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THIRD HANKEL DETERMINANTS FOR STARLIKE AND CONVEX FUNCTIONS OF ORDER ALPHA

Halit Orhan and Paweł Zaprawa

Abstract. In this paper we obtain the bounds of the third Hankel determinants for the classes $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{K}(\alpha)$ of convex functions of order α. Moreover,we derive the sharp bounds for functions in these classes which are additionally 2-fold or 3-fold symmetric.

1. Introduction

Let Δ be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be the family of all functions f analytic in Δ normalized by the condition $f(0) = f'(0) - 1 = 0$. It means that f has the expansion $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Pommerenke (see, [12, 13]) defined the q-th Hankel determinant for a function f as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \dots & \dots & \dots & \dots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}, \tag{1}$$

where $n, q \in \mathbb{N}$.

Following Pommerenke, many authors focused on the investigating of the second Hankel determinant $H_2(2) = a_2a_4 - {a_3}^2$ (see, e.g. [6–8, 10, 11]). Only a few papers have been devoted to the third Hankel determinant

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix},$$

(see, [2, 3, 16, 19, 20]). The results from these papers are far from accurate. In [21] it was proved that

Theorem 1. 1. *If $f \in \mathcal{S}^*$, then $|H_3(1)| \leq 1$,*
2. *If $f \in \mathcal{K}$, then $|H_3(1)| \leq \frac{49}{540} = 0.090\dots$.*

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Moreover, in [21] the sharp bounds for 2-fold and 3-fold symmetric starlike functions or convex functions were obtained. Recall that for a given class $A \subset \mathcal{A}$, a function $f \in A$ is said to be n-fold symmetric if $f(\varepsilon z) = \varepsilon f(z)$ holds for all $z \in \Delta$, where $\varepsilon = \exp(2\pi i/n)$ means the principal n-th root of 1. The set of all n-fold symmetric functions belonging to A is denoted by $A^{(n)}$. If $f \in A^{(n)}$, then f has the Taylor series expansion $f(z) = z + a_{n+1}z^{n+1} + a_{2n+1}z^{2n+1} + \cdots$. Certainly, the set $A^{(2)}$ consists of all functions in A which are odd. The definition of an n-fold symmetric function can be extended on functions f normalized by $f(0) = 1$.

The main aim of this paper is to discuss the third Hankel determinants for the classes $\mathcal{S}^*(\alpha)$ of starlike functions of order α and $\mathcal{K}(\alpha)$ of convex functions of order α.

Let us start with recalling the definitions. Let f, g be univalent and $\alpha < 1$. Then

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,$$

$$g \in \mathcal{K}(\alpha) \Leftrightarrow \operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > \alpha.$$

Obviously, $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$ are the classes of starlike functions and convex functions, respectively. Two particular choices of α are also interesting. For $\alpha = 1/2$ we know that $\mathcal{S}^*(1/2)$ contains $\mathcal{K}$ (see, [9,17]). The class $\mathcal{S}^*(1/2)$ plays important role in solving some differential equations (see, [5]). On the other hand, taking $\alpha = -1/2$ we obtain the class $\mathcal{K}(-1/2)$ consisting of functions which are close-to-convex, but not necessarily starlike. Umezawa proved ([18]) that functions in this class are convex in one direction. In [4], Bshouty and Lyzzaik showed the importance of $\mathcal{K}(-1/2)$ in the theory of harmonic functions. For other results for this class, see for example [1,15].

From (1) it follows that $f \in \mathcal{S}^*(\alpha)$ can be written in the form

$$\frac{zf'(z)}{f(z)} = \alpha + (1-\alpha)p(z), \tag{2}$$

where p belongs to the class $\mathcal{P}$ consisting of functions analytic in Δ for which $\operatorname{Re} p(z) > 0$.

Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ and $p(z) = 1 + p_1z + p_2z^2 + \cdots$ be in $\mathcal{S}^*$ and $\mathcal{P}$, respectively. Applying the correspondence (2), we can write

$$(n-1)a_n = (1-\alpha)\sum_{j=1}^{n-1} a_j p_{n-j}. \tag{3}$$

From (3) it follows that

$$\begin{aligned}
a_2 &= (1-\alpha)p_1,\\
a_3 &= \tfrac{1}{2}(1-\alpha)\left[p_2 + (1-\alpha){p_1}^2\right],\\
a_4 &= \tfrac{1}{3}(1-\alpha)\left[p_3 + \tfrac{3}{2}(1-\alpha)p_1p_2 + \tfrac{1}{2}(1-\alpha)^2{p_1}^3\right],
\end{aligned}$$

$$a_5 = \tfrac{1}{4}(1-\alpha)\left[p_4 + \tfrac{4}{3}(1-\alpha)p_1p_3 + \tfrac{1}{2}(1-\alpha){p_2}^2 + (1-\alpha)^2{p_1}^2p_2 + \tfrac{1}{6}(1-\alpha)^3{p_1}^4\right].$$

2. Preliminaries

To obtain our results we need the following sharp inequalities for functions $p \in \mathcal{P}$.

Lemma 1 ([14])**.** *If $p \in \mathcal{P}$, then the sharp estimate $|p_n| \le 2$ holds for $n = 1, 2, \ldots$.*

Lemma 2 ([6])**.** *If $p \in \mathcal{P}$, then the following estimate holds for $n, k = 1, 2, \ldots$, $n > k$*

$$|p_n - \mu p_k p_{n-k}| \le \begin{cases} 2 & \mu \in [0,1] \\ 2|2\mu - 1| & \mu \ge 1. \end{cases}$$

3. Bounds of $|H_3(1)|$ for $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$

At the beginning, observe that $H_3(1)$ can be written in the form

$$H_3(1) = (a_3a_5 - {a_4}^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - {a_3}^2). \tag{4}$$

Now, with help of (3), we can express $H_3(1)$ for $f \in \mathcal{S}^*$ as a polynomial of four variables: p_1, p_2, p_3, p_4 in the form

$$\begin{aligned} F(p_1,p_2,p_3,p_4) = \frac{(1-\alpha)^2}{144}\big[&-(1-\alpha)^4{p_1}^6 + 3(1-\alpha)^3{p_1}^4p_2 \\ &+8(1-\alpha)^2{p_1}^3p_3 - 9(1-\alpha)^2{p_1}^2{p_2}^2 - 18(1-\alpha){p_1}^2p_4 \\ &+24(1-\alpha)p_1p_2p_3 - 9(1-\alpha){p_2}^3 + 18p_2p_4 - 16{p_3}^2\big]. \end{aligned} \tag{5}$$

According to the Alexander theorem, $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. Therefore, if $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}^*$ and $g(z) = zf'(z) = z + b_2z^2 + b_3z^3 + \cdots \in \mathcal{K}$, then $nb_n = a_n$. Putting it into the definition of $H_3(1)$ for a convex function and applying the formulae obtained from (3) lead to $H_3(1) = G(p_1, p_2, p_3, p_4)$, where

$$\begin{aligned} G(p_1,p_2,p_3,p_4) = \frac{(1-\alpha)^2}{8640}\big[&-(1-\alpha)^4{p_1}^6 + 6(1-\alpha)^3{p_1}^4p_2 \\ &+12(1-\alpha)^2{p_1}^3p_3 - 21(1-\alpha)^2{p_1}^2{p_2}^2 - 36(1-\alpha){p_1}^2p_4 \\ &+36(1-\alpha)p_1p_2p_3 - 4(1-\alpha){p_2}^3 + 72p_2p_4 - 60{p_3}^2\big]. \end{aligned} \tag{6}$$

Now we can prove:

Theorem 2. 1. *If $f \in \mathcal{S}^*(\alpha)$, then*

$$|H_3(1)| \le \begin{cases} \frac{(1-\alpha)^2(18-\alpha)}{18} & \alpha \in [0,1) \\ \frac{(1-\alpha)^2(1-2\alpha)^2}{18}(18 - 3\alpha + 2\alpha^2) & \alpha \le 0, \end{cases}$$

