Bull. Korean Math. Soc. **55** (2018), No. 1, pp. 115–137
https://doi.org/10.4134/BKMS.b160864
pISSN: 1015-8634 / eISSN: 2234-3016

CYCLIC CODES OVER THE RING $\mathbb{F}_p[u,v,w]/\langle u^2, v^2, w^2, uv - vu, vw - wv, uw - wu\rangle$

Pramod Kumar Kewat and Sarika Kushwaha

Abstract. Let $R_{u^2,v^2,w^2,p}$ be a finite non chain ring $\mathbb{F}_p[u,v,w]\langle u^2, v^2, w^2, uv - vu, vw - wv, uw - wu\rangle$, where p is a prime number. This ring is a part of family of Frobenius rings. In this paper, we explore the structures of cyclic codes over the ring $R_{u^2,v^2,w^2,p}$ of arbitrary length. We obtain a unique set of generators for these codes and also characterize free cyclic codes. We show that Gray images of cyclic codes are 8-quasicyclic binary linear codes of length $8n$ over $\mathbb{F}_p$. We also determine the rank and the Hamming distance for these codes. At last, we have given some examples.

1. Introduction

Cyclic codes are a key family of linear codes because of their lavish algebraic structures and practical accomplishment. A considerable attention has been paid to cyclic codes over rings in the early 1990's because of their affluent applications to design error-correcting coding schemes for wireless communication system. The classification of cyclic codes of length n over chain rings as well as over some non chain rings, when $\gcd(n,p) = 1$, where p is the characteristic of the ring, has been completely discussed in the literature. On the other hand, the classication of codes when p divides n is still not complete. Cyclic codes have been extensively studied over various finite chain rings in [1–5, 10]. More latterly, cyclic codes over finite non-chain rings have also been contemplated. However, the analysis on non-chain rings seems to be challenging as the algebraic structure does not allow to give nice and compact presentation of linear codes over these rings. Yildiz and Karadeniz in [12] characterized cyclic codes of odd length over the non-chain ring $\mathbb{F}_2[u,v]/\langle u^2, v^2, uv - vu\rangle$ and obtained some good binary codes under two Gray maps. Sobhani and Molakarimi in [11] characterized the cyclic codes over the ring $F_{2^m}[u,v]/\langle u^2, v^2, uv-vu\rangle$, they also determined $\mathbb{F}_{2^m}$-basis as well as the mass the formula for the number of these codes. The authors of [8] extended these studies in more general way for cyclic

Received October 26, 2016; Accepted March 7, 2017.
2010 *Mathematics Subject Classification.* 94B15, 94B05.
Key words and phrases. cyclic codes, Hamming distance, Gray map.

codes over the ring $\mathbb{F}_p[u,v]/\langle u^2, v^2, uv-vu\rangle$ and presented some ternary optimal codes as gray images. In [6], S. T. Dougherty et al. considered a family of rings $F_2[u_1,u_2,\ldots,u_k]/\langle u_1^2, u_2^2,\ldots,u_k^2, u_iu_j-u_ju_i\rangle$ and studied one-generator cyclic codes.

The purpose of this paper is to obtain the structure theorems for cyclic codes over the non-chain $R_{u^2,v^2,w^2,p}$ ring in more general setting. The paper is organized as follows: In Section 2, we find a generating set of polynomials along with the conditions on these generators for the cyclic codes over the ring $R_{u^2,v^2,w^2,p}$ of length n. We also discuss here generating polynomials for cases of free cyclic codes and n relatively prime to p. In Section 3, we prove some lemmas, in which we express certain types of polynomials over $R_{u^2,v^2,w^2,p}$ as linear combinations of generators of cyclic codes over $R_{u^2,v^2,w^2,p}$. By help of these lemmas, we derive a minimal spanning set, we also calculate ranks of these codes. In Section 4, we obtain the minimum distance of corresponding codes of length p^l. In Section 5.1, we will show that gray images of cyclic codes are 8-quasicyclic binary linear code of length $8n$ over $\mathbb{F}_p$. In Section 6, we discuss some examples of cyclic codes over the ring $R_{u^2,v^2,w^2,p}$ of length 3, 4, 5 over $\mathbb{F}_3, \mathbb{F}_2, \mathbb{F}_5$ respectively.

2. The structures of cyclic codes over the ring $R_{u^2,v^2,w^2,p}$

Let $R_{u^2,v^2,w^2,p} = \mathbb{F}_p[u,v,w]/\langle u^2,v^2,w^2,uv-vu,wv-vw,uw-wu\rangle$, where p is a prime number and n is a positive integer. We can write $R_{u^2,v^2,w^2,p}$ as $R_{u^2,v^2,w^2,p} = R_{u^2,v^2,p} + wR_{u^2,v^2,p}, w^2 = 0$, where $R_{u^2,v^2,p} = \mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$ and $u^2 = 0, v^2 = 0$. Let $R_{u^2,v^2,w^2,p,n} = R_{u^2,v^2,w^2,p}[x]/\langle x^n - 1\rangle$. Let C be a cyclic code of length n over the ring $R_{u^2,v^2,w^2,p}$. We can also consider C as an ideal in the ring $R_{u^2,v^2,w^2,p,n}$. We define the map $\psi : R_{u^2,v^2,w^2,p} \to R_{u^2,v^2,p}$ by $\psi(\alpha + w\beta) = \alpha$, where $\alpha, \beta \in R_{u^2,v^2,p}$. Clearly the map ψ is a surjective ring homomorphism. Let $R_{u^2,v^2,p,n} = R_{u^2,v^2,p}[x]/\langle x^n - 1\rangle$. We extend this homomorphism to a homomorphism ϕ from C to the ring $R_{u^2,v^2,p,n}$ defined by

$$\phi(c_0 + c_1x + \cdots + c_{n-1}x^{n-1}) = \psi(c_0) + \psi(c_1)x + \cdots + \psi(c_{n-1})x^{n-1}, \tag{2.1}$$

where $c_i \in R_{u^2,v^2,w^2,p}$. Let $J = \{r(x) \in R_{u^2,v^2,p,n}[x] : wr(x) \in \ker\phi\}$. We see that J is an ideal of $R_{u^2,v^2,p,n}$. Hence, we can consider J as a cyclic code over $R_{u^2,v^2,p}$. We know from Theorem 3.1 of [8] that any ideal of $R_{u^2,v^2,p,n}$ is of the form $\langle g(x) + up_1(x) + vq_1(x) + uvr_1(x), ua_1(x) + vq_2(x) + uvr_2, va_2(x) + uvr_3(x), uva_3(x)\rangle$. Now we assume that $B_1 = g(x) + up_1(x) + vq_1(x) + uvr_1(x)$, $B_2 = ua_1(x) + vq_2(x) + uvr_2$, $B_3 = va_2(x) + uvr_3(x)$, $B_4 = uva_3(x)$. So $J = \langle B_1, B_2, B_3, B_4\rangle$. Therefore, we can write $\ker\phi = \langle wB_1, wB_2, wB_3, wB_4\rangle$. Since ϕ is a surjective homomorphism, the image $\mathrm{Im}\phi$ is an ideal of $R_{u^2,v^2,p,n}$. Hence, $\mathrm{Im}\phi$ is a cyclic code over $R_{u^2,v^2,p}$. Again we can write $\mathrm{Im}\phi$ as above. That is, $\mathrm{Im}\phi = \langle B_1', B_2', B_3', B_4'\rangle$. Therefore, the code C over the ring $R_{u^2,v^2,w^2,p}$ can be written as $C = \langle A_1, A_2, \ldots, A_8\rangle$, where, A_i's are defined as follows:

$$A_1 = f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x) + wf_{1,5}(x) + uwf_{1,6}(x)$$

$$\begin{aligned}
&\quad + vwf_{1,7}(x) + uvwf_{1,8}(x),\\
A_2 &= uf_2(x) + vf_{2,3}(x) + uvf_{2,4}(x) + wf_{2,5}(x) + uwf_{2,6}(x) + vwf_{2,7}(x)\\
&\quad + uvwf_{2,8}(x),\\
A_3 &= vf_3(x) + uvf_{3,4}(x) + wf_{3,5}(x) + uwf_{3,6}(x) + vwf_{3,7}(x) + uvwf_{3,8}(x),\\
A_4 &= uvf_4(x) + wf_{4,5}(x) + uwf_{4,6}(x) + vwf_{4,7}(x) + uvwf_{4,8}(x),\\
A_5 &= wf_5(x) + uwf_{5,6}(x) + vwf_{5,7}(x) + uvwf_{5,8}(x),\\
A_6 &= uwf_6(x) + vwf_{6,7}(x) + uvwf_{6,8}(x),\\
A_7 &= vwf_7(x) + uvwf_{7,8}(x),\\
A_8 &= uvwf_8(x).
\end{aligned}$$

Throughout this paper we use $A_1, A_2, \ldots, A_8$ for above polynomials.

