6. One of key steps in Yau's method for the proof of gradient estimates of harmonic function is the Bochner formula involving the Riemannian Ricci curvature tensor. In the CR analogue of the Yau's gradient estimate, the crucial step is then the CR Bochner formula 2.2. Note that the right-hand side involves a term $\langle J\nabla_b u, \nabla_b u_0\rangle$ that has no analogue in the Riemannian case, so we have to overcome this difficulty.

2. Preliminaries

We first introduced some basic materials in a pseudohermitian $(2n+1)$-manifold (see [6], [7] for more details). Let (M, ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contract structure ξ. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so are $[JX, Y] + [X, JY]$ and $J([JX, Y] + [X, JY]) = [JX, Y] - [X, Y]$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM \otimes C$, where Z_α is any local frame of $T_{1,0}$, $Z_{\bar{\alpha}} = \bar{Z}_\alpha \in T_{0,1}$ and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies $d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}$ for some positive definite hermitian matrix of function $(h_{\alpha\bar{\beta}})$. Actually we can always choose Z_α such that $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$; hence, through this note, we assume $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$.

The Levi form $\langle , \rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z, W \rangle_{L_\theta} = -i\langle d\theta, Z \wedge \bar{W} \rangle$. We can extend $\langle , \rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle \bar{Z}, \bar{W} \rangle_{L_\theta} = \langle Z, \bar{W} \rangle_{L_\theta}$ for all $Z, W\ inT_{0,1}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle , \rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form over M with respect to the volume form $d\mu = \theta \wedge (d\theta)^n$, we get an inner product on the space of sections of each tensor bundle. We denote the inner product by the notation $\langle , \rangle$.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes C$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \theta_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \theta_{\bar{\alpha}}^{\bar{\beta}} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where θ_α^β are the 1-forms uniquely determined by the following equations:

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \theta_\alpha^\beta + \theta \wedge \tau^\beta, \\ 0 &= \tau_\alpha \wedge \theta^\alpha, \\ 0 &= \theta_\alpha^\beta + \theta_{\bar{\beta}}^{\bar{\alpha}}. \end{aligned}$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma} = A_{\gamma\alpha}$. The curvature of Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar{\alpha}}\}$, is

$$\begin{aligned} \Pi_\beta^\alpha &= \overline{\Pi_{\bar{\beta}}^{\bar{\alpha}}} = d\omega_\beta^\alpha - \omega_\beta^\gamma \wedge \omega_\gamma^\alpha, \\ \Pi_0^\alpha &= \Pi_\alpha^0 = \Pi_0^{\bar{\beta}} = \Pi_{\bar{\beta}}^0 = \Pi_0^0 = 0. \end{aligned}$$

Webster showed that Π_β^α can be written

$$\Pi_\beta^\alpha = R_{\beta\rho\bar{\sigma}}^\alpha \theta^\rho \wedge \theta^{\bar{\sigma}} + W_{\beta\rho}^\alpha \theta^\rho \wedge \theta - W_{\beta\bar{\rho}}^\alpha \theta^{\bar{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy

$$R_{\beta\bar{\alpha}\rho\bar{\sigma}} = \overline{R_{\alpha\bar{\beta}\sigma\bar{\rho}}} = R_{\bar{\alpha}\beta\bar{\sigma}\rho} = R_{\rho\bar{\alpha}\beta\bar{\sigma}}, \quad W_{\beta\bar{\alpha}\gamma} = W_{\gamma\bar{\alpha}\beta}.$$

Here $R_{\delta\alpha\bar{\beta}}^\gamma$ is the pseudohermitian curvature tensor, $R_{\alpha\bar{\beta}} = R_{\gamma\alpha\bar{\beta}}^\gamma$ is the pseudohermitian Ricci curvature tensor and $A_{\alpha\beta}$ is the torsion tensor. We define

Ric and Tor by

$$Ric(X,Y) = R_{\alpha\bar{\beta}}X^{\alpha}Y^{\bar{\beta}}, \quad Tor(X,Y) = i(A_{\bar{\alpha}\bar{\beta}}X^{\bar{\alpha}}Y^{\bar{\beta}} - A_{\alpha\beta}X^{\alpha}Y^{\beta})$$

for $X = X^{\alpha}Z_{\alpha}$, $Y = Y^{\beta}Z_{\beta}$ on $T_{1,0}$.

