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A STABILITY RESULT FOR P-CENTROID BODIES

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Abstract. In this paper, we prove a stability result for p-centroid bodies with respect to the Hausdorff distance. As its application, we show that the symmetric convex body is determined by its p-centroid body.

1. Introduction

The centroid body ΓK of a convex body $K \in \mathbb{R}^d$ is a classical notion from geometry (see e.g. [3, 7, 9, 11, 12, 23, 24]) that has attracted much attention in recent years. The name centroid body was first given and investigated by Petty [22], but the concept had previously appeared in work of Dupin, in connection with problems for floating bodies (see e.g., the book of Schneider [23], Section 7.4). If K is an origin symmetric convex body, it turns out that ΓK is bounded by the locus of the centroids of all the halves of K obtained by cutting K with hyperplanes through the origin.

The concept of a centroid body had a natural extension in what became known as the L_p Brunn-Minkowski theory and its dual (see e.g., [13, 14]). For each real number $p \geq 1$, the p-centroid body $\Gamma_p K$ of a convex body K is defined by its support function:

$$h_{\Gamma_p K}(x) = \Big(\frac{1}{V(K)} \int_K |x \cdot y|^p dy\Big)^{\frac{1}{p}}, \tag{1}$$

where the integration is with respect to Lebesgue measure. This body is homothetic to the p-centroid body defined by Lutwak and Zhang in [18] (see also [15]).

For $p = 1$, the set $\Gamma_1 K$ is known in the literature as the centroid body $\Gamma K$of K.

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For $p = 2$, $\Gamma_2 K$ is homothetic to the Legendre ellipsoid of K, which arises in classical mechanics in connection with the moments of inertia of K (see e.g., [19]).

$\Gamma_\infty K$ is interpreted as a limit of (1), as $p \to \infty$, then $\Gamma_\infty K = conv(K \cup (-K))$, where *conv* stands for the convex hull. Such a body was investigated by Fáry and Rédei [3] in the framework of affine inequalities related to the geometry of numbers.

The p-centroid bodies have recently been studied by different authors (see e.g. [1,2,4,5,8,15,16,18,20,21,26,27] etc.). In [5], Gardner and Giannopoulos established an inclusion for p-cross-section bodies and applied this to disprove a conjecture of Makai and Martini. In [18], Lutwak and Zhang established the centro-affine inequality involving the volumes of K and its polar p-centroid body. The L_p Busemann-Petty centroid inequality was established by Lutwak, Yang and Zhang in [15] with an independent approach presented by Campi and Gronchi [1]. In [4], Fleury, Guédon and Paouris proved a type of stability result for p-centroid bodies with respect to the geometric distance. The Orlicz Brunn-Minkowski theory originated with the work of Lutwak, Yang and Zhang in 2010 and the topics of Orlicz centroid bodies are treated in [17,25].

The geometric distance between two symmetric convex bodies K and L is defined by

$$d(K,L) = \inf\{ab \mid a, b > 0 \text{ and } \frac{1}{a}K \subset L \subset bK\}.$$

In [4], Fleury, Guédon and Paouris proved the following stability result for origin symmetric convex bodies.

Theorem A ([4]). *There exists a constant $c > 0$ such that for every integer d greater than 3 and any odd integer $p \le d$, we have the following property: if K is a symmetric convex body in $\mathbb{R}^d$ such that for some $a > 1$ and $\varepsilon \in (0, (ca)^{-2d^3})$,*

$$d(K, B^d) \le a \quad \textit{and} \quad d(\Gamma_p \widetilde{K}, \Gamma_p \widetilde{B^d}) \le 1 + \varepsilon,$$

where $\widetilde{K} = V(K)^{-\frac{1}{d}} K$ and $\widetilde{B^d} = V(B^d)^{-\frac{1}{d}} B^d$, then

$$d(K, B^d) \le 1 + h(\varepsilon) \quad \textit{and} \quad (1 - h(\varepsilon))\Gamma_p \widetilde{B^d} \subset \Gamma_p \widetilde{K} \subset (1 + h(\varepsilon))\Gamma_p \widetilde{B^d},$$

where $h(\varepsilon) = (ca)^{d+p+1} \varepsilon^{\frac{1}{d^2}}$.