2. *If* $f \in \mathcal{K}(\alpha)$, *then*

$$|H_3(1)| \le \begin{cases} \frac{(1-\alpha)^2(49-16\alpha)}{540} & \alpha \in [0,1) \\ \frac{(1-\alpha)^2}{540}(49-102\alpha+40\alpha^2-8\alpha^3) & \alpha \in [-3,0] \\ \frac{(1-\alpha)^2}{540}(46-88\alpha+21\alpha^2-4\alpha^3+4\alpha^4) & \alpha \le -3. \end{cases}$$

Proof. From (5),

$$\begin{aligned} F(p_1,p_2,p_3,p_4) = \frac{(1-\alpha)^2}{144}&\left[10(p_2-(1-\alpha){p_1}^2)(p_4-(1-\alpha){p_2}^2)\right.\\ &+8(p_2-(1-\alpha){p_1}^2)(p_4-(1-\alpha)p_1p_3)\\ &\left.+(1-\alpha)(p_2-(1-\alpha){p_1}^2)^3-16(p_3-(1-\alpha)p_1p_2)^2\right]. \end{aligned} \tag{7}$$

The triangle inequality and Lemma 2 lead to the declared bound for $f \in \mathcal{S}^*$.

If $f \in \mathcal{K}$, then, from (6),

$$\begin{aligned} G(p_1,p_2,p_3,p_4) = \frac{(1-\alpha)^2}{2160}&\left[4(1-\alpha){p_2}^3+6p_4(p_2-(1-\alpha){p_1}^2)\right.\\ &+9p_2(p_4-(1-\alpha){p_2}^2)\\ &+3(p_2-(1-\alpha){p_1}^2)(p_4-(1-\alpha)p_1p_3)\\ &-12p_3(p_3-(1-\alpha)p_1p_2)+3(1-\alpha){p_2}^2(p_2-(1-\alpha){p_1}^2)\\ &\left.-3{p_3}^2+(1-\alpha)(p_2-(1-\alpha){p_1}^2)^2(p_2-\tfrac{1}{4}(1-\alpha){p_1}^2)\right]. \end{aligned} \tag{8}$$

As above, it is enough to apply the triangle inequality and Lemma 1 and Lemma 2. □

Consequently,

Corollary 1. 1. $|H_3(1)| \le 35/144$ *for all* $f \in \mathcal{S}^*(1/2)$,
2. $|H_3(1)| \le 1$ *for all* $f \in \mathcal{S}^*$,
3. $|H_3(1)| \le 10$ *for all* $f \in \mathcal{S}^*(-1/2)$.

Corollary 2. 1. $|H_3(1)| \le 41/2160$ *for all* $f \in \mathcal{K}(1/2)$,
2. $|H_3(1)| \le 49/540$ *for all* $f \in \mathcal{K}$,
3. $|H_3(1)| \le 37/80$ *for all* $f \in \mathcal{K}(-1/2)$.

The authors of [3] proved that $|H_3(1)| \le 3.608\ldots$ for $f \in \mathcal{K}(-1/2)$. The estimate in Corollary 2, point 3, substantially improves this result.

4. Bounds of $|H_3(1)|$ for 2-fold and 3-fold symmetric functions

The results in Theorem 2 are not sharp. It is possible to derive sharp bounds considering functions satisfying an additional condition of n-fold symmetry. Observe that if $f \in A^{(3)}$, then $f(z) = z + a_4z^4 + a_7z^7 + \cdots$, and consequently $H_3(1) = -{a_4}^2$. Similarly, if $f \in A^{(2)}$, then $f(z) = z + a_3z^3 + a_5z^5 + \cdots$, so $H_3(1) = a_3(a_5 - {a_3}^2)$.

Theorem 3. 1. *If* $f \in \mathcal{S}^*(\alpha)^{(3)}$, *then* $|H_3(1)| \le \frac{4}{9}(1-\alpha)^2$,

2. *If* $f \in \mathcal{K}(\alpha)^{(3)}$*, then* $|H_3(1)| \le \frac{1}{36}(1-\alpha)^2$.
The bounds are sharp.