We will denote components of covariant derivatives with indices preceded by comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0,\alpha,\bar{\alpha}\}$ indices derivatives with respect to $\{T, Z_{\alpha}, Z_{\bar{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma, for instance, $u_{\alpha} = Z_{\alpha}u$, $u_{\alpha\bar{\beta}} = Z_{\bar{\beta}}Z_{\alpha}u - \omega_{\alpha}^{\gamma}(Z_{\bar{\beta}})Z_{\gamma}u$. For a real function u, the subgradient ∇_b is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u\rangle = du(Z)$ for all vector fields Z tangent to contact plane. Locally $\nabla_b u = \sum_{\alpha} u_{\bar{\alpha}}Z_{\alpha} + \sum_{\alpha} u_{\alpha}Z_{\bar{\alpha}}$. We can use the connection to define the subhessian as the complex linear map $(\nabla^H)^2u : T_{1,0}\oplus T_{0,1} \to T_{1,0}\oplus T_{0,1}$ by

$$(\nabla^H)^2u(Z) = \nabla_Z\nabla_b u.$$

In particular, $|\nabla_b u|^2 = 2u_{\alpha}u_{\bar{\alpha}}$, $|\nabla_b^2 u|^2 = 2(u_{\alpha\beta}u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}}u_{\bar{\alpha}\beta})$ and $\triangle_b u = \sum_{\alpha}(u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha})$.

We also need the following definition:

Definition 2.1. Let (M, J, θ) be a complete noncompact pseudohermtian $(2n+1)$-manifold with

$$2Ric(X,X) - (n-2)Tor(X,X) \geq -2k|X|$$

for all $Z \in T_{1,0}$ and k is an nonnegative constant. We say that (M, J, θ) satisfies the CR sub-Lapacian comparison property if there exists a positive constant $C_0 = C_0(k, n)$ such that

$$\triangle_b r \leq C_0(\frac{1}{r} + \sqrt{k}) \tag{5}$$

in the sense of distributions. Here $r(x)$ is the Carnot-Cartheodory distance from a fixed point $x_0 \in M$.

We need the following lemmas.

Lemma 2.2 ([5])**.** *For a smooth real-valued function u and any $\nu > 0$, we have*

$$\begin{aligned}\triangle_b|\nabla_b u|^2 = {} & 2|(\nabla^H)^2\varphi|^2 + 2\langle\nabla_b u, \nabla_b\triangle_b u\rangle \\ & + 2(2Ric - (n-2)Tor)((\nabla_b u)_C, (\nabla_b u)_C) + 4\langle J\nabla_b u, \nabla_b u_0\rangle,\end{aligned} \tag{6}$$

where $(\nabla_b u)_C = u_{\overline{\alpha}}Z_{\alpha}$ is the corresponding complex $(1,0)$-vector of $\nabla_b u$.

From Lemma 2.2, the authors in [2, 3] obtained the following:

Lemma 2.3 ([2, 3])**.** *For a smooth real-valued function u and any $\nu > 0$, we have*

$$\begin{aligned}\triangle_b|\nabla_b u|^2 \geq {} & \frac{1}{n}(\triangle_b u)^2 + nu_0^2 + 2\langle\nabla_b u, \nabla_b\triangle_b u\rangle \\ & + 2(2Ric - (n-2)Tor)((\nabla_b u)_C, (\nabla_b u)_C)\end{aligned} \tag{7}$$

$$- 2v|\nabla_b u_0|^2 - \frac{2}{\nu}|\nabla_b u|^2,$$

where $(\nabla_b u)_C = u_{\overline{\alpha}} Z_\alpha$ *is the corresponding complex* $(1,0)$*-vector of* $\nabla_b u$*.*

Lemma 2.4. *Let* (M, J, θ) *be a pseudohermitian* $(2n+1)$*-manifold with* $[\triangle_b, T] = 0$*. If* $u(x)$ *is the positive solution of*

$$\triangle_b u + au \log u = 0, \tag{8}$$

then $f = \log u$ *satisfies*

$$\triangle_b f_0 = -af_0 - 2\langle \nabla_b f_0, \nabla_b f\rangle. \tag{9}$$

Proof. Since u is the solution of (8), we have

$$\triangle_b f = -af - |\nabla_b f|^2. \tag{10}$$

From $[\triangle_b, T] = 0$ and (15), we have

$$\triangle_b f_0 = [\triangle_b f]_0 = -af_0 - 2\langle \nabla_b f_0, \nabla_b f\rangle. \qquad \square$$