For an ideal C of the ring $R_{u^2,v^2,w^2,p,n} = R_{u^2,v^2,w^2,p}[x]/\langle x^n-1\rangle$, we define the residue and the torsion of an ideal C as

$$\begin{aligned}
&\mathrm{Res}(C) = \{a \in R_{u^2,v^2,p,n} \mid \exists\, b \in R_{u^2,v^2,p,n} : a + wb \in C\} \text{ and}\\
&\mathrm{Tor}(C) = \{a \in R_{u^2,v^2,p,n} \mid wa \in C\}.
\end{aligned}$$

It can be easily shown that when C is an ideal of $R_{u^2,v^2,w^2,p,n}$, $\mathrm{Res}(C)$ and $\mathrm{Tor}(C)$ both are ideals of $R_{u^2,v^2,p,n}$. And it is easy to show that $\mathrm{Res}(C) = \mathrm{Im}\phi$ and $\mathrm{Tor}(C) = J$. Now we define eight ideals associated to C.

$$\begin{aligned}
(2.2)\qquad C_1 &= \mathrm{Res}(\mathrm{Res}(\mathrm{Res}(C)))\\
&= C \bmod \langle u, v, w\rangle = \langle f_1(x)\rangle,\\
(2.3)\qquad C_2 &= \mathrm{Tor}(\mathrm{Res}(\mathrm{Res}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid uf(x) \in C \bmod \langle v, w\rangle\} = \langle f_2(x)\rangle,\\
(2.4)\qquad C_3 &= \mathrm{Res}(\mathrm{Tor}(\mathrm{Res}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid vf(x) \in C \bmod \langle uv, w\rangle\} = \langle f_3(x)\rangle,\\
(2.5)\qquad C_4 &= \mathrm{Tor}(\mathrm{Tor}(\mathrm{Res}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid uvf(x) \in C \bmod \langle w\rangle\} = \langle f_4(x)\rangle,\\
(2.6)\qquad C_5 &= \mathrm{Res}(\mathrm{Res}(\mathrm{Tor}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid wf(x) \in C \bmod \langle uw, vw\rangle\} = \langle f_5(x)\rangle,\\
(2.7)\qquad C_6 &= \mathrm{Tor}(\mathrm{Res}(\mathrm{Tor}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid uwf(x) \in C \bmod \langle vw\rangle\} = \langle f_6(x)\rangle,\\
(2.8)\qquad C_7 &= \mathrm{Res}(\mathrm{Tor}(\mathrm{Tor}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid vwf(x) \in C \bmod \langle uvw\rangle\} = \langle f_7(x)\rangle,\\
(2.9)\qquad C_8 &= \mathrm{Tor}(\mathrm{Tor}(\mathrm{Tor}(C)))\\
&= \{f(x) \in \mathbb{F}_p[x] \mid uvwf(x) \in C\} = \langle f_8(x)\rangle.
\end{aligned}$$

These are ideals of $\mathbb{F}_p[x]/\langle x^n-1\rangle$, hence principal ideals. Throughout this paper we use $C_1,\ C_2,\ldots,C_8$ for above ideals.

Theorem 1. *Any ideal C of the ring $R_{u^2,v^2,w^2,p,n}$ is uniquely generated by the polynomials $A_1,A_2,\ldots,A_8$ with $f_{i,j}(x)=0$ or $\deg(f_{i,j}(x))<\deg(f_j(x))$, where A_i, f_i and $f_{i,j}$ are defined as above.*

Proof. Proof is similar to the proof of Theorem 1 [11]. □

Theorem 2. *Let $C=\langle A_1,A_2,\ldots,A_8\rangle$ be an ideal of the ring $R_{u^2,v^2,w^2,p,n}$. Then we must have*

(1) $f_8(x)|f_i(x)$ *for* $1\le i\le 7$; $f_j(x)|f_1(x)|(x^n-1)$ *for* $2\le j\le 7$;

(2) $f_4(x)|f_2(x)$; $f_4(x)|f_3(x)$; $f_6(x)|f_5(x)$; $f_6(x)|f_2(x)$; $f_7(x)|f_5(x)$; $f_7(x)|f_3(x)$;

(3) $f_{i+1}(x)|f_{i,i+1}(x)\left(\frac{x^n-1}{f_i(x)}\right)$ *for* $1\le i\le 7$;

(4) *For a fix j, $1\le j\le 7$,* $f_{i+j}(x)|\frac{x^n-1}{f_i(x)}\frac{x^n-1}{f_{i+1}(x)}\cdots\frac{x^n-1}{f_{i+j-1}(x)}f_{i,i+j}(x)$ *for* $1\le i\le 8-j$;

(5) $f_i(x)|\frac{x^n-1}{f_{i-2}(x)}\left(f_{i-2,i}(x)-\frac{f_{i-2,i-1}(x)}{f_{i-1}(x)}f_{i-1,i}(x)\right)$ *for* $3\le i\le 8$;

(6)
$$f_i(x)|\frac{x^n-1}{f_{i-3}(x)}\left(f_{i-3,i}(x)-\frac{f_{i-3,i-2}(x)}{f_{i-2}(x)}f_{i-2,i}(x)-\frac{f_{i-3,i-1}(x)-\frac{f_{i-3,i-2}(x)}{f_{i-2}(x)}f_{i-2,i-1}(x)}{f_{i-1}(x)}f_{i-1,i}(x)\right)$$
for $4\le i\le 8$;

(7) $f_i(x)|\frac{x^n-1}{f_{i-4}(x)}\left(f_{i-4,i}(x)-\frac{f_{i-4,i-3}(x)}{f_{i-3}(x)}f_{i-3,i}(x)-Af_{i-2,i}(x)-Bf_{i-1,i}(x)\right)$ *for* $i\in(5,6,7,8)$, *where*
$$A=\left(\frac{f_{i-4,i-2}(x)-\frac{f_{i-4,i-3}(x)}{f_{i-3}(x)}f_{i-3,i-2}(x)}{f_{i-2}(x)}\right)\ \textit{and}$$
$$B=\left(\frac{f_{i-4,i-1}(x)-\frac{f_{i-4,i-3}(x)}{f_{i-3}(x)}f_{i-3,i-1}(x)-Af_{i-2,i-1}(x)}{f_{i-1}(x)}\right).$$

(8)
$$f_i(x)|\frac{x^n-1}{f_{i-5}(x)}\left(f_{i-5,i}(x)-\frac{f_{i-5,i-4}(x)}{f_{i-4}(x)}f_{i-4,i}(x)-Af_{i-3,i}(x)-Bf_{i-2,i}(x)-Df_{i-1,i}(x)\right)$$
for $i\in(6,7,8)$, *where*
$$A=\left(\frac{f_{i-5,i-3}(x)-\frac{f_{i-5,i-4}(x)}{f_{i-4}(x)}f_{i-4,i-3}(x)}{f_{i-3}(x)}\right),$$
$$B=\left(\frac{f_{i-5,i-2}(x)-\frac{f_{i-5,i-4}(x)}{f_{i-4}(x)}f_{i-4,i-2}(x)-Af_{i-3,i-2}(x)}{f_{i-2}(x)}\right)\ \text{and}$$
$$D=\left(\frac{f_{i-5,i-1}(x)-\frac{f_{i-5,i-4}(x)}{f_{i-4}(x)}f_{i-4,i-1}(x)-Af_{i-3,i-1}(x)-Bf_{i-2,i-1}(x)}{f_{i-1}}\right).$$

(9)
$$f_i(x)|\frac{x^n-1}{f_{i-6}(x)}\left(f_{i-6,i}(x)-Af_{i-5,i}(x)-Bf_{i-4,i}(x)-Df_{i-3,i}(x)-Ef_{i-2,i}(x)-Ff_{i-1,i}(x)\right)$$
for $i\in(7,8)$, *where*
$$A=\left(\frac{f_{i-6,i-5}(x)}{f_{i-5}(x)}\right),$$
$$B=\left(\frac{f_{i-6,i-4}(x)-Af_{i-5,i-4}(x)}{f_{i-4}(x)}\right),$$

$D = \left(\frac{f_{i-6,i-3}(x) - Af_{i-5,i-3}(x) - Bf_{i-4,i-3}(x)}{f_{i-3}(x)}\right)$,
$E = \left(\frac{f_{i-6,i-2}(x) - Af_{i-5,i-2}(x) - Bf_{i-4,i-2}(x) - Df_{i-3,i-2}(x)}{f_{i-2}(x)}\right)$ *and*
$F = \left(\frac{f_{i-6,i-1}(x) - Af_{i-5,i-1}(x) - Bf_{i-4,i-1}(x) - Df_{i-3,i-1}(x) - Ef_{i-2,i-1}(x)}{f_{i-1}(x)}\right)$.

(10) $f_8(x) | \frac{x^n-1}{f_1(x)}\Big(f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x) - Ff_{6,8}(x) - Gf_{7,8}(x)\Big)$, *where*
$A = \left(\frac{f_{1,2}(x)}{f_2(x)}\right)$,
$B = \left(f_{1,3}(x) - A\frac{f_{2,3}(x)}{f_3(x)}\right)$,
$D = \left(\frac{f_{1,4}(x) - Af_{2,4}(x) - Bf_{3,4}(x)}{f_4(x)}\right)$,
$E = \left(\frac{f_{1,5}(x) - Af_{2,5}(x) - Bf_{3,5}(x) - Df_{4,5}(x)}{f_5(x)}\right)$,
$F = \left(\frac{f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) - Df_{4,6}(x) - Ef_{5,6}(x)}{f_6(x)}\right)$ *and*
$G = \left(\frac{f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)}{f_7(x)}\right)$.

(11) $f_i(x) | f_{i-2,i-1}(x)$ *for* $i \in (4, 6, 8)$;
(12) $f_i(x) | \Big(f_{1,2}(x) - \frac{f_1(x)}{f_{i-1}(x)} f_{i-1,i}(x)\Big)$ *for* $i \in (4, 6, 8)$;
(13) $f_i(x) | \Big(f_{i-5,i-4}(x) - \frac{f_{i-5}(x)}{f_{i-1}(x)} f_{i-1,i}(x)\Big)$ *for* $i \in (7, 8)$;
(14) $f_i(x) | \Big(f_{i-6,i-4}(x) - \frac{f_{i-6}(x)}{f_{i-2}(x)} f_{i-2,i}(x) + \frac{f_{i-6,i-5}(x) - \frac{f_{i-6}(x)}{f_{i-2}(x)} f_{i-2,i-1}(x)}{f_{i-1}(x)} f_{i-1,i}(x)\Big)$ *for* $i \in (7, 8)$;
(15) $f_7(x) | f_{4,5}(x)$ *and* $f_7(x) | f_{3,5}(x)$;
(16) $f_8(x) | f_{2,5}(x)$;
(17) $f_8(x) | \Big(f_{3,6}(x) - \frac{f_{3,5}(x)}{f_7(x)} f_{7,8}(x)\Big)$;
(18) $f_8(x) | \Big(f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)} f_{7,8}(x)\Big)$;
(19) $f_8(x) | \Big(f_{5,6}(x) - \frac{f_5(x)}{f_7(x)} f_{7,8}(x)\Big)$;
(20) $f_8(x) | \Big(f_{1,4}(x) - \frac{f_1(x)}{f_5(x)} f_{5,8}(x) - Af_{6,8}(x) - Bf_{7,8}(x)\Big)$, *where*
$A = \left(\frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)} f_{5,6}(x)}{f_6(x)}\right)$ *and* $B = \left(\frac{f_{1,3}(x) - \frac{f_1(x)}{f_5(x)} f_{5,7}(x) - (A)f_{6,7}(x)}{f_6(x)}\right)$.