Proof. 1. Let $\tilde{f}(z) = \sqrt[3]{f(z^3)}$. Since

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)} = \frac{z^3 f'(z^3)}{f(z^3)},$$

it follows that

$$f \in \mathcal{S}^*(\alpha) \Leftrightarrow \tilde{f} \in \mathcal{S}^*(\alpha)^{(3)}.$$

Assuming that $f(z) = z + a_2 z^2 + \cdots$ and $\tilde{f}(z) = z + b_4 z^4 + \cdots$ we have $b_4 = a_2/3$. Hence, for $\tilde{f} \in \mathcal{S}^*(\alpha)^{(3)}$,

$$|H_3(1)| = |b_4|^2 = \frac{1}{9}|a_2|^2 = \frac{1}{9}(1-\alpha)^2|p_1|^2 \le \frac{4}{9}(1-\alpha)^2.$$

Equality holds for rotations of

$$\tilde{f}_0(z) = \frac{z}{(1-z^3)^{2(1-\alpha)/3}} = z + \tfrac{2}{3}(1-\alpha)z^4 + \cdots .$$

For this function,

$$\frac{z\tilde{f}_0{}'(z)}{\tilde{f}_0(z)} = \tilde{p}_0(z), \quad \tilde{p}_0(z) = \frac{1+(1-2\alpha)z^3}{1-z^3}.$$

2. Taking into account the relation $z\tilde{g}'(z) = \tilde{f}(z)$ valid for $\tilde{f} \in \mathcal{S}^*(\alpha)^{(3)}$ and $\tilde{g} \in \mathcal{K}(\alpha)^{(3)}$, we obtain the expansion $\tilde{g}(z) = z + \frac{b_4}{4}z^4 + \cdots$. Then for $\tilde{g} \in \mathcal{K}(\alpha)^{(3)}$,

$$|H_3(1)| = \frac{1}{16}|b_4|^2 = \frac{1}{144}|a_2|^2 \le \frac{1}{36}(1-\alpha)^2,$$

with equality for

$$\tilde{g}_0(z) = \int_0^z (1-\zeta^3)^{-2(1-\alpha)/3}\, d\zeta = z + \tfrac{1}{6}(1-\alpha)z^4 + \cdots .$$

Obviously,

$$1 + \frac{z\tilde{g}_0{}'(z)}{\tilde{g}_0(z)} = \tilde{p}_0(z).$$

□

In particular,

Corollary 3. 1. $|H_3(1)| \le 1/9$ *for all* $f \in \mathcal{S}^*(1/2)^{(3)}$,
2. $|H_3(1)| \le 4/9$ *for all* $f \in \mathcal{S}^{*(3)}$,
3. $|H_3(1)| \le 1$ *for all* $f \in \mathcal{S}^*(-1/2)^{(3)}$.

Corollary 4. 1. $|H_3(1)| \le 1/144$ *for all* $f \in \mathcal{K}(1/2)^{(3)}$,
2. $|H_3(1)| \le 1/36$ *for all* $f \in \mathcal{K}^{(3)}$,
3. $|H_3(1)| \le 1/16$ *for all* $f \in \mathcal{K}(-1/2)^{(3)}$.

Now, we turn to the case $n = 2$. For $f(z) = z + \alpha_3 z^3 + \alpha_5 z^5 + \cdots \in \mathcal{A}^{(2)}$ and real μ, let us define

$$\Phi_f(\mu) \equiv \left|\alpha_3\left(\alpha_5 - \mu{\alpha_3}^2\right)\right|. \tag{9}$$

It is clear that

$$|H_3(1)| = \Phi_f(1).$$

For a given f and real δ let us define $f_\delta(z) = e^{-i\delta} f(e^{i\delta} z)$. Then $f_\delta(z) = z + \alpha_3 e^{2i\delta} z^3 + \alpha_5 e^{4i\delta} z^5 + \cdots$ and

$$\Phi_{f_\delta}(\mu) = \left|e^{2i\delta}\alpha_3\left(e^{4i\delta}\alpha_5 - \mu e^{4i\delta}{\alpha_3}^2\right)\right| = \Phi_f(\mu).$$

It means that Φ_f is invariant under rotation.