3. Subgradient estimates for a nonlinear subelliptic equation

In this section, we obtain the following results:

Theorem 3.1. *Let* (M, J, θ) *be a complete pseudohermitian* $(2n+1)$*-manifold. Suppose that*

$$2Ric(X, X) - (n-2)Tor(X, X) \geq -2k|X| \tag{11}$$

for all $X \in T_{1,0} \oplus T_{0,1}$ *and* $k > 0$*. Furthermore, we assume that* (M, J, θ) *satisfies the CR sub-Laplacian comparison property* (5)*. If* u *is the positive solution of*

$$\triangle_b u + au \log u = 0 \tag{12}$$

with $[\triangle_b, T] = 0$ *on* M*, let* $f(x) = \log u(x)$*. Then we have*

$$\begin{aligned} &\frac{|\nabla_b u|^2}{u^2} + a\frac{n + a\beta + 4 + \gamma + 2\beta k}{n + a\beta} \ln u + \beta\frac{u_0^2}{u^2} \\ &< \frac{(n + a\beta + 4 + \gamma + 2\beta k)^2}{2(4 + \gamma + 2\beta k)^2}[2k + \frac{4}{\beta} - a\frac{4 + \gamma + 2\beta k}{n + a\beta} + \frac{C}{R}] \end{aligned}$$

on the ball $B_{x_0}(R)$ *of large enough radius* R *which depends only on* a, β, γ, k*, where* $a \leq 0$*,* $\beta \geq 0$ *and* $\gamma > 0$ *are constants such that* $n + a\beta > 0$*.*

Proof. Since u is the positive solution of (12), we have

$$\triangle_b f(x,t) = -af(x,t) - |\nabla_b f(x,t)|^2. \tag{13}$$

Now we define a real-valued function $F(x, t, R, \alpha, \beta) : M \times [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^* \times \mathbb{R}^+ \to \mathbb{R}$ by

$$F(x, t, \alpha, \beta) = t(|\nabla_b f|^2 - \alpha af + \beta t\eta(x) f_0^2), \tag{14}$$

where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$, $\mathbb{R}^+ = (0,\infty)$ and $\eta : M \to [0,1]$ is a smooth cut-off function defined by

$$\eta(x) = \eta(r(x)) = \begin{cases} 1 & \text{if } x \in B_{x_0}(R), \\ 0 & \text{if } x \in M\backslash B_{x_0}(2R), \end{cases}$$

such that $-\frac{C}{R}\eta^{\frac{1}{2}} \leq \eta' \leq 0$ and $|\eta''| \leq \frac{C}{R^2}$, where we denote $\frac{\partial}{\partial}\eta$ by η' and $r(x)$ is the Carnot-Carathéodory distance to a fixed point x_0. By CR sub-Laplcian comparison property (5), we have

$$\triangle_b \eta = \eta'' + \eta' \triangle_b r \geq -\frac{C}{R^2} - \frac{C}{R}(\frac{C_1}{R} + C_2) \geq -\frac{C}{R}, \tag{15}$$

where C is a positive constant.

From Lemma 2.3, the assumption (11) and (15), we have

$$\begin{aligned}
\triangle_b F &= t(\triangle_b|\nabla_b f|^2 - \alpha a \triangle_b f + \beta t \triangle_b(\eta f_0^2)) \\
&\geq t[n f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - 2(k + \frac{1}{\nu})|\nabla_b f|^2 - 2\nu|\nabla_b f_0|^2 \\
&\quad - \alpha a \triangle_b f + \beta t(\eta \triangle_b f_0^2 + f_0^2 \triangle_b \eta + 4 f_0 \langle \nabla_b \eta, \nabla_b f_0\rangle)] \\
&\geq t[n f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - 2(k + \frac{1}{\nu})|\nabla_b f|^2 - 2\nu|\nabla_b f_0|^2 \\
&\quad - \alpha a \triangle_b f + 2\beta t \eta f_0 \triangle_b f_0 + 2\beta t \eta |\nabla_b f_0|^2 - \frac{C\beta t}{R} f_0^2 + 4\beta t f_0 \langle \nabla_b \eta, \nabla_b f_0\rangle] \\
&\geq t[n f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - 2(k + \frac{1}{\nu})|\nabla_b f|^2 - 2\nu|\nabla_b f_0|^2 \\
&\quad - \alpha a \triangle_b f + 2\beta t \eta f_0 \triangle_b f_0 + 2\beta t \eta |\nabla_b f_0|^2 - \frac{C\beta t}{R} f_0^2 - \beta t \eta |\nabla_b f_0|^2 \\
&\quad - 4\beta t \eta^{-1} |\nabla_b \eta|^2 f_0^2].
\end{aligned}$$