Proof. (1) We have $vwA_2 \in C$. Therefore, $uvwf_2(x) \in C$. This gives $f_2(x) \in C_8 = \langle f_8(x) \rangle$. Thus, $f_8(x) | f_2(x)$. Similarly, if we take uwA_3, wA_4, uvA_5, vA_6 and uA_7 we get $f_8(x) | f_i(x)$ for $3 \leq i \leq 7$.

(2) We have $vA_2 \in C$. Therefore, $uvf_2(x) \in C$ mod w. This gives $f_2(x) \in C_4 = \langle f_4(x) \rangle$. Thus, $f_4(x) | f_2(x)$. Similarly, if we take uA_3, uA_5,wA_2, vA_5, wA_3 and take mod by w, vw, vw, uvw, uvw respectively, we get the other conditions of (2).

(3) For $1 \leq i \leq 7$, we have $\frac{x^n-1}{f_i(x)} A_i \in C$. Therefore, $\frac{x^n-1}{f_i(x)} f_{i,i+1}(x) \in C_{i+1} = \langle f_{i+1}(x) \rangle$. Hence, $f_{i+1}(x) | \frac{x^n-1}{f_i(x)} f_{i,i+1}(x)$.

(4) For $j = 1$, Condition 4 is reduced to Condition 3. For $j = 2$ and for $1 \leq i \leq 6$, we have $\frac{x^n-1}{f_i(x)} \frac{x^n-1}{f_{i+1}(x)} A_i \in C$. This with Condition 3 gives $\frac{x^n-1}{f_i(x)} \frac{x^n-1}{f_{i+1}(x)} f_{i,i+2} \in C_{i+2} = \langle f_{i+2}(x) \rangle$. Hence, $f_{i+2}(x) | \frac{x^n-1}{f_i(x)} \frac{x^n-1}{f_{i+1}(x)} f_{i,i+2}$. This proves the condition for $j = 2$. Similarly for other value of j we can prove Condition 4.

(5) For $i = 3$, we have

$$\begin{aligned} &\left(\frac{x^n-1}{f_1(x)} A_1 - \frac{x^n-1}{f_1(x)} \frac{f_{1,2}(x)}{f_2(x)} A_2 \right) \\ &= v \frac{x^n-1}{f_1(x)} (f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)) + uv \frac{x^n-1}{f_1(x)} (f_{1,4}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,4}(x)) \\ &\quad + w \frac{x^n-1}{f_1(x)} (f_{1,5}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,5}(x)) + uw \frac{x^n-1}{f_1(x)} (f_{1,6}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,6}(x)) \\ &\quad + vw \frac{x^n-1}{f_1(x)} (f_{1,7}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,7}(x)) + uvw \frac{x^n-1}{f_1(x)} (f_{1,8}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,8}(x)) \\ &\in C. \end{aligned}$$

Since $v \frac{x^n-1}{f_1(x)} (f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)) \in C \bmod \langle uv, w \rangle$, therefore,

$$\frac{x^n-1}{f_1(x)} (f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)) \in C_3 \Rightarrow f_3(x) \Big| \left(\frac{x^n-1}{f_1(x)} (f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)) \right).$$

Similarly we get the results for rest of the values of i.

(6) For $i = 4$, we have

$$\left(\frac{x^n-1}{f_1(x)} A_1 - \frac{x^n-1}{f_1(x)} \frac{f_{1,2}(x)}{f_2(x)} A_2 + \frac{x^n-1}{f_1(x) f_3(x)} \left(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x) \right) A_3 \right) \in C.$$

Since $uv \frac{x^n-1}{f_1(x)} \left(f_{1,4}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,4}(x) + \frac{(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x))}{f_3(x)} f_{3,4}(x) \right) \in C \bmod w$, therefore,

$$\begin{aligned} &\frac{x^n-1}{f_1(x)} \left(f_{1,4}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,4}(x) + \frac{f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)}{f_3(x)} f_{3,4}(x) \right) \in C_4 \\ &\Rightarrow f_4(x) | \frac{x^n-1}{f_1(x)} \left(f_{1,4}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,4}(x) + \frac{f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x)}{f_3(x)} f_{3,4}(x) \right). \end{aligned}$$

Similarly we get the results for rest of the values of i.

(7) For $i = 5$, we have

$$\begin{aligned} &\frac{x^n-1}{f_1(x)} \left(A_1 - f_{1,2}(x) \frac{A_2}{f_2(x)} + \left(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x) \right) \frac{A_3}{f_3(x)} \right) \\ &- \left(\frac{x^n-1}{f_1(x)} \left(f_{1,4}(x) - f_{1,2}(x) \frac{A_2}{f_2(x)} + \left(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x) \right) \frac{A_3}{f_3(x)} \right) \frac{A_4}{f_4(x)} \right) \in C. \end{aligned}$$

Since

$$\begin{aligned} &w \frac{x^n-1}{f_1(x)} \left(f_{1,5}(x) - f_{1,2}(x) \frac{f_{2,5}(x)}{f_2(x)} + \left(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x) \right) \frac{f_{3,5}(x)}{f_3(x)} \right) \\ &- \frac{x^n-1}{f_1(x)} \left(f_{1,4}(x) - f_{1,2}(x) \frac{f_{2,4}(x)}{f_2(x)} + \left(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)} f_{2,3}(x) \right) \frac{f_{3,4}(x)}{f_3(x)} \right) \frac{f_{4,5}(x)}{f_4(x)} \end{aligned}$$

$\in C \bmod (uw, vw)$, therefore,

$$\frac{x^n-1}{f_1(x)}\Big(f_{1,5}(x) - f_{1,2}(x)\frac{f_{2,5}(x)}{f_2(x)} + \Big(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)\Big)\frac{f_{3,5}(x)}{f_3(x)}\Big)$$
$$-\frac{x^n-1}{f_1(x)}\Big(f_{1,4}(x) - f_{1,2}(x)\frac{f_{2,4}(x)}{f_2(x)} + \Big(f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)\Big)\frac{f_{3,4}(x)}{f_3(x)}\Big)\frac{f_{4,5}(x)}{f_4(x)}$$
$\in C_5$.

Similarly, we get the results for rest of the values of i.

(8) For $i = 6$, we have $\Big(\frac{x^n-1}{f_1(x)}A_1 - AA_2 - BA_3 + DA_4 + EA_5\Big) \in C$. Since $uw\Big(\frac{x^n-1}{f_1(x)}f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) + Df_{4,6}(x) + Ef_{5,6}(x)\Big) \in C \bmod vw$, therefore,

$$\frac{x^n-1}{f_1(x)}\Big(f_{1,6}(x) - \frac{f_{1,2}(x)}{f_2(x)}f_{2,6}(x) - Af_{3,6}(x) - Bf_{4,6}(x) - Df_{5,6}(x)\Big) \in C_6$$
$$\Rightarrow f_6(x) | \frac{x^n-1}{f_1(x)}\Big(f_{1,6}(x) - \frac{f_{1,2}(x)}{f_2(x)}f_{2,6}(x) - Af_{3,6}(x) - Bf_{4,6}(x) - Df_{5,6}(x)\Big),$$

where,

$$A = \frac{x^n-1}{f_1(x)}\frac{f_{1,2}(x)}{f_2(x)},$$
$$B = \frac{x^n-1}{f_1(x)}\Big(\frac{f_{1,3}(x) - \frac{f_{1,2}(x)}{f_2(x)}f_{2,3}(x)}{f_3(x)}\Big),$$
$$C = \frac{x^n-1}{f_1(x)}\Big(\frac{-f_{1,4}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,4}(x) + Af_{3,4}(x)}{f_4(x)}\Big) \text{ and}$$
$$D = \frac{x^n-1}{f_1(x)}\Big(\frac{-f_{1,5}(x) + \frac{f_{1,2}(x)}{f_2(x)}f_{2,5}(x) + Af_{3,5}(x) + Bf_{4,5}(x)}{f_5(x)}\Big).$$

Similarly, we get the results for rest of the values of i.

(9) For $i = 7$, we have $\frac{x^n-1}{f_1(x)}\Big(A_1 - AA_2 - BA_3 - DA_4 - EA_5 - FA_6\Big) \in C$. Since

$$vw\frac{x^n-1}{f_1(x)}\Big(f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)\Big)$$
$\in C \bmod uvw$, therefore,

$$\frac{x^n-1}{f_1(x)}\Big(f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)\Big) \in C_7$$
$$\Rightarrow f_7(x) | \frac{x^n-1}{f_1(x)}\Big(f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)\Big),$$

where

$$A = \Big(\frac{f_{1,2}(x)}{f_2(x)}\Big),$$
$$B = \Big(f_{1,3}(x) - A\frac{f_{2,3}(x)}{f_3(x)}\Big),$$
$$D = \Big(\frac{f_{1,4}(x) - Af_{2,4}(x) - Bf_{3,4}(x)}{f_4(x)}\Big),$$

$$E = \left(\frac{f_{1,5}(x) - Af_{2,5}(x) - Bf_{3,5}(x) - Df_{4,5}(x)}{f_5(x)}\right) \text{ and}$$
$$F = \left(\frac{f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) - Df_{4,6}(x) - Ef_{5,6}(x)}{f_6(x)}\right).$$