In this paper, we will prove a stability result for the convex bodies from their p-centroid bodies with respect to the Hausdorff distance $\delta(K, L)$. Let $\mathscr{K}_e^d(r, R)$ be the set of origin symmetric convex bodies K satisfying $rB^d \subset K \subset RB^d$.

Our main result is the following theorem.

Theorem 1. *Let $K, L \in \mathscr{K}_e^d(r, R)$ such that $V(K) = V(L)$, $p \ge 1$, $p \ne 2k$, $k \in \mathbb{N}$. If, for some $\varepsilon \ge 0$,*

$$\delta(\Gamma_p K, \Gamma_p L) \le \varepsilon,$$

then

$$\delta(K,L) \le c(d,r,R,p)\varepsilon^{\frac{2}{(d+1)(d+4)}}$$

with an explicit constant $c(d,r,R,p)$ *depending only on* d,r,R,p.

From Theorem 1, we obtain the following corollary.

Corollary. *Let* $K, L \in \mathscr{K}_e^d(r,R)$ *such that* $V(K) = V(L)$. *If for some* $p \ge 1$, $p \ne 2k$, $k \in \mathbb{N}$, $\Gamma_p K = \Gamma_p L$, *then* $K = L$.

2. Background and materials

For quick later reference, we collect in this section background materials regarding convex bodies (see e.g., the book of Schneider [23]). We also state some known facts about spherical harmonics (see e.g., the book of Groemer [6]).

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space with corresponding Euclidean norm $|\cdot|$. Let B^d denote the origin centered standard unit ball in $\mathbb{R}^d$. The set S^{d-1} is the unit sphere of $\mathbb{R}^d$ and σ is its spherical Lebesgue measure. Write κ_d for $V(B^d)$ the volume of B^d and ω_d for $\sigma(S^{d-1})$ the surface area of B^d.

A set K in $\mathbb{R}^n$ is a star-shaped (about the origin) if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin. The radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, of a compact, star-shaped (about the origin) $K \in \mathbb{R}^n$, is defined by

$$\rho_K(v) := \max\{\lambda \ge 0 : \lambda v \in K\} \qquad \text{for } v \in \mathbb{R}^d \setminus \{0\}.$$

If ρ_K is positive and continuous, K is called a star body (about the origin).

A convex body K is a compact, convex set with non-empty interiors. Let $V(K)$ denote the volume of K. Associated with a convex body K is its support function h_K defined for $x \in \mathbb{R}^d$ by $h_K(x) := \max\{x \cdot y : y \in K\}$. The function h_K is positively homogeneous of degree 1. We will usually be concerned with the restriction of the support function to the unit sphere S^{d-1}.

The Hausdorff distance between convex bodies K, L is defined by

$$\delta(K,L) := \min\{\lambda \ge 0 | K \subset L + \lambda B^d, L \subset K + \lambda B^d\}.$$

In terms of the support function, the Hausdorff distance between convex bodies K, L can also be expressed as follows,

$$\delta(K,L) = \max_{u \in S^{d-1}} |h_K(u) - h_L(u)|. \tag{2}$$

If f is a real valued function on S^{d-1}, let

$$f^+(u) := \frac{1}{2}(f(u) + f(-u)), \qquad f^-(u) := \frac{1}{2}(f(u) - f(-u)).$$

Then, $f = f^+ + f^-$, and f^+ is an even function and f^- is an odd function on S^{d-1}.

For each real number $p \geq 1$, let $\mathscr{C}_p$ denote the p-cosine transformation on S^{d-1}, i.e., for each bounded integrable function f on S^{d-1}, let $\mathscr{C}_p f$ be the function defined by

$$\mathscr{C}_p(f)(u) = \int_{S^{d-1}} |u \cdot v|^p f(v) d\sigma(v)$$

for $u \in S^{d-1}$.