From the definition of η, we have $\eta^{-1}|\nabla_b \eta|^2 \leq \frac{C}{R^2}$. Now we take $v = \frac{\beta t \eta}{2}$, we have

$$\begin{aligned}
\triangle_b F \geq t[(n - \frac{C\beta t}{R}) f_0^2 + \frac{1}{n}(\triangle_b f)^2 + 2\langle \nabla_b f, \nabla_b \triangle_b f\rangle \\
- 2(k + \frac{2}{\beta t \eta})|\nabla_b f|^2 - \alpha a \triangle_b f + 2\beta t \eta f_0 \triangle_b f_0].
\end{aligned} \tag{16}$$

From Lemma 2.4 and the definition of F, we have

(17)

$$\begin{aligned}
&2\langle \nabla_b f, \nabla_b \triangle_b f\rangle - \alpha a \triangle_b f + 2\beta t \eta f_0 \triangle_b f_0 \\
= &2\langle \nabla_b f, \nabla_b[-af - |\nabla_b f|^2]\rangle - \alpha a \triangle_b f + 2\beta t \eta f_0[-a f_0 - 2\langle \nabla_b f, \nabla_b f_0\rangle] \\
= &-2a|\nabla_b f|^2 - 2\langle \nabla_b f, \nabla_b |\nabla_b f|^2\rangle - \alpha a \triangle_b f - 2a\beta t \eta f_0^2 - 4\beta t \eta f_0 \langle \nabla_b f, \nabla_b f_0\rangle \\
= &-2a|\nabla_b f|^2 - 2\langle \nabla_b f, \nabla_b(\frac{F}{t} + \alpha a f - \beta \eta t f_0^2)\rangle \\
&- \alpha a[\frac{F}{\alpha t} - (1 + \frac{1}{\alpha})|\nabla_b f|^2 - \frac{\beta t \eta}{\alpha} f_0^2] - 2a\beta t \eta f_0^2 - 4\beta t \eta f_0 \langle \nabla_b f, \nabla_b f_0\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -(1+\alpha)a|\nabla_b f|^2 - \frac{2}{t}\langle\nabla_b, \nabla_b F\rangle - \frac{aF}{t} + 2\beta t f_0^2\langle\nabla_b f, \nabla_b\eta\rangle - a\beta t\eta f_0^2\\
&\geq -(1+\alpha)a|\nabla_b f|^2 - \frac{2}{t}\langle\nabla_b, \nabla_b F\rangle - \frac{aF}{t} - 2\beta t f_0^2|\nabla_b f||\nabla_b\eta| - a\beta t\eta f_0^2\\
&\geq -(1+\alpha)a|\nabla_b f|^2 - \frac{2}{t}\langle\nabla_b, \nabla_b F\rangle - \frac{aF}{t} - \frac{2C\beta t}{R} f_0^2|\nabla_b f|\eta^{\frac{1}{2}} - a\beta t\eta f_0^2\\
&\geq -(1+\alpha)a|\nabla_b f|^2 - \frac{2}{t}\langle\nabla_b, \nabla_b F\rangle - \frac{aF}{t} - \frac{C\beta t}{R} f_0^2 - \frac{C\beta t}{R}\eta f_0^2|\nabla_b f|^2\\
&\quad - a\beta t\eta f_0^2.
\end{aligned}
$$