In [21] the bounds of $\Phi_f(\mu)$ for ${\mathcal{S}^*}^{(2)}$ were found.

Theorem 4. *If $f \in {\mathcal{S}^*}^{(2)}$, then*

$$\Phi_f(\mu) \le \begin{cases} 1-\mu & \mu \le 2/3 \\ \frac{1}{3\sqrt{3(2\mu-1)}} & \mu \in [2/3, 1] \\ \frac{1}{3\sqrt{3(3-2\mu)}} & \mu \in [1, 4/3] \\ \mu - 1 & \mu \ge 4/3. \end{cases} \tag{10}$$

The estimate is sharp.

In order to find the analog of Theorem 4 for $\mathcal{S}^*(\alpha)^{(2)}$ we need to establish the correspondence between the coefficients of a function $f \in {\mathcal{S}^*}^{(2)}$ and a function $\tilde{f} \in \mathcal{S}^*(\alpha)^{(2)}$. Let

$$\tilde{f}(z) = z\left(\frac{f(z)}{z}\right)^{1-\alpha}, \ \alpha < 1. \tag{11}$$

From

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)} = \alpha + (1-\alpha)\frac{zf'(z)}{f(z)}, \tag{12}$$

we conclude that

$$f \in \mathcal{S}^* \Leftrightarrow \tilde{f} \in \mathcal{S}^*(\alpha).$$

This equivalence is valid also for the corresponding subclasses consisting of odd functions.

If $f(z) = z + a_3 z^3 + \cdots$ and $\tilde{f}(z) = z + b_3 z^3 + \cdots$, then, comparing the coefficients of both sides of

$$\begin{aligned} &\left(z + 3b_3 z^3 + 5b_5 z^5 + \cdots\right)\left(z + a_3 z^3 + a_5 z^5 + \cdots\right) \\ = \ &\left(z + b_3 z^3 + b_5 z^5 + \cdots\right)\left(z + (3-2\alpha)a_3 z^3 + (5-4\alpha)a_5 z^5 + \cdots\right), \end{aligned} \tag{13}$$

leads to

$$\begin{aligned} b_3 &= (1-\alpha)a_3, \\ b_5 &= (1-\alpha)a_5 - \tfrac{1}{2}\alpha(1-\alpha){a_3}^2. \end{aligned}$$

Hence, for $\tilde{f} \in \mathcal{S}^*(\alpha)^{(2)}$,

$$|H_3(1)| = \left|b_3\left(b_5 - {b_3}^2\right)\right| = (1-\alpha)^2 \left|a_3\left(a_5 - \tfrac{1}{2}(2-\alpha){a_3}^2\right)\right|.$$

Now, it is enough to apply Theorem 4 with $\mu = (2-\alpha)/2$. In this way we get the following theorem.

Theorem 5. *If $f \in \mathcal{S}^*(\alpha)^{(2)}$, then*

$$|H_3(1)| \le \begin{cases} \frac{1}{2}\alpha(1-\alpha)^2 & \alpha \in [2/3, 1) \\ \frac{1}{3\sqrt{3(1-\alpha)}}(1-\alpha)^2 & \alpha \in [0, 2/3] \\ \frac{1}{3\sqrt{3(1+\alpha)}}(1-\alpha)^2 & \alpha \in [-2/3, 0] \\ -\frac{1}{2}\alpha(1-\alpha)^2 & \alpha \le -2/3. \end{cases}$$

The estimate is sharp.