Let $L_2(S^{d-1})$ denote the class of all real valued Lebesgue integral functions f on S^{d-1} with the property that $\int_{S^{d-1}} f^2(u) d\sigma(u) < \infty$. If $f, g \in L_2(S^{d-1})$, the inner product $\langle f, g\rangle$ is defined by $\langle f, g\rangle = \int_{S^{d-1}} f(u)g(u)d\sigma(u)$. Let $\|\cdot\|$ denote the norm derived from this inner product. Two functions f, g from $L_2(S^{d-1})$ are said to be orthogonal if $\langle f, g\rangle = 0$. A sequence $H_0, H_1, \ldots$ with $H_i \in L_2(S^{d-1})$ and $\|H_i\| \neq 0$, for all i, will be called an orthogonal sequence if $\langle H_i, H_j\rangle = 0$ whenever $i \neq j$. If $f \in L_2(S^{d-1})$ and $H_0, H_1, \ldots$ is a given orthogonal sequence, then $\sum_{i=0}^{\infty} \alpha_i H_i$ is called the Fourier series of f, where the numbers $\alpha_i = \frac{\langle f, H_i\rangle}{\|H_i\|^2}$. To indicate that $\sum_{i=0}^{\infty} \alpha_i H_i$ is the Fourier series of a given function f, we write

$$f \sim \sum_{i=0}^{\infty} \alpha_i H_i. \tag{3}$$

Let ∇ denote the gradient. If f is a function whose domain is a subject of $\mathbb{R}^d$ that contains S^{d-1}, we write $f^{\wedge}$ for the restriction of f to S^{d-1}. On the other hand, if f is defined on S^{d-1}, we let $f^{\vee}$ denote the radial extension of f to $\mathbb{R}^d \setminus \{0\}$. This means that $f^{\vee}(x) = f(\frac{x}{|x|})$. Using the above extension procedure one can transfer the gradient to the operator acting on functions on S^{d-1}. We define ∇_o by $\nabla_o f = (\nabla f^{\vee})^{\wedge}$. So $\nabla_o f$ exists if f is, respectively, twice or once differentiable.

Let $\mathscr{H}_n^d$ denote the space of all spherical harmonics of degree n in d variables and $\mathscr{H}^d$ the space of all finite sums of spherical harmonics of dimension d. If $H_0, H_1, \ldots$ is a standard sequence of d-dimensional spherical harmonics, the relation of (3) will also be expressed by saying that $\sum_{i=0}^{\infty} \alpha_i H_i$ is a harmonic expansion of f. We define $Q_n = \sum_{\chi(H_i)=n} \alpha_i H_i$, where $\chi(H_i)$ is the order of the spherical harmonic H_i, and call $\sum_{n=0}^{\infty} Q_n$ the condensed harmonic expansion, or simply the condensed expansion of f and write again

$$f \sim \sum_{n=0}^{\infty} Q_n. \tag{4}$$

From the definition of the Fourier coefficients, it follows immediately that

$$\|f - \sum_{i=0}^{m} \alpha_i H_i\|^2 = \|f\|^2 - \sum_{i=0}^{m} \alpha_i^2 \|H_i\|^2. \tag{5}$$

An obvious consequence of (5) is the fact that equality $\lim_{m\to\infty}\|f-\sum_{i=0}^{m}\alpha_i H_i\| = 0$ holds if and only if $\|f\|^2=\sum_{i=0}^{\infty}\alpha_i^2\|H_i\|^2$. The latter relation is called Parseval's equation.

If f is twice differentiable, from (4) and Parseval's equation, one can have the following classical conclusion

$$\|\nabla_o f\|^2=\sum_{n=1}^{\infty} n(n+d-2)\|Q_n\|^2. \tag{6}$$

Lemma 2.1 ([6]). *Let f and g be two continuous functions on S^{d-1} with respective contensed harmonic expansions*

$$f\sim\sum_{n=0}^{\infty}Q_n,\qquad g\sim\sum_{n=0}^{\infty}\xi_n Q_n,$$

and assume that $Q_n=0$ whenever $\xi_n=0$. If ξ signifies the sequence $\xi_0,\xi_1,\dots$ and if $t>0$ is given, let $\Gamma(f,\xi,t)$ be defined by

$$\Gamma(f,\xi,t)=\sum_{\xi_n\neq 0}|\xi_n|^{-t}\|Q_n\|^2, \tag{7}$$

provided this series converges. Then

$$\|f\|\le\Gamma(f,\xi,t)^{\frac{1}{t+2}}\|g\|^{\frac{t}{t+2}}. \tag{8}$$

3. Proof of the main result

The following result is crucial to prove our stability result for p-centroid bodies. A simple proof is given by Kolodobsky [10].