From (16) and (17), we have

$$
\begin{aligned}
\triangle_b F \geq\ & -2\langle\nabla_b f, \nabla_b F\rangle - aF + t[\frac{1}{n}(\triangle_b f)^2 + (n - \frac{C\beta t}{R} - a\beta t\eta) f_0^2\\
& -2[k + \frac{2}{\beta t\eta} + a(1+\alpha)]|\nabla_b f|^2 - \frac{C\beta t}{R}\eta f_0^2|\nabla_b f|^2].
\end{aligned}
\tag{18}
$$

From (15) and (18), we have

$$
\begin{aligned}
\triangle_b(\eta F) &= \triangle_b\eta F + 2\langle\nabla_b\eta, \nabla_b F\rangle + \eta\triangle_b F\\
&\geq -\frac{C}{R}F + 2\langle\nabla_b\eta, \nabla_b F\rangle - 2\eta\langle\nabla_b f, \nabla_b F\rangle - a\eta F\\
&\quad + t\eta[\frac{1}{n}(\triangle_b f)^2 + (n - \frac{C\beta t}{R} - a\beta t\eta) f_0^2\\
&\quad - 2[k + \frac{2}{\beta t\eta} + a(1+\alpha)]|\nabla_b f|^2 - \frac{C\beta t}{R}\eta f_0^2|\nabla_b f|^2].
\end{aligned}
\tag{19}
$$

From the definition of F, we have

(20)

$$
\begin{aligned}
(\triangle_b f)^2 &= (\frac{F}{\alpha t} - \frac{1+\alpha}{\alpha}|\nabla_b f|^2 - \frac{\beta t\eta}{\alpha} f_0^2)^2\\
&\geq \frac{F^2}{\alpha^2 t^2} - \frac{2(1+\alpha)}{\alpha^2 t}F|\nabla_b f|^2 - \frac{2\beta\eta}{\alpha^2}F f_0^2 + \frac{(1+\alpha)^2}{\alpha^2}|\nabla_b f|^4 + \frac{\beta^2 t^2\eta^2}{\alpha^2} f_0^4.
\end{aligned}
$$

From (19) and (20), we have

$$
\begin{aligned}
\triangle_b(\eta F) &\geq \frac{\eta F^2}{n\alpha^2 t} - \frac{C}{R}F + 2\langle\nabla_b\eta, \nabla_b F\rangle - 2\eta\langle\nabla_b f, \nabla_b F\rangle - a\eta F\\
&\quad + [n - \frac{C\beta t}{R} - a\beta t\eta - \frac{2\beta\eta}{n\alpha^2}F]t\eta f_0^2\\
&\quad + [-\frac{2(1+\alpha)}{n\alpha^2}\eta F - 2kt\eta - \frac{4}{\beta} - a(1+\alpha)t\eta]|\nabla_b f|^2\\
&\quad + [\frac{(1+\alpha)^2}{\alpha^2}|\nabla_b f|^4 + \frac{\beta^2 t^2\eta^2}{\alpha^2} f_0^4]\frac{t\eta}{n} - t\eta\frac{C\beta t}{R}\eta f_0^2|\nabla_b f|^2\\
&\geq \frac{\eta F^2}{n\alpha^2 t} - \frac{C}{R}F + 2\langle\nabla_b\eta, \nabla_b F\rangle - 2\eta\langle\nabla_b f, \nabla_b F\rangle - a\eta F
\end{aligned}
$$

$$
\begin{aligned}
&+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta\eta}{n\alpha^2}F]t\eta f_0^2\\
&+[-\frac{2(1+\alpha)}{n\alpha^2}\eta F-2kt\eta-\frac{4}{\beta}-a(1+\alpha)t\eta]|\nabla_b f|^2\\
&+t\eta\beta\frac{\beta^2t^2\eta^2}{n\alpha^2}[1-\frac{n^2\alpha^4}{4(1+\alpha)^2}\frac{C^2}{R^2}]f_0^4\\
\geq{}&\frac{\eta F^2}{n\alpha^2 t}-\frac{C}{R}F+2\langle\nabla_b\eta,\nabla_b F\rangle-2\eta\langle\nabla_b f,\nabla_b F\rangle-a\eta F\\
&+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta\eta}{n\alpha^2}F]t\eta f_0^2\\
&+[-\frac{2(1+\alpha)}{n\alpha^2}\eta F-2kt\eta-\frac{4}{\beta}-a(1+\alpha)t\eta]|\nabla_b f|^2
\end{aligned}
$$