Similarly, we get the results for rest of the values of i.
(10) We have

$$\begin{aligned}
&\tfrac{x^n-1}{f_1(x)}\Big(A_1 - AA_2 - BA_3 - DA_4 - EA_5 - FA_6 - GA_7\Big)\\
&= uvw\tfrac{x^n-1}{f_1(x)} * \Big(f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x)\\
&\qquad - Ff_{6,8}(x) - Gf_{7,8}(x)\Big) \in C.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\tfrac{x^n-1}{f_1(x)}\Big(f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x)\\
&\qquad - Ff_{6,8}(x) - Gf_{7,8}(x)\Big) \in C_8\\
&\Rightarrow f_8(x) | \tfrac{x^n-1}{f_1(x)} * \Big(f_{1,8}(x) - Af_{2,8}(x) - Bf_{3,8}(x) - Df_{4,8}(x) - Ef_{5,8}(x)\\
&\qquad - Ff_{6,8}(x) - Gf_{7,8}(x)\Big),
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\tfrac{f_{1,2}(x)}{f_2(x)}\right),\\
B &= \left(f_{1,3}(x) - A\tfrac{f_{2,3}(x)}{f_3(x)}\right),\\
D &= \left(\tfrac{f_{1,4}(x) - Af_{2,4}(x) - Bf_{3,4}(x)}{f_4(x)}\right),\\
E &= \left(\tfrac{f_{1,5}(x) - Af_{2,5}(x) - Bf_{3,5}(x) - Df_{4,5}(x)}{f_5(x)}\right),\\
F &= \left(\tfrac{f_{1,6}(x) - Af_{2,6}(x) - Bf_{3,6}(x) - Df_{4,6}(x) - Ef_{5,6}(x)}{f_6(x)}\right) \text{ and}\\
G &= \left(\tfrac{f_{1,7}(x) - Af_{2,7}(x) - Bf_{3,7}(x) - Df_{4,7}(x) - Ef_{5,7}(x) - Ff_{6,7}(x)}{f_7(x)}\right).
\end{aligned}$$

(11) For $i = 4$, we have $uA_3 = uvf_{2,3}(x) + uwf_{2,5}(x) + uvwf_{2,7}(x) \in C$. Therefore, $uvf_{2,3}(x) \in C \bmod w \Rightarrow f_{2,3}(x) \in C_4 \Rightarrow f_4(x) | f_{2,3}(x)$. Similarly we get the results for rest of the values of i.
(12) For $i = 4$, we have $\left(vA_1 - \frac{f_1(x)}{f_3(x)}A_3\right) \in C$. Since $uv\left(f_{1,2}(x) - \frac{f_1(x)}{f_3(x)}f_{3,4}(x)\right) \in C \bmod w$, therefore,

$$\left(f_{1,2}(x) - \tfrac{f_1(x)}{f_3(x)}f_{3,4}(x)\right) \in C_4 \Rightarrow f_4(x) | \left(f_{1,2}(x) - \tfrac{f_1(x)}{f_3(x)}f_{3,4}(x)\right).$$

Similarly, we get the results for rest of the values of i.

(13) For $i = 7$, we have $\Big(wA_2 - \frac{f_2(x)}{f_6(x)}A_6\Big) \in C$. Since $vw\Big(f_{2,3}(x) - \frac{f_2(x)}{f_6(x)}f_{6,7}(x)\Big) \in C \bmod w$, therefore, $f_7(x)|\Big(f_{2,3}(x) - \frac{f_2(x)}{f_6(x)}f_{6,7}(x)\Big)$. Similarly we get the result for $i = 8$.

(14) For $i = 7$, we have $\Big(wA_1 - \frac{f_1(x)}{f_5(x)}A_5 - \frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)}f_{5,6}(x)}{f_6(x)}A_6\Big) \in C$. Since $vw\Big(f_{1,3}(x) - \frac{f_1(x)}{f_5(x)}f_{5,7}(x) - \frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)}f_{5,6}(x)}{f_6(x)}f_{6,7}(x)\Big) \in C \bmod uvw$, therefore,

$$\Big(f_{1,3}(x) - \frac{f_1(x)}{f_5(x)}f_{5,7}(x) - \frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)}f_{5,6}(x)}{f_6(x)}f_{6,7}(x)\Big) \in C_7$$
$$\Rightarrow f_7(x)|\Big(f_{1,3}(x) - \frac{f_1(x)}{f_5(x)}f_{5,7}(x) - \frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)}f_{5,6}(x)}{f_6(x)}f_{6,7}(x)\Big).$$

Similarly, we get the results for rest of the values of i.

(15) We have $vA_4 \in C$. Since $vwf_{4,5}(x) \in C \bmod uvw$, therefore, $f_{4,5}(x) \in C_7 \Rightarrow f_7(x)|f_{4,5}(x)$. Similarly by taking vA_3 we can show $f_7(x)|f_{3,5}(x)$.

(16) We have $uvA_2 = uvwf_{2,5}(x) \in C$. Therefore,

$$f_{2,5}(x) \in C_8 \Rightarrow f_8(x)|f_{2,5}(x).$$

(17) We have $vA_3 - \frac{f_{3,5}(x)}{f_7(x)}A_7 = uvw\Big(f_{3,6}(x) - \frac{f_{3,5}(x)}{f_7(x)}f_{7,8}(x)\Big) \in C_8$. Therefore,

$$\Big(f_{3,6}(x) - \frac{f_{3,5}(x)}{f_7(x)}f_{7,8}(x)\Big) \in C_8 \Rightarrow f_8(x)|\Big(f_{3,6}(x) - \frac{f_{3,5}(x)}{f_7(x)}f_{7,8}(x)\Big).$$

(18) We have $vA_4 - \frac{f_{4,5}(x)}{f_7(x)}A_7 = uvw\Big(f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)}f_{7,8}(x)\Big) \in C$. Therefore,

$$\Big(f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)}f_{7,8}(x)\Big) \in C_8 \Rightarrow f_8(x)|\Big(f_{4,6}(x) - \frac{f_{4,5}(x)}{f_7(x)}f_{7,8}(x)\Big).$$

(19) We have $vA_5 - \frac{f_5(x)}{f_7(x)}A_7 = uvw\Big(f_{5,6}(x) - \frac{f_5(x)}{f_7(x)}f_{7,8}(x)\Big) \in C$. Therefore,

$$\Big(f_{5,6}(x) - \frac{f_5(x)}{f_7(x)}f_{7,8}(x)\Big) \in C_8 \Rightarrow f_8(x)|\Big(f_{5,6}(x) - \frac{f_5(x)}{f_7(x)}f_{7,8}(x)\Big).$$

(20) We have $wA_1 - \frac{f_1(x)}{f_5(x)}A_5 - AA_6 - BA_7 = uvw\Big(f_{1,4}(x) - \frac{f_1(x)}{f_5(x)}f_{5,8}(x) - Af_{6,8}(x) - Bf_{7,8}(x)\Big) \in C$. Therefore,

$$\Big(f_{1,4}(x) - \frac{f_1(x)}{f_5(x)}f_{5,8}(x) - Af_{6,8}(x) - Bf_{7,8}(x)\Big) \in C_8$$
$$\Rightarrow f_8(x)|\Big(f_{1,4}(x) - \frac{f_1(x)}{f_5(x)}f_{5,8}(x) - Af_{6,8}(x) - Bf_{7,8}(x)\Big),$$

where $A = \Big(\frac{f_{1,2}(x) - \frac{f_1(x)}{f_5(x)}f_{5,6}(x)}{f_6(x)}\Big)$ and $B = \Big(\frac{f_{1,3}(x) - \frac{f_1(x)}{f_5(x)}f_{5,7}(x) - Af_{6,7}(x)}{f_6(x)}\Big)$. □

Theorem 3. *If $C = \langle A_1, A_2, \ldots, A_8\rangle$ is a cyclic code over the ring $R_{u^2,v^2,w^2,p}$, then C is a free cyclic code if and only if $f_1(x) = f_8(x)$. In this case, we have $C = \langle A_1\rangle$ and $A_1|(x^n - 1)$ in $R_{u^2,v^2,w^2,p}[x]$.*

Proof. Let $f_1(x) = f_8(x)$. Since $f_8(x)|f_4(x)|f_2(x)|f_1(x)$, $f_4(x)|f_3(x)|f_1(x)$, $f_6(x)|f_2(x)|f_1(x)$, $f_8(x)|f_6(x)|f_5(x)|f_1(x)$, $f_8(x)|f_7(x)|f_3(x)|f_1(x)$ and $f_7(x)|f_5(x)|f_1(x)$, therefore, we get $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f_5(x) = f_6(x) = f_7(x) = f_8(x)$. Let $B_1 = f_1(x) + uf_{1,2}(x) + vf_{1,3}(x) + uvf_{1,4}(x), B_2 = uf_2(x)+vf_{2,3}(x)+uvf_{2,4}, B_3 = vf_3(x)+uvf_{3,4}(x), B_4 = uvf_4(x)$. Then we have $\mathrm{Im}\phi = \langle B_1, B_2, B_3, B_4\rangle$ and $\mathrm{Ker}\phi = w\langle B_5, B_6, B_7, B_8\rangle$, where $B_5 = f_5(x) + uf_{5,6}(x) + vf_{5,7}(x) + uvf_{5,8}(x)$, $B_6 = uf_6(x) + vf_{6,7}(x) + uvf_{6,8}$, $B_7 = vf_7(x) + uvf_{7,8}(x)$, $B_8 = uvf_8(x)$. From [8, Proposition 3.3], we get $\mathrm{Im}\phi = \langle B_1\rangle$ and $\mathrm{Ker}\phi = w\langle B_5\rangle$. Therefore, we have $C = \langle f_1(x)+uf_{1,2}(x)+vf_{1,3}(x)+uvf_{1,4}(x)+wf_{1,5}(x)+uwf_{1,6}(x)+vwf_{1,7}(x)+uvwf_{1,8}(x), wf_5(x)+uwf_{5,6}(x)+vwf_{5,7}(x)+uvwf_{5,8}(x)\rangle$. Now to show $C = \langle A_1\rangle$ we show that $f_{1,2}(x) = f_{5,6}(x)$, $f_{1,3}(x) = f_{5,7}(x)$ and $f_{1,4}(x) = f_{5,8}(x)$. Since $wA_1 - A_5 = uw(f_{1,2}(x) - f_{5,6}(x)) + vw(f_{1,3}(x)-f_{5,7}(x))+uvw(f_{1,4}(x)-f_{5,8}(x)) \in C$, this gives $(f_{1,2}(x)-f_{5,6}(x)) \in C_6 = \langle f_6(x)\rangle$. Therefore, $f_6(x)|(f_{1,2}(x) - f_{5,6}(x))$. Since $(\deg(f_{1,2}(x)), \deg(f_{5,6}(x))) < \deg(f_6(x))$, this implies that $f_{1,2}(x) - f_{5,6}(x) = 0$. Thus, $f_{1,2}(x) = f_{5,6}(x)$. Hence, we get $wA_1-A_5 = vw(f_{1,3}(x)-f_{5,7}(x))+uvw(f_{1,4}(x) -f_{5,8}(x)) \in C$. This gives $(f_{1,3}(x) - f_{5,7}(x)) \in C_7 = \langle f_7(x)\rangle$. Therefore, $f_7(x)|(f_{1,3}(x) - f_{5,7}(x)$. Since $(\deg(f_{1,3}(x)), \deg(f_{5,7}(x))) < \deg(f_7(x))$, this implies that $(f_{1,3}(x) - f_{5,7}(x)) = 0$. Therefore, $f_{1,3}(x) = f_{5,7}(x)$. Finally, we have $wA_1 - A_5 = uvw(f_{1,4}(x) - f_{5,8}(x)) \in C$. This gives $(f_{1,4}(x) - f_{5,8}(x)) \in C_8 = \langle f_8(x)\rangle$. Therefore, $f_8(x)|(f_{1,4}(x) - f_{5,8}(x))$. Since