Lemma 3.1. *If $H\in\mathscr{H}_n^d$ $(d\ge 2)$ and $p\ge 1$, $p\neq 2k$, $k\in\mathbb{N}$, then, for all $u\in S^{d-1}$,*

$$\mathscr{C}_p(H)(u)=\int_{S^{d-1}}|u\cdot v|^p H(v)d\sigma(v)=\lambda_{d,n}H(u),$$

where

$$\lambda_{d,n}=\frac{\pi^{\frac{d}{2}-1}\Gamma(p+1)\sin(\frac{\pi(n-p)}{2})\Gamma(\frac{n-p}{2})}{2^{p-1}\Gamma(\frac{d+n+p}{2})}$$

for even n and $\lambda_{d,n}=0$ for odd n.

The following result was obtained by Greomer [6] for $p=1$. Along the same approach, we extend it to all $p\ge 1$, $p\neq 2k$, $k\in\mathbb{N}$. For reader's convenience, we present the proof it here.

Lemma 3.2. *If f_1 and f_2 are twice continuously differentiable functions on $S^{d-1}(d\ge 2)$ and $p\ge 1$, $p\neq 2k$, $k\in\mathbb{N}$, then*

$$\|f_1^+-f_2^+\|^2\le\psi(f_1,f_2)\|\mathscr{C}_p(f_1)-\mathscr{C}_p(f_2)\|^{\frac{4}{d+4}}, \tag{9}$$

with $\psi(f_1,f_2)=\lambda_{d,0}^{-2-\frac{2}{d+4}}\|\mathscr{C}_pf_1-\mathscr{C}_pf_2\|^2+\frac{2}{c_p}(\nabla_o\|f_1\|^2+\nabla_o\|f_2\|^2)$ and $c_p>0$ is a constant.

Proof. If we write the condensed expansion of $f_i (i = 1, 2)$ in the form

$$f_i \sim \sum_{n=0}^{\infty} Q_n^i,$$

then

$$f_1^+ - f_2^+ \sim \sum_{n=2k, k\in\mathbb{N}} (Q_n^1 - Q_n^2). \tag{10}$$

To prove (9), we let λ denote the sequence $\lambda_{d,0}, \lambda_{d,2}, \lambda_{d,4}, \ldots$ with $\lambda_{d,n}$ as in Lemma 3.1. From (10) and (7), it follows that

(11)

$$\Gamma(f_1^+ - f_2^+, \lambda, \frac{4}{d+2}) = \lambda_{d,0}^{-\frac{4}{d+2}} \|Q_0^1 - Q_0^2\|^2 + \sum_{n=2k, k\in\mathbb{N}^+} |\lambda_{d,n}|^{-\frac{4}{d+2}} \|Q_n^1 - Q_n^2\|^2.$$

Combining $\langle \mathscr{C}_p f, H\rangle = \langle f, \mathscr{C}_p H\rangle$ with Lemma 3.1, one sees that

$$\mathscr{C}_p(f) \sim \sum_{n=0}^{\infty} \lambda_{d,n} Q_n, \tag{12}$$

with $\lambda_{d,n}$ as in Lemma 3.1.