when R is large enough such that $1-\frac{n^2\alpha^4}{4(1+\alpha)^2}\frac{C^2}{R^2}>0$. For $t\leq 1$ and $\eta\leq 1$, we have

$$
\begin{aligned}
t\eta\triangle_b(\eta F)\geq{}&\frac{(\eta F)^2}{n\alpha^2}-\frac{C}{R}(\eta F)+2t\eta\langle\nabla_b\eta,\nabla_b F\rangle-2t\eta^2\langle\nabla_b f,\nabla_b F\rangle-at\eta(\eta F)\\
&+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)]t^2\eta^2 f_0^2\\
&+[-\frac{2(1+\alpha)}{n\alpha^2}(\eta F)-2k-\frac{4}{\beta}-a(1+\alpha)t\eta]t\eta|\nabla_b f|^2.
\end{aligned}
\tag{21}
$$

Since $(\eta F)(x(t),t,\alpha,\beta)=\max_{x\in B_{x_0}(2R)}(\eta F)(x,t,\alpha,\beta)$ at a critical point $(x(t),$ $t)$ of (ηF), we have

$$
\nabla_b(\eta F)(x(t),t,\alpha,\beta)=0\quad\text{and}\quad\triangle_b(\eta F)(x(t),t,\alpha,\beta)\leq 0.
\tag{22}
$$

We claim that if $(\eta F)(x(t),t,\alpha,\beta)\leq 0$, the theorem is true. So we assume that $(\eta F)(x(t),t,\alpha,\beta)\geq 0$ at the point $(x(t),t)$. From (22), we have at the point $(x(t),t)$,

$$
2t\eta\langle\nabla_b\eta,\nabla_b F\rangle=-2tF|\nabla_b\eta|^2\geq-\frac{2tC}{R^2}\eta F\geq-\frac{2C}{R}\eta F
\tag{23}
$$

and

$$
\begin{aligned}
-2t\eta^2\langle\nabla_b f,\nabla_b F\rangle&=2t\eta F\langle\nabla_b f,\nabla_b\eta\rangle\\
&\geq-2t(\eta F)|\nabla_b f||\nabla_b\eta|\\
&\geq-2t\frac{C}{R}\eta^{\frac{1}{2}}(\eta F)|\nabla_b f|\\
&\geq-\frac{Ct}{R}(\eta F)^2-\frac{Ct}{R}\eta|\nabla_b f|^2.
\end{aligned}
\tag{24}
$$

From (21), (22), (23) and (24), we have at the point $(x(t),t)$

$$
0\geq(\frac{1}{n\alpha^2}-\frac{C}{R})(\eta F)^2-\frac{3C}{R}(\eta F)-at\eta(\eta F)
$$

$$\begin{aligned}
&+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)]t^2\eta^2 f_0^2 \\
&+[-\frac{2(1+\alpha)}{n\alpha^2}(\eta F)-2k-\frac{4}{\beta}-a(1+\alpha)t\eta-\frac{C}{R}]t\eta|\nabla_b f|^2.
\end{aligned} \tag{25}$$

Now we claim at $t=1$,

$$(\eta F)(x(1),1,\alpha,\beta)<\frac{n\alpha^2}{-2(1+\alpha)}[2k+\frac{4}{\beta}+a(1+\alpha)+\frac{C}{R}] \tag{26}$$

for a large enough R which to be determined later. Here $(1+\alpha)<0$ for some α to be chosen later.

We prove it by contradiction. Suppose not, that is,

$$(\eta F)(x(1),1,\alpha,\beta)\geq\frac{n\alpha^2}{-2(1+\alpha)}[2k+\frac{4}{\beta}+a(1+\alpha)+\frac{C}{R}].$$

Since $(\eta F)(x(t),t,\alpha,\beta)$ is continuous in the variable t and $(\eta F)(x(0),0,\alpha,\beta)=0$, by intermediate-value theorem there exists a $t_0\in(0,1]$ such that

$$(\eta F)(x(t_0),t_0,\alpha,\beta)=\frac{n\alpha^2}{-2(1+\alpha)}[2k+\frac{4}{\beta}+a(1+\alpha)+\frac{C}{R}]. \tag{27}$$