$$(\deg(f_{1,4}(x)), \deg(f_{5,8}(x))) < \deg(f_8(x)),$$

this implies that $(f_{1,4}(x) - f_{5,8}(x)) = 0$. Therefore, $f_{1,4}(x) = f_{5,8}(x)$. This shows that $wA_1 = A_5$. Hence $C = \langle A_1\rangle$ and $C \simeq R^{n-\deg(f_1(x))}_{u^2,v^2,w^2,p}$. Conversely, if C is a free cyclic code, we must have $C = \langle A_1\rangle$. Since $uvwf_8(x) \in C$, we have $uvwf_8(x) = uvw\alpha f_1(x)$ for some $\alpha \in \mathbb{F}_p$. Note that $f_8(x)|f_1(x)$, hence by comparing the coefficients both sides, we get $f_1(x) = f_8(x)$. Now for the second condition, by the division algorithm, we have $x^n - 1 = A_1q(x) + r(x)$, where $r(x) = 0$ or $\deg(r(x)) < \deg(f_1(x))$. This implies that $r(x) = (x^n - 1) - A_1q(x) \in C$. Since A_1 is the lowest degree polynomial in C, so $r(x) = 0$. Hence, $A_1|(x^n - 1)$ in $R_{u^2,v^2,w^2,p}[x]$. □

2.1. When $\gcd(n, p) = 1$

Let n be a positive integer relatively prime to p. First, we slightly refine Theorem 3.4 of [8], which gives the structure of a cyclic code over the ring $R_{u^2,v^2,p} = \mathbb{F}_p[u, v]/\langle u^2, v^2, uv-vu\rangle$. Let $C_{u,v}$ be a cyclic code over the ring $R_{u^2,v^2,p}$. From Theorem 3.4 of [8], we have $C_{u,v} = \langle g(x)+ua_1(x)+uvr_1(x), va_2(x)+uva_3(x)\rangle$ with $a_3|a_1(x)|g(x)|(x^n-1)$ and $a_3(x)|a_2(x)|g(x)|(x^n-1)$. Thus, $\frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}(g(x) +ua_1(x) + uvr_1(x)) = uv\frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}r_1(x) \in C_{u,v}$. This gives $\frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}r_1(x) \in$

$C_4 = \mathrm{Tor}(\mathrm{Tor}(C_{u,v})) = \langle a_3(x)\rangle$ (see Page 165 of [8]). Hence, $a_3(x)|\frac{x^n-1}{g(x)}\frac{x^n-1}{a_1(x)}r_1(x)$. Since n is relatively prime to p, x^n-1 can be uniquely factored as product of distinct irreducible factors. Therefore, we must have g.c.d.$\left(a_3(x), \frac{x^n-1}{g(x)}\right) =$ g.c.d.$\left(a_3(x), \frac{x^n-1}{a_1(x)}\right) = 1$. This gives $a_3(x)|r_1(x)$. But, from Theorem 3.1 of [8], we have $\deg(r_1(x)) < \deg(a_3(x))$. This gives $r_1(x) = 0$. Thus we have proved the following theorem.

Theorem 4. *Let $C_{u,v}$ be a cyclic code over the ring $R_{u^2,v^2,p}$ of length n. If n is relatively prime to p, then we have $C_{u,v} = \langle f_1(x)+uf_2(x), vf_3(x)+uvf_4(x)\rangle$ with $f_4(x)|f_2(x)|f_1(x)|(x^n-1)$ and $f_4(x)|f_3(x)|f_1(x)|(x^n-1)$.*

If $C = \langle A_1, A_2, \ldots, A_8\rangle$ is a cyclic code of length n over the ring $R_{u^2,v^2,w^2,p}$, then we have $\mathrm{Im}\phi = \langle A_1, A_2, A_3, A_4\rangle$ and $\ker\phi = w\langle A_5, A_6, A_7, A_8\rangle$. (See Equation 2.1 for the definition of ϕ.) Note that we can consider $\mathrm{Im}\phi$ and $\ker\phi$ as cyclic codes over the ring $R_{u^2,v^2,p}$. Since n is relatively prime to p, from the above theorem, we have $\mathrm{Im}\phi = \langle f_1(x) + uf_2(x), vf_3(x) + uvf_4(x)\rangle$ and $\ker\phi = w\langle f_5(x) + uf_6(x), vf_7(x) + uvf_8(x)\rangle$ with $f_4(x)|f_2(x)|f_1(x)|(x^n-1)$, $f_4(x)|f_3(x)|f_1(x)|(x^n-1)$, $f_8(x)|f_6(x)|f_5(x)|(x^n-1)$ and $f_8(x)|f_7(x)|f_5(x)|(x^n-1)$. We also have the conditions $f_5(x)|f_1(x)$, $f_6(x)|f_2(x)$ and $f_7(x)|f_3(x)$ (from Conditions 1 and 2 of Theorem 2). Therefore, the code C can be written as $C = \langle f_1(x)+uf_2(x)+w(f_{1,5}(x)+uf_{1,6}(x)+vf_{1,7}(x)+uvf_{1,8}(x)),\ f_3(x)+uf_4(x)+w(f_{3,5}(x)+uf_{3,6}(x)+vf_{3,7}(x)+uvf_{3,8}(x)),\ w(f_5(x)+uf_6(x)),\ w(vf_7(x)+uvf_8(x))\rangle$ with the same conditions as above on $f_i(x)$'s. From Condition 4 of Theorem 2, for $i = 1$ and $4 \le j \le 7$, we get $f_{1+j}(x)|\frac{x^n-1}{f_1(x)}\frac{x^n-1}{f_2(x)}\cdots\frac{x^n-1}{f_j(x)}f_{1,1+j}(x)$. Since n is relatively prime to p, x^n-1 can be uniquely factored as product of distinct irreducible factors. Therefore, we must have g.c.d.$\left(f_{1+j}(x), \frac{x^n-1}{f_k(x)}\right) = 1$ for $1 \le k \le j$. This gives $f_{1+j}(x)|f_{1,1+j}(x)$. From Theorem 1, we have $\deg(f_{1,j+1}(x)) < \deg(f_{1+j}(x))$ for $4 \le j \le 7$. This gives $f_{1,1+j}(x) = 0$ for $4 \le j \le 7$. Similarly, from Condition 4 of Theorem 2, for $i = 3$ and $2 \le j \le 5$, we can show that $f_{3,3+j}(x) = 0$. Thus we have proved the following theorem.

Theorem 5. *Let $C = \langle A_1, A_2, \ldots, A_8\rangle$ be a cyclic code over the ring $R_{u^2,v^2,w^2,p}$ of length n. If n is relatively prime to p, then we have*

$$C = \langle f_1(x)+uf_2(x), vf_3(x)+uvf_4(x), w(f_5(x)+uf_6(x)), w(vf_7(x)+uvf_8(x))\rangle$$

with the conditions:

$$f_4(x)|f_2(x)|f_1(x)|(x^n-1), f_4(x)|f_3(x)|f_1(x), f_8(x)|f_6(x)|f_5(x)|(x^n-1),$$
$$f_8(x)|f_7(x)|f_5(x)|f_1(x), f_6(x)|f_2(x) \text{ and } f_7(x)|f_3(x).$$

3. Ranks and minimal spanning sets

We follow Dougherty and Shiromoto [7, page 401] for the definition of the rank of a code C. We first prove the number of lemmas that we use to find the rank and the minimal spanning set of cyclic codes over $R_{u^2,v^2,w^2,p}$.

Lemma 1. *Let C be a cyclic code over the ring $R_{u^2,v^2,w^2,p}$. If $C = \langle A_1, A_2, \ldots, A_8\rangle$, then polynomials in C in the following forms can be written as follows*:

(1) $w(p_0(x)+up_1(x)+vp_2(x)+uvp_3(x)) = q_5(x)A_5+q_6(x)A_6+q_7(x)A_7+q_8(x)A_8$,
(2) $w(up_1(x)+vp_2(x)+uvp_3(x)) = q_6(x)A_6+q_7(x)A_7+q_8(x)A_8$,
(3) $w(vp_2(x)+uvp_3(x)) = q_7(x)A_7+q_8(x)A_8$,
(4) $w(uvp_3(x)) = q_8(x)A_8$,

for some $q_i(x) \in \mathbb{F}_p[x]$, $5 \le i \le 8$.