This equation and Parseval's equation show that

$$\begin{aligned}\|Q_0^1 - Q_0^2\|^2 &\le \lambda_{d,0}^{-2} \sum_{n=2k, k\in\mathbb{N}} |\lambda_{d,n}|^2 \|Q_n^1 - Q_n^2\|^2 \\ &= \lambda_{d,0}^{-2} \|\mathscr{C}_p f_1 - \mathscr{C}_p f_2\|^2. \end{aligned} \tag{13}$$

Let

$$c_p = \min\{\min_{p>n, n \text{ even}} \{|\lambda_{d,n}|^{\frac{4}{d+2}} n(n+d-2)\}, \min_{p<n, n \text{ even}} \{|\lambda_{d,n}|^{\frac{4}{d+2}} n(n+d-2)\}\}.$$

From (11), (13), the fact that $\|Q_n^1 - Q_n^2\|^2 \le 2(\|Q_n^1\|^2 + \|Q_n^2\|^2)^2$ and (6), we have

(14)

$$\begin{aligned}&\Gamma(f_1^+ - f_2^+, \lambda, \frac{4}{d+2}) \\ &\le \lambda_{d,0}^{-2-\frac{2}{d+4}} \|\mathscr{C}_p f_1 - \mathscr{C}_p f_2\|^2 + \sum_{n=2k, k\in\mathbb{N}^+} |\lambda_{d,n}|^{-\frac{4}{d+2}} \|Q_n^1 - Q_n^2\|^2 \\ &\le \lambda_{d,0}^{-2-\frac{2}{d+4}} \|\mathscr{C}_p f_1 - \mathscr{C}_p f_2\|^2 \\ &\quad + \sum_{n=2k, k\in\mathbb{N}^+} \frac{2}{|\lambda_{d,n}|^{\frac{4}{d+2}} n(n+d-2)} \Big[n(n+d-2)\|Q_n^1\|^2 + n(n+d-2)\|Q_n^2\|^2\Big] \\ &\le \lambda_{d,0}^{-2-\frac{2}{d+4}} \|\mathscr{C}_p f_1 - \mathscr{C}_p f_2\|^2 \\ &\quad + \sum_{n=2k, k\in\mathbb{N}^+} \frac{2}{c_p} \Big[n(n+d-2)\|Q_n^1\|^2 + n(n+d-2)\|Q_n^2\|^2\Big]\end{aligned}$$

$$\leq \lambda_{d,0}^{-2-\frac{2}{d+4}} \|\mathscr{C}_p f_1 - \mathscr{C}_p f_2\|^2 + \frac{2}{c_p}(\nabla_o \|f_1\|^2 + \nabla_o \|f_2\|^2).$$

From Lemma 2.1 and (14), we obtain (9). □

Proof of Theorem 1. We assume that the assumptions are satisfied and K and L have twice continuously differentiable radial functions. If the Theorem is proved under this assumption, then the general case follows by approximation.

It follows from the definitions of the p-centroid body and p-cosine transformation that

$$\begin{aligned} h^p_{\Gamma_p K}(u) &= \frac{1}{(d+p)V(K)} \int_{S^{d-1}} |u \cdot v| \rho_K^{d+p}(v) d\sigma(v) \\ &= \frac{1}{(d+p)V(K)} \mathscr{C}_p(\rho_K^{d+p})(u) \end{aligned} \tag{15}$$

and

$$h^p_{\Gamma_p L}(u) = \frac{1}{(d+p)V(L)} \mathscr{C}_p(\rho_L^{d+p})(u). \tag{16}$$

Since $K, L \in \mathscr{K}_e^d(r, R)$, it is clear that

$$r^d \kappa_d \leq V(K), V(L) \leq R^d \kappa_d$$

and

$$\max_{u \in S^{d-1}} \{h_{\Gamma_p K}(u), h_{\Gamma_p L}(u)\} \leq \Big(\frac{1}{(d+p) r^d \kappa_d} \kappa_{d-1} R^{d+p}\Big)^{\frac{1}{p}} =: c_1(d, r, R, p),$$

where $c_1(d, r, R, p)$ is an explicit constant depending only on d, r, R, p.