From (25) and (27), we have

$$\begin{aligned}
0 &\geq (\frac{1}{n\alpha^2}-\frac{C}{R})(\eta F)^2(x(t_0),t_0)-\frac{3C}{R}(\eta F)(x(t_0),t_0)-at\eta(\eta F)(x(t_0),t_0)\\
&\quad+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0)]t^2\eta^2 f_0^2\\
&\quad+[-\frac{2(1+\alpha)}{n\alpha^2}(\eta F)(x(t_0),t_0)-2k-\frac{4}{\beta}-a(1+\alpha)t\eta-\frac{C}{R}]t\eta|\nabla_b f|^2\\
&\geq (\frac{1}{n\alpha^2}-\frac{C}{R})(\eta F)^2(x(t_0),t_0)-\frac{3C}{R}(\eta F)(x(t_0),t_0)-at\eta(\eta F)(x(t_0),t_0)\\
&\quad+[n-\frac{C\beta t}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0)]t^2\eta^2 f_0^2\\
&\quad+[-\frac{2(1+\alpha)}{n\alpha^2}(\eta F)(x(t_0),t_0)-2k-\frac{4}{\beta}-a(1+\alpha)-\frac{C}{R}]t\eta|\nabla_b f|^2\\
&= (\frac{1}{n\alpha^2}-\frac{C}{R})(\eta F)^2(x(t_0),t_0)-\frac{3C}{R}(\eta F)(x(t_0),t_0)-at\eta(\eta F)(x(t_0),t_0)\\
&\quad+[n-\frac{C\beta}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0)]t^2\eta^2 f_0^2
\end{aligned}$$

that is

$$\begin{aligned}
0 &\geq (\frac{1}{n\alpha^2}-\frac{C}{R})(\eta F)^2(x(t_0),t_0)-\frac{3C}{R}(\eta F)(x(t_0),t_0)-at\eta(\eta F)(x(t_0),t_0)\\
&\quad+[n-\frac{C\beta}{R}-a\beta t\eta-\frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0)]t^2\eta^2 f_0^2.
\end{aligned} \tag{28}$$

From (28), we have

$$0 \geq (\frac{1}{n\alpha^2} - \frac{C}{R})(\eta F)^2(x(t_0),t_0) - \frac{3C}{R}(\eta F)(x(t_0),t_0) - at\eta(\eta F)(x(t_0),t_0)$$
$$+ [n - \frac{C\beta}{R} - a\beta t\eta - \frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0)]t^2\eta^2 f_0^2. \tag{29}$$

Moreover, we compute

$$\begin{aligned}
&(\frac{1}{n\alpha^2} - \frac{C}{R})(\eta F)(x(t_0),t_0) - \frac{3C}{R} - at_0\eta \\
&= (\frac{1}{n\alpha^2} - \frac{C}{R})(\frac{n\alpha^2}{-2(1+\alpha)})[2k + \frac{4}{\beta} + a(1+\alpha) + \frac{C}{R}] - \frac{3C}{R} - at_0\eta \\
&= \frac{-1}{2(1+\alpha)}(2k + \frac{4}{\beta} + a(1+\alpha)) - \frac{C}{R}[\frac{n\alpha^2}{-2(1+\alpha)}(2k + \frac{4}{\beta} \\
&\quad + a(1+\alpha) + \frac{C}{R}) + \frac{1}{2(1+a)} + 3] - at_0\eta \\
&\geq \frac{-1}{2(1+\alpha)}(2k + \frac{4}{\beta} + a(1+\alpha)) - \frac{C}{R}[\frac{n\alpha^2}{-2(1+\alpha)}(2k + \frac{4}{\beta} \\
&\quad + a(1+\alpha) + \frac{C}{R}) + \frac{1}{2(1+a)} + 3],
\end{aligned} \tag{30}$$

where we use the condition $a \leq 0$.

$$\begin{aligned}
&n - \frac{C\beta}{R} - a\beta t_0\eta - \frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0) \\
&\geq n - \frac{C\beta}{R} - \frac{2\beta}{n\alpha^2}(\eta F)(x(t_0),t_0) \\
&= n - \frac{C\beta}{R} - \frac{2\beta}{n\alpha^2}(\frac{n\alpha^2}{-2(1+\alpha)})[2k + \frac{4}{\beta} + a(1+\alpha) + \frac{C}{R}] \\
&= n - \frac{C\beta}{R} + \frac{\beta}{1+\alpha}[2k + \frac{4}{\beta} + a(1+\alpha) + \frac{C}{R}] \\
&= [n + \frac{\beta}{1+\alpha}[2k + \frac{4}{\beta} + a(1+\alpha)]] - \frac{C}{R}\frac{\alpha\beta}{1+\alpha}.
\end{aligned}$$