According to Theorem 4 and the correspondence (11), the extremal functions in Theorem 5 are: $f(z) = \frac{z}{(1-z^2)^{1-\alpha}}$ for $\alpha \in [2/3, 1)$ and $\alpha \le -2/3$, $f(z) = \frac{z}{[(1-z^2)^t(1+z^2)^{1-t}]^{1-\alpha}}$, $t = (1 + 1/\sqrt{3(1-\alpha)})/2$ for $\alpha \in [0, 2/3]$ and $f(z) = \frac{z}{(1-2tz^2+z^4)^{(1-\alpha)/2}}$, $t = 1/\sqrt{3(1+\alpha)}$ for $\alpha \in [-2/3, 0]$.

The similar theorem, but for $\mathcal{K}(\alpha)^{(2)}$, holds.

Theorem 6. *If $f \in \mathcal{K}(\alpha)^{(2)}$, then*

$$|H_3(1)| \le \begin{cases} \frac{8+\alpha}{270}(1-\alpha)^2 & \alpha \in [-2, 1) \\ \frac{1}{15\sqrt{3(1-\alpha)}}(1-\alpha)^2 & \alpha \in [-8, -2] \\ \frac{1}{15\sqrt{3(17+\alpha)}}(1-\alpha)^2 & \alpha \in [-14, -8] \\ -\frac{8+\alpha}{270}(1-\alpha)^2 & \alpha \le -14. \end{cases}$$

The estimate is sharp.

Proof. From equivalence

$$\tilde{g} \in \mathcal{K}(\alpha)^{(2)} \Leftrightarrow \tilde{f}(z) = z\tilde{g}'(z) \in \mathcal{S}^*(\alpha)^{(2)}, \tag{14}$$

where $\tilde{f}(z) = z + b_3 z^3 + \cdots$, $\tilde{g}(z) = z + c_3 z^3 + \cdots$ it follows that

$$c_3 = \tfrac{1}{3}b_3, \quad c_5 = \tfrac{1}{5}b_5,$$

so, for $\tilde{g} \in \mathcal{K}(\alpha)^{(2)}$, there is

$$\begin{aligned} H_3(1) = \left|c_3\left(c_5 - {c_3}^2\right)\right| &= \frac{1}{15}\left|b_3\left(b_5 - \tfrac{5}{9}{b_3}^2\right)\right| \\ &= \frac{1}{15}(1-\alpha)^2 \left|a_3\left(a_5 - \tfrac{1}{18}(10-\alpha){a_3}^2\right)\right|, \end{aligned}$$

where a_k are coefficients of $f \in \mathcal{S}^{*(2)}$ described above.

Applying Theorem 4 the claimed bound follows. The extremal functions can be derived from Theorem 5 and (14). □

From Theorem 5 and Theorem 6 we obtain what follows.

Corollary 5. 1. $|H_3(1)| \leq \sqrt{6}/36$ *for all* $f \in \mathcal{S}^*(1/2)^{(2)}$,
2. $|H_3(1)| \leq \sqrt{3}/9$ *for all* $f \in \mathcal{S}^{*(2)}$,
3. $|H_3(1)| \leq \sqrt{6}/4$ *for all* $f \in \mathcal{S}^*(-1/2)^{(2)}$.

Corollary 6. 1. $|H_3(1)| \leq 17/2160$ *for all* $f \in \mathcal{K}(1/2)^{(2)}$,
2. $|H_3(1)| \leq 4/135$ *for all* $f \in \mathcal{K}^{(2)}$,
3. $|H_3(1)| \leq 1/16$ *for all* $f \in \mathcal{K}(-1/2)^{(2)}$.

The estimates given in Corollaries 1-6 for $\mathcal{S}^{*(n)}$ and $\mathcal{K}^{(n)}$, $n = 1, 2, 3$ coincide with those results proved in [21] (Theorem 3.1 and Theorem 3.3).

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Halit Orhan
Department of Mathematics
Faculty of Science
Atatürk University
Erzurum, Turkey
Email address: horhan@atauni.edu.tr

Paweł Zaprawa
Department of Mathematics
Lublin University of Technology
Nadbystrzycka 38D, 20-618 Lublin, Poland
Email address: p.zaprawa@pollub.pl