Proof. (1) Let $A' = w(p_0(x)+up_1(x)+vp_2(x)+uvp_3(x)) \in C$. Thus, $p_0(x) \in C_5 = \langle f_5(x)\rangle$. This gives $p_0(x) = q_5(x)f(x)$ for some $q_5(x) \in \mathbb{F}_p[x]$. Therefore, $A' - q_5(x)A_5 = w((p_1(x)-q_5(x)f_{5,6}(x))+u\ (p_2(x)-q_5(x)f_{5,7}(x))+uv(p_3(x)-q_5(x)f_{5,8}(x))) \in C$. Thus, $p_1(x) - q_5(x)f_{5,6}(x) \in C_6 = \langle f_6(x)\rangle$. Therefore, $(p_1(x) - q_5(x)f_{5,6}(x)) = q_6(x)f_6(x)$ for some $q_6(x) \in \mathbb{F}_p[x]$. Again,

$$\begin{aligned}A' - q_5(x)A_5 - q_6(x)A_6 = w(v(&p_2(x) - q_5(x)f_{5,7}(x) - q_1(x)f_{6,7}(x)) \\ &+ uv(p_3(x) - q_5(x)f_{5,8}(x) - q_1(x)f_{6,8}(x))) \in C.\end{aligned}$$

Thus, $(p_2(x)-q_5(x)f_{5,7}(x)-q_1(x)f_{6,7}(x)) \in C_7 = \langle f_7(x)\rangle$. Therefore, $(p_2(x)-q_5(x)f_{5,7}(x) - q_6(x)f_{6,7}(x)) = q_7(x)f_7(x)$ for some $q_7(x) \in \mathbb{F}_p[x]$. Again,

$$\begin{aligned}&A' - q_5(x)A_5 - q_6(x)A_6 - q_7(x)A_7 \\ &= w(uv(p_3(x) - q_5(x)f_{5,8}(x) - q_6(x)f_{6,8}(x) - q_7(x)f_{7,8}(x)) \in C.\end{aligned}$$

Thus, $(p_3(x)-q_5(x)f_{5,8}(x)-q_6(x)f_{6,8}(x)-q_7(x)f_{7,8}(x)) \in C_8 = \langle f_8(x)\rangle$. Therefore, $(p_3(x) - q_5(x)f_{5,8}(x) - q_6(x)f_{6,8}(x) - q_7(x)f_{7,8}(x)) = q_8(x)f_8(x)$ for some $q_8(x) \in \mathbb{F}_p[x]$. That is, $A' - q_5(x)A_5 - q_6(x)A_6 - q_7(x)A_7 - q_8(x)A_8 = 0 \Rightarrow A' = q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8$. This proves Statement (1). The proof of other cases are similar to the proof of Statement (1). $\square$

Lemma 2. *Let C be a cyclic code over the ring $R_{u^2,v^2,w^2,p}$. If $C = \langle A_1, A_2, \ldots, A_8\rangle$ and $\deg(f_i(x)) = t_i, 1 \le i \le 8$, then the following conditions hold*:

(1) $x^{t_i-t_8}A_8 = c_iu_{8-i}A_i + q_8(x)A_8$, $1 \le i \le 7$, *where* $\deg(q_8(x)) < t_i - t_8$, $u_1 = u, u_2 = v, u_3 = uv, u_4 = w, u_5 = uw, u_6 = vw$ *and* $u_7 = uvw$,
(2) $x^{t_5-t_7}A_7 = c_5vA_5 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_7(x)) < t_5 - t_7$,
(3) $x^{t_3-t_7}A_7 = c_3wA_3 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_7(x)) < t_3 - t_7$,
(4) $x^{t_1-t_7}A_7 = c_1vwA_1 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_7(x)) < t_1 - t_7$,
(5) $x^{t_5-t_6}A_6 = c_5uA_5 - q'_6(x)A_6 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_6(x)) < t_5 - t_6$,
(6) $x^{t_2-t_6}A_6 = c_2wA_2 - q'_6(x)A_6 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_6(x)) < t_2 - t_6$,
(7) $x^{t_1-t_6}A_6 = c_1uwA_1 - q'_6(x)A_6 - q'_7(x)A_7 - q'_8(x)A_8$, *where* $\deg(q'_6(x)) < t_1 - t_6$, *and*
(8) $x^{t_1-t_5}A_5 = c_1wA_1 + q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8$, *where* $\deg(q_5(x)) < t_1 - t_5$,

$c_i \in \mathbb{F}_p$ *and* $q_i(x), q'_i(x) \in \mathbb{F}_p[x]$.

Proof. (1) From Condition (1) of Theorem 2, we have $f_8(x)|f_i(x)$, $1 \le i \le 7$. Thus, $f_i(x) = s_i(x)f_8(x)$ for some $s_i(x) \in \mathbb{F}_p[x]$. This can be written as, $f_i(x) = (s_{i0} + xs_{i1} + \cdots + x^{t_i - t_8}s_{i(t_i-t_8)})f_8(x)$, where $s_{ij} \in \mathbb{F}_p$. Clearly $s_{i(t_i-t_8)} \neq 0$. Therefore, $u_{8-i}A_i - s_i(x)A_8 = uvw(f_i(x) - s_i(x)f_8(x)) = 0$. This gives $x^{t_i-t_8}A_8 = s^{-1}_{i(t_i-t_8)}u_{8-i}A_i - s^{-1}_{i(t_i-t_8)}(s_{i0} + xs_{i1} + \cdots + x^{t_i-t_8-1}s_{i(t_i-t_8-1)})A_8$. Hence, $x^{t_i-t_8}A_8 = c_iu_{8-i}A_i + q_8(x)A_8$, where $\deg(q_8(x)) < t_i - t_8$.

(2) From Condition (2) of Theorem 2, we have $f_7(x)|f_5(x)$. Thus, $f_5(x) = s_5(x)f_7(x)$ for some $s_5(x) \in F_p[x]$. This can be written as $f_5(x) = (s_{50} + xs_{51} + \cdots + x^{t_5-t_7}s_{5(t_5-t_7)})f_7(x)$. This together with Condition (4) of Lemma 1, we get $vA_5 - s_5(x)A_7 = w(uv(f_{5,6}(x) - s_5(x)f_{7,8}(x))) = q_8(x)A_8$. Thus,

$$s_5(x)A_7(x) = vA_5 - q_8(x)A_8. \tag{3.1}$$

This can be written as $x^{t_5-t_7}A_7 = s^{-1}_{5(t_5-t_7)}vA_5 - s^{-1}_{5(t_5-t_7)}(s_{50} + xs_{51} + \cdots + x^{t_5-t_7-1}\ s_{5(t_5-t_7-1)})\ A_7(x) - s^{-1}_{5(t_5-t_7)}q_8(x)A_8$. Thus,

$$x^{t_5-t_7}A_7 = c_5vA_5 - q'_7(x)A_7 - q'_8(x)A_8, \tag{3.2}$$

where $\deg(q'_7(x)) = t_5 - t_7 - 1 < t_5 - t_7$.

(3) The proof is similar to Condition 2.

(4) The proof is similar to Condition 2.

(5) From Condition (2) of Theorem 2, we have $f_6(x)|f_5(x)$. Thus, $f_5(x) = s_5(x)f_6(x)$ for some $s_5(x) \in F_p[x]$. This can be written as, $f_5(x) = (s_{50} + s_{51}x + \cdots + s_{5(t_5-t_6)}x^{t_5-t_6})f_6(x)$, where $s_{5i} \in \mathbb{F}_p$. This together with Condition (3) of Lemma 1, we get $uA_5 - s_5(x)A_6(x) = w(v(-s_5(x)f_{6,7}(x)) + uv(f_{5,7}(x) - s_5(x)f_{6,8}(x))) = q_7(x)A_7 + q_8(x)A_8 \in C$. Thus,

$$s_5(x)A_6(x) = uA_5 - q_7(x)A_7 - q_8(x)A_8. \tag{3.3}$$

This can be written as $x^{t_5-t_6}A_6 = s^{-1}_{5(t_5-t_6)}uA_5 - s^{-1}_{5(t_5-t_6)}(s_{50} + xs_{51} + \cdots + x^{t_5-t_6-1}\ s_{5(t_5-t_6-1)})A_6 - s^{-1}_{5(t_5-t_6)}q_7(x)A_7 - s^{-1}_{5(t_5-t_6)}q_8(x)A_8$, i.e.,

$$x^{t_5-t_6}A_6 = c_5uA_5 - q'_6(x)A_6 - q'_7(x)A_7 - q'_8(x)A_8, \tag{3.4}$$

where $\deg(q'_6(x)) = t_5 - t_6 - 1 < t_5 - t_6$.

(6) The proof is similar to Condition 5.

(7) The proof is similar to Condition 5.