Now we choose α such that

$$(1+\alpha) < -\frac{4 + 2\beta k}{n + a\beta},$$

where we choose a such that $-a\beta < n$, so have

$$n + \frac{\beta}{1+\alpha}[2k + \frac{4}{\beta} + a(1+\alpha)] > 0.$$

Now we let

$$(1+\alpha) = -\frac{4 + \gamma + 2\beta k}{n + a\beta} \tag{31}$$

for some constant $\gamma > 0$. Then for $R = R(a, \beta, \gamma, k)$ large enough, we have

$$(\frac{1}{n\alpha^2} - \frac{C}{R})(\eta F)(x(t_0), t_0) - \frac{3C}{R} - at_0\eta > 0$$

and

$$n - \frac{C\beta}{R} - a\beta t_0\eta - \frac{2\beta}{n\alpha^2}(\eta F)(x(t_0), t_0) > 0.$$

This leads to a contradiction with (29). From (26) and (31), we have

$$(\eta F)(x(1), 1, \alpha, \beta) < \frac{(n+a\beta+4+\gamma+2\beta k)^2}{2(4+\gamma+2\beta k)^2}[2k + \frac{4}{\beta} - a\frac{4+\gamma+2\beta k}{n+a\beta} + \frac{C}{R}].$$

This implies

$$\begin{aligned} &\max_{x \in B_{x_0}(2R)}(|\nabla_b f|^2 + a\frac{n + a\beta + 4 + \gamma + 2\beta k}{n + a\beta} f + \beta\eta f_0^2) \\ < &\frac{(n + a\beta + 4 + \gamma + 2\beta k)^2}{2(4 + \gamma + 2\beta k)^2}[2k + \frac{4}{\beta} - a\frac{4 + \gamma + 2\beta k}{n + a\beta} + \frac{C}{R}]. \end{aligned}$$

When we fix on the set $x \in B_{x_0}(R)$, we have

$$\begin{aligned} &|\nabla_b f|^2 + a\frac{n + a\beta + 4 + \gamma + 2\beta k}{n + a\beta} f + \beta f_0^2 \\ < &\frac{(n + a\beta + 4 + \gamma + 2\beta k)^2}{2(4 + \gamma + 2\beta k)^2}[2k + \frac{4}{\beta} - a\frac{4 + \gamma + 2\beta k}{n + a\beta} + \frac{C}{R}] \end{aligned}$$

on $B_{x_0}(R)$. □

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References

[1] S.-C. Chang, T.-J. Kuo, and S.-H. Lai, *Li-Yau grandient estimate and entropy formula for the CR heat equation in a closed pseudohermitian manifold 3-manifold*, J. Differential Geometry **89** (2011), 185–216.

[2] ———, *CR Li-Yau gradient estimate and linear entropy formula for Witten Laplacian via Bakry-Emery pseudohermitian Ricci curature*, Preprint.

[3] S.-C. Chang, T.-J. Kuo, and J. Z. Tie, *Yau's gradient estimate and Liouville theroem for positive pseudoharmonic functions in a complete pseudohermitian manifold*, arXiv: 1506.03270v1[math.Ap], 10 Jun 2015.

[4] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975), no. 3, 333–354.

[5] A. Greenleaf, *The first eigenvalue of a sublaplacian on a pseudohermitian manifold*, Comm. Partial Differential Equations **10** (1985), no. 2, 191–217.

[6] J. M. Lee, *Pseudo-Einstein structure on CR Manifold*, Amer. J. Math. **110** (1988), no. 1, 157–178.

[7] ———, *The Fefferman metric and pseudohermitian invariants*, Trans. Amer. Math. Soc. **296** (1986), no. 1, 411–429.

[8] L. Ma, *Gradient estimate for a simple elliptic equation on complete noncompact Riemannian manifold*, J. Funct. Anal. **241** (2006), no. 1, 374–382.

[9] S. T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

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