Hence

$$\begin{aligned} &\|\mathscr{C}_p(\rho_K^{d+p}) - \mathscr{C}_p(\rho_L^{d+p})\|^2 \\ &\leq \big((d+p)R^d\kappa_d\big)^2 \int_{S^{d-1}} |h^p_{\Gamma_p K}(u) - h^p_{\Gamma_p L}(u)|^2 d\sigma(u) \\ &= \big((d+p)R^d\kappa_d\big)^2 \int_{S^{d-1}} |h_{\Gamma_p K}(u) - h_{\Gamma_p L}(u)|^2 \Big(\sum_{i=0}^{p-1} h^i_{\Gamma_p K}(u) h^{p-1-i}_{\Gamma_p L}(u)\Big)^2 d\sigma(u) \\ &\leq \big((d+p)R^d\kappa_d\big)^2 \omega_d \delta(\Gamma_p K, \Gamma_p L)^2 p^2 c_1(d, r, R, p)^{2(p-1)} \\ &\leq c_2(d, r, R, p)\varepsilon^2, \end{aligned} \tag{17}$$

where $c_2(d, r, R, p) := \big((d+p)R^d\kappa_d\big)^2 \omega_d p^2 c_1(d, r, R, p)^{2(p-1)}$ is an explicit constant depending only on d, r, R, p.

A special case of the estimate (see e.g., [6, p. 243]) for $K \in \mathscr{K}_e^d(r, R)$ gives

$$\|\nabla_o \rho_K^m\| \leq m\sqrt{(d-1)\omega_d} \frac{R^{m+1}}{r} \tag{18}$$

for $m > 0$.

From standard computations, involving Lemma 3.2, (18) and the fact $\rho_K = \rho_K^+, \rho_L = \rho_L^+$, we can have

$$\begin{aligned}\|\rho_K^{d+p} - \rho_L^{d+p}\|^2 &= \|(\rho_K^{d+p})^+ - (\rho_L^{d+p})^+\|^2 \\ &\leq c_3(d,r,R,p)\|\mathscr{C}_p(\rho_K^{d+p}) - \mathscr{C}_p(\rho_L^{d+p})\|^{\frac{4}{d+4}} \qquad (19)\end{aligned}$$

with an explicit constant $c_3(d,r,R,p)$ depending only on d,r,R,p.

Denote $c_4(d,r,R,p) = c_2(d,r,R,p)^{\frac{2}{d+4}} c_3(d,r,R,p)$, from (17) and (19), we get

$$\|\rho_K^{d+p} - \rho_L^{d+p}\|^2 \leq c_4(d,r,R,p)\varepsilon^{\frac{4}{d+4}}, \qquad (20)$$

with an explicit constant $c_4(d,r,R,p)$ depending only on d,r,R,p.

It follows from $K, L \in \mathscr{K}_e^d(r,R)$ that

$$\begin{aligned}&\|\rho_K^{d+p} - \rho_L^{d+p}\|^2 \\ &= \int_{S^{d-1}} |\rho_K^{d+p}(u) - \rho_L^{d+p}(u)|^2 d\sigma(u) \\ &= \int_{S^{d-1}} \Big|\big(\rho_K(u) - \rho_L(u)\big)\big(\sum_{i=0}^{d+p-1} \rho_K^i(u)\rho_L^{d+p-1-i}(u)\big)\Big|^2 d\sigma(u) \\ &\geq (d+p)^2 r^{2(d+p-1)} \int_{S^{d-1}} \big|\rho_K(u) - \rho_L(u)\big|^2 d\sigma(u) \\ &= (d+p)^2 r^{2(d+p-1)} \|\rho_K - \rho_L\|^2. \qquad (21)\end{aligned}$$

For convex bodies $K, L \in \mathscr{K}_e^d(r,R)$, the Hausdorff distance $\delta(K,L)$ can be estimated in terms of the radial L_2-metric by

$$\delta(K,L) \leq c_d R^2 r^{-\frac{d+3}{d+1}} \|\rho_K - \rho_L\|^{\frac{2}{d+1}} \qquad (22)$$

with an explicit constant c_d depending only on the dimension d (see e.g., Groemer [6], Lemma 2.3.2).

The conclusion can be obtained from (20), (21) and (22). □

If $\varepsilon = 0$, from Theorem 1 we can obtain the following result.

Corollary 3.1. *Let K, L be two origin-symmetric convex bodies in $\mathbb{R}^d$ such that $V(K) = V(L)$. If for some $p \geq 1$, $p \neq 2k$, $k \in \mathbb{N}$, $\Gamma_p K = \Gamma_p L$, then $K = L$.*

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