(8) By the division algorithm, we have

$$\begin{aligned} &x^{t_1-t_5}(f_5(x) + uf_{5,6}(x) + vf_{5,7}(x) + uvf_{5,8}(x)) \\ &= c_1A_1 + (p_0(x) + up_1(x) + vp_2(x) + uvp_3(x) + wp_4(x) + uwp_5(x) \\ &\quad + vwp_6(x) + uvwp_7(x)), \end{aligned} \tag{3.5}$$

where $\deg(p_0(x)) < \deg(f_1(x)) = t_1$. Multiplying Eq. (3.5) by w and applying Condition 1 of Lemma 1 gives $x^{t_1-t_5}A_5 - c_1wA_1 = w(p_0(x) + up_1(x) + vp_2(x) + uvp_3(x)) = q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8$. That is,

$$x^{t_1-t_5}A_5 = c_1wA_1 + q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8. \tag{3.6}$$

We have $p_0(x) = q_5(x)f_5(x)$, thus, $\deg(q_5(x))+ \deg(f_5(x)) = \deg(p_0(x)) < t_1$. Hence, $\deg(q_5(x)) < t_1 - t_5$. $\square$

Theorem 6. *Let C be a cyclic code of length n over $R_{u^2,v^2,w^2,p}$. If $C = \langle A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\rangle$ with $t_i = \deg(f_i(x))$, $1 \le i \le 8$, $t'_4 = \min\{t_2, t_3\}$, $t'_6 = \min\{t_2, t_5\}$, $t'_7 = \min\{t_3, t_5\}$ and $t'_8 = \min\{t_4, t_6, t_7\}$, then C has rank $n+2t_1+t'_4+t'_6+t'_7+t'_8-t_2-t_3-t_4-t_5-t_6-t_7-t_8$. The minimal spanning set B of the code C is $B = \{A_1, xA_1, \ldots, x^{n-t_1-1}A_1,\ A_2, xA_2, \ldots, x^{t_1-t_2-1}A_2, A_3, xA_3, \ldots, x^{t_1-t_3-1}A_3, A_4, xA_4, \ldots, x^{t'_4-t_4-1}A_4,\ A_5,\ xA_5, \ldots, x^{t_1-t_5-1}A_5,\ A_6,\ xA_6, \ldots, x^{t'_6-t_6-1}A_6, A_7, xA_7, \ldots, x^{t'_7-t_7-1}A_7, A_8, xA_8, \ldots, x^{t'_8-t_8-1}A_8\}$.*

Proof. It is sufficient to show that B spans the set $B' = \{A_1, xA_1, \ldots, x^{n-t_1-1}A_1$, $A_2, xA_2 \cdots x^{n-t_2-1}A_2$, A_3, xA_3, …, $x^{n-t_3-1}A_3$, A_4, xA_4, …, $x^{n-t_4-1}A_4$, A_5, xA_5, …, $x^{n-t_5-1}A_5$, A_6, xA_6, …, $x^{n-t_6-1}A_6$, A_7, $xA_7, \ldots$, $x^{n-t_7-1}A_7$, A_8, $xA_8, \ldots, x^{n-t_8-1}A_8\}$. To show B spans B', we write the set B' as $B' = B_1 \cup B_2$, where $B_1 = \{A_1, xA_1, \ldots x^{n-t_1-1}A_1$, A_2, xA_2, …, $x^{n-t_2-1}A_2$, A_3, xA_3, …, $x^{n-t_3-1}A_3$, A_4, xA_4, …, $x^{n-t_4-1}A_4\}$ and $B_2 = \{A_5$, xA_5, …, $x^{n-t_5-1}A_5$, A_6, xA_6, …, $x^{n-t_6-1}A_6$, A_7, xA_7, …, $x^{n-t_7-1}A_7$, A_8, xA_8, …, $x^{n-t_8-1}A_8\}$. First we show that B spans B_2 and then we show that B spans B_1. To show B spans B_2 we divide the proof in twelve cases.

Case (1). Let $t'_8 = t_7$, $t'_7 = t_5$ and $t'_6 = t_5$. We first show that the element $x^{t_7-t_8}A_8 \in B_2 - B$ is linear combinations of some elements of B and then we show that other elements of the set $B_2 - B$ are linear combinations of elements of B. From Statement 1 of Lemma 2,

$$x^{t_7-t_8}A_8 = c_7uA_7 + q_8(x)A_8, \tag{3.7}$$

where $\deg(q_8(x)) < t_7 - t_8$. Therefore, $x^{t_7-t_8}A_8 \in \mathrm{Span}(B)$. Multiplying Equation (3.7) by x, x^2, x^3, …, $x^{t_5-t_7-1}$ and then putting the value of $x^{t_7-t_8}A_8$ in the equation obtained, we can show that $x^{t_7-t_8+1}A_8$, $x^{t_7-t_8+2}A_8$, …, $x^{t_5-t_8-1}A_8 \in \mathrm{Span}(B)$. From Statement 1 of Lemma 2, we have

$$x^{t_5-t_8}A_8 = c_5uvA_5 + q_8(x)A_8, \tag{3.8}$$

where $\deg(q_8(x)) < t_5 - t_8$. Therefore, $x^{t_5-t_8}A_8 \in \mathrm{Span}(B)$. Arguing as above, we can show that the terms $x^{t_5-t_8+1}A_8, x^{t_5-t_8+2}A_8, \ldots, x^{t_1-t_8-1}A_8 \in \mathrm{Span}(B)$. Again, from Statement 1 of Lemma 2, we have

$$x^{t_1-t_8}A_8 = c_5uvwA_1 + q_8(x)A_8, \tag{3.9}$$

where $\deg(q_8(x)) < t_1 - t_8$. As above, we can show that $x^{t_1-t_8}A_8$, $x^{t_1-t_8+1}A_8$, …, $x^{n-t_8-1}A_8 \in \mathrm{Span}(B)$. Now we show that $x^{t_5-t_7}A_7 \in \mathrm{Span}(B)$. From Statement 2 of Lemma 2, we have

$$x^{t_5-t_7}A_7 = c_5vA_5 - q'_7(x)A_7 - q'_8(x)A_8, \tag{3.10}$$

where $\deg(q'_7(x)) < (t_5 - t_7)$. In the above discussion, we have shown that $x^iA_8 \in \mathrm{Span}(B)$ for $1 \le i \le n - t_8 - 1$. Clearly, $q'_8(x)A_8 \in \mathrm{Span}(B)$. And, also the term c_5vA_5, $q'_7(x)A_7 \in \mathrm{Span}(B)$ (since $\deg(q_7(x)) < (t_5 - t_7)$). Therefore,

$x^{t_5-t_7}A_7 \in \mathrm{Span}(B)$. As above, after putting the value of $x^{t_5-t_7}A_7$ in the equation obtained by multiplying Equation (3.10) by x, x^2, …, $x^{t_1-t_5-1}$, successively, we can show that $x^{t_5-t_7+1}A_7$, $x^{t_5-t_7+2}A_7$, …, $x^{t_1-t_7-1}A_7 \in \mathrm{Span}(B)$. From Statement 4 of Lemma 2, we have

$$x^{t_1-t_7}A_7 = c_1vwA_1 - q_7'(x)A_7 - q_8'(x)A_8, \tag{3.11}$$

where $\deg(q_7'(x)) < t_1 - t_7$. As above, we can show that $x^{t_1-t_7}A_7$, $x^{t_1-t_7+1}A_7$, …, $x^{n-t_7-1}A_7 \in \mathrm{Span}(B)$. Now we show that $x^{t_5-t_6}A_6 \in \mathrm{Span}(B)$. From Statement 5 of Lemma 2, we have

$$x^{t_5-t_6}A_6 = c_5uA_5 - q_6'(x)A_6 - q_7'(x)A_7 - q_8'(x)A_8, \tag{3.12}$$

where $\deg(q_6'(x)) < (t_5 - t_6)$. In the above discussion, we have shown that $x^iA_j \in \mathrm{Span}(B)$ for $1 \le i \le n - t_j - 1$, $j = 7, 8$. Clearly, $q_7'(x)A_7, q_8'(x)A_8 \in \mathrm{Span}(B)$. Therefore, $x^{t_5-t_6}A_6 \in \mathrm{Span}(B)$. In a similar way, as above, after putting the value of $x^{t_5-t_6}A_6$ in the equation obtained by multiplying Equation (3.12) by x, x^2, x^3, …, $x^{t_1-t_5-1}$, successively, we can show that $x^{t_5-t_6+1}A_6$, $x^{t_5-t_6+2}A_6, \ldots,$ $x^{t_1-t_6-1}A_6 \in \mathrm{Span}(B)$. From Statement 7 of Lemma 2, we have

$$x^{t_1-t_6}A_6 = c_1uwA_1 - q_6'(x)A_6 - q_7'(x)A_7 - q_8'(x)A_8, \tag{3.13}$$

where $\deg(q_6'(x)) < t_1 - t_6$. Again as above, we can show that $x^{t_1-t_6+1}A_6$, $x^{t_1-t_6+2}A_6, \ldots, x^{n-t_6-1}A_6 \in \mathrm{Span}(B)$. Now we show that the next term $x^{t_1-t_5}A_5 \in \mathrm{Span}(B)$. From Statement 8 of Lemma 2, we have

$$x^{t_1-t_5}A_5 = c_1wA_1 + q_5(x)A_5 + q_6(x)A_6 + q_7(x)A_7 + q_8(x)A_8, \tag{3.14}$$

where $\deg(q_5(x)) < t_1 - t_5$. In the above discussion, we have shown that $x^iA_j \in \mathrm{Span}(B)$ for $1 \le i \le n - t_j - 1$, $j = 6, 7, 8$. Clearly, $q_6(x)A_6, q_7(x)A_7, q_8(x)A_8 \in \mathrm{Span}(B)$. Therefore, $x^{t_1-t_5}A_5 \in \mathrm{Span}(B)$ (since $\deg(q_5(x)) < t_1 - t_5$). Multiplying Equation (3.14) by x, x^2, $x^3, \ldots,$ x^{n-t_1-1} and then putting the value of $x^{t_1-t_5}A_5$ in the equation obtained, we can show that the terms $x^{t_1-t_5+1}A_5$, $x^{t_1-t_5+1}A_5, \ldots,$ $x^{n-t_5-1}A_5 \in \mathrm{Span}(B)$.

Case (2A). Let $t_8' = t_4$, $t_7' = t_3$ and $t_6' = t_2$. Let $t_4' = t_3$. As in Case 1, by using Statement 1 of Lemma 2 for $i = 4, 3$ and 1, successively, we can show that $x^{t_4-t_8}A_8, x^{t_4-t_8+1}A_8, \ldots, x^{n-t_8-1}A_8 \in \mathrm{Span}(B)$. Similarly, as in Case 1, by using Statements 3 and 4 of Lemma 2, successively, we can show that $x^{t_3-t_7}A_7$, $x^{t_3-t_7+1}A_7, \ldots,$ $x^{n-t_7-1}A_7 \in \mathrm{Span}(B)$. Again, as in Case 1, by using Statement 6 and then Statement 7 of Lemma 2, successively, we can show that $x^{t_2-t_6}A_6, x^{t_2-t_6+1}A_6, \ldots, x^{n-t_6-1}A_6 \in \mathrm{Span}(B)$. In a similar fashion, as in Case 1, by using Statement 8 of Lemma 2, we can show that $x^{t_1-t_5}A_5, x^{t_1-t_5+1}A_5, \ldots, x^{n-t_5-1}A_5 \in \mathrm{Span}(B)$.

Case (2B). Let $t_8' = t_4$, $t_7' = t_3$, $t_6' = t_2$ and $t_4' = t_2$. As in Case 1, by using Statement 1 of Lemma 2 for $i = 4, 2$ and 1, successively, we can show that $x^{t_4-t_8}A_8, x^{t_4-t_8+1}A_8, \ldots, x^{n-t_8-1}A_8 \in \mathrm{Span}(B)$. Similarly, as in Case 1, by using Statements 3, 4, 6, 7 and then 8 of Lemma 2, successively, we can show that the rest of elements belongs to $\mathrm{Span}(B)$.

Case (3). Let $t'_8 = t_6$, $t'_7 = t_3$ and $t'_6 = t_5$. As in Case 1, by using Statement 1 of Lemma 2 for $i = 6, 5$ and 1, successively, we can show that $x^{t_6-t_8}A_8, x^{t_6-t_8+1}A_8, \ldots, x^{n-t_8-1}A_8 \in \mathrm{Span}(B)$. Similarly, as in Case 1, by using Statements 3 and 4 of Lemma 2, successively, we can show that $x^{t_3-t_7}A_7$, $x^{t_3-t_7+1}A_7, \ldots, x^{n-t_7-1}A_7 \in \mathrm{Span}(B)$. Again, as in Case 1, by using Statement 5 and then Statement 7 of Lemma 2, successively, we can show that $x^{t_5-t_6}A_6, x^{t_5-t_6+1}A_6, \ldots, x^{n-t_6-1}A_6 \in \mathrm{Span}(B)$. In a similar fashion, as in Case 1, by using Statement 8 of Lemma 2, we can show that $x^{t_1-t_5}A_5, x^{t_1-t_5+1}A_5, \ldots, x^{n-t_5-1}A_5 \in \mathrm{Span}(B)$.

The remaining cases are as follows:
Case (4). If $t'_8 = t_7$, $t'_7 = t_3$ and $t'_6 = t_5$.
Case (5). If $t'_8 = t_7$, $t'_7 = t_5$ and $t'_6 = t_2$.
Case (6). If $t'_8 = t_7$, $t'_7 = t_5$ and $t'_6 = t_5$.
Case (7). If $t'_8 = t_6$, $t'_7 = t_3$ and $t'_6 = t_2$.
Case (8). If $t'_8 = t_6$, $t'_7 = t_5$ and $t'_6 = t_2$.
Case (9). If $t'_8 = t_6$, $t'_7 = t_5$ and $t'_6 = t_5$.
Case (10). If $t'_8 = t_4$, $t'_7 = t_5$ and $t'_6 = t_5$, **(10A):** $t'_4 = t_3$, **(10B):** $t'_4 = t_2$.
Case (11). If $t'_8 = t_4$, $t'_7 = t_3$ and $t'_6 = t_5$, **(11A):** $t'_4 = t_3$, **(11B):** $t'_4 = t_2$.
Case (12). If $t'_8 = t_4$, $t'_7 = t_5$ and $t'_6 = t_2$, **(12A):** $t'_4 = t_3$, **(12B):** $t'_4 = t_2$.
In a similar way as above, by using Lemma 2, we can show that B spans B_2 in these cases.

Now we show that B spans B_1. From Equation (2.1), we have a homomorphism $\phi : C \to R_{u^2,v^2,p,n}$. Therefore, $C/\mathrm{Ker}\phi \simeq \phi(C)$ and $\phi(C)$ is a cyclic code over the ring $R_{u^2,v^2,p}$. Thus, we have $C/\mathrm{Ker}\phi$ as a cyclic code over $R_{u^2,v^2,p}$. Therefore, from Theorem 4.1 of [8], the minimal spanning set B_ϕ of the code $C/\mathrm{Ker}\phi$ is $\{A_1+\mathrm{Ker}\phi,\ xA_1+\mathrm{Ker}\phi, \ldots, x^{n-t_1-1}A_1+\mathrm{Ker}\phi,\ A_2+\mathrm{Ker}\phi,\ xA_2+\mathrm{Ker}\phi, \ldots, x^{t_1-t_2-1}A_2+\mathrm{Ker}\phi,\ A_3+\mathrm{Ker}\phi,\ xA_3+\mathrm{Ker}\phi, \ldots, x^{t_1-t_3-1}A_3+\mathrm{Ker}\phi,\ A_4+\mathrm{Ker}\phi,\ xA_4+\mathrm{Ker}\phi, \ldots, x^{t'_4-t_4-1}A_4+\mathrm{Ker}\phi\}$. To show B spans B_1, we only show that $x^{t_1-t_2}A_2 \in \mathrm{Span}(B)$. In a similar way, we can show that $x^{t_1-t_2+1}A_2, \ldots, x^{n-t_2-1}A_2, \ldots, x^{t'_4-t_4}A_4, \ldots, x^{n-t_4-1}A_4 \in \mathrm{Span}(B)$. Since B_ϕ spans $C/\mathrm{Ker}\phi$, we can write $x^{t_1-t_2}A_2 + \mathrm{Ker}\phi$ as a $R_{u^2,v^2,p}$ linear combination of the elements of B_ϕ, i.e., $x^{t_1-t_2}A_2 + \mathrm{Ker}\phi = \sum_{i=0}^{n-t_1-1}\alpha_{i1}(x^iA_1 + \mathrm{Ker}\phi) + \cdots + \sum_{i=0}^{t'_4-t_4-1}\alpha_{i4}(x^iA_4 + \mathrm{Ker}\phi)$, where $c_{ij} \in R_{u^2,v^2,p}$. Thus, $x^{t_1-t_2}A_2 - (\sum_{i=0}^{n-t_1-1}\alpha_{i1}(x^iA_1)+\cdots+\sum_{i=0}^{t'_4-t_4-1}\alpha_{i4}(x^iA_4)) \in \mathrm{Ker}\phi$. Since $\mathrm{Ker}\phi = \mathrm{Span}(B_2)$ and B spans B_2, we get $x^{t_1-t_2}A_2 \in \mathrm{Span}(B)$. Similarly, we can show that $x^{t_1-t_2+1}A_2, \ldots, x^{n-t_2-1}A_2, \ldots, x^{t'_4-t_4}A_4, \ldots, x^{n-t_4-1}A_4 \in \mathrm{Span}(B)$. This shows that B spans B_1. It is easy to see that any elements of the spanning set B can not be written as the linear combination of its preceding elements and other elements in the spanning set B. Here we only show that $x^{t_1-t_3-1}A_2$ can not be written as linear combinations of others element of spanning set B. The proof is similar for the rest. Suppose, if possible $x^{t_1-t_2-1}A_2$ can be written as linear combinations of the other element of the spanning set B. Then we have

$$\begin{aligned} x^{t_1-t_3-1}A_3 = & \sum_{i=0}^{n-t_1-1} \alpha_{1i}x^iA_1 + \sum_{i=0}^{t_1-t_2-1} \alpha_{2i}x^iA_2 + \sum_{i=0}^{t_1-t_3-2} \alpha_{3i}x^iA_3 \\ & + \sum_{i=0}^{t_4'-t_4-1} \alpha_{4i}x^iA_4 + \sum_{i=0}^{t_1-t_5-1} \alpha_{5i}x^iA_5 + \sum_{i=0}^{t_6'-t_6-1} \alpha_{6i}x^iA_6 \\ & + \sum_{i=0}^{t_7'-t_7-1} \alpha_{7i}x^iA_7 + \sum_{i=0}^{t_8'-t_8-1} \alpha_{8i}x^iA_8, \end{aligned}$$

where $\alpha_{ji} = \beta_{j1}^{(i)} + u\beta_{j2}^{(i)} + v\beta_{j3}^{(i)} + uv\beta_{j4}^{(i)} + w\beta_{j5}^{(i)} + uw\beta_{j6}^{(i)} + vw\beta_{j7}^{(i)} + uvw\beta_{j8}^{(i)} \in \mathbb{F}_p$. (Note that i is not a power of β it is a notation.) We have

$$\begin{aligned} & x^{t_1-t_3-1}(vf_3(x)+uvf_{3,4}(x)+wf_{3,5}(x)+uwf_{3,6}(x)+vwf_{3,7}(x)+uvwf_{3,8}(x)) \\ = & f_1(x)\sum_{i=0}^{n-t_1-1} \beta_{11}^{(i)}x^i + uf_1(x)\sum_{i=0}^{n-t_1-1} \beta_{12}^{(i)}x^i + uf_{1,2}(x)\sum_{i=0}^{n-t_1-1} \beta_{11}^{(i)}x^i \\ & + uf_2(x)\sum_{i=0}^{t_1-t_2-2} \beta_{21}^{(i)}x^i + vf_1(x)\sum_{i=0}^{n-t_1-1} \beta_{13}^{(i)}x^i + vf_{1,3}(x)\sum_{i=0}^{n-t_1-1} \beta_{11}^{(i)}x^i \\ & + vf_{2,3}(x)\sum_{i=0}^{t_1-t_2-1} \beta_{21}^{(i)}x^i + vf_3(x)\sum_{i=0}^{t_1-t_3-2} \beta_{31}^{(i)}x^i \\ & + uvm_2(x) + wm_3(x) + uw(m_4(x) + vwm_5(x) + uvwm_6(x)), \end{aligned}$$

where $m_2(x), \dots, m_6(x)$ are polynomials in $\mathbb{F}_p[x]$. By comparing both sides, we have $\beta_{11}^{(i)} = 0$, $\beta_{12}^{(i)} = 0$ for $0 \le i \le n - t_1 - 1$, $\beta_{21}^{(i)} = 0$ for $0 \le i \le t_1 - t_2 - 1$, and $x^{t_1-t_3-1}f_3(x) = f_1(x)\sum_{i=0}^{n-t_1-1} \beta_{13}^{(i)}x^i + f_3(x)\sum_{i=0}^{t_1-t_3-2} \beta_{31}^{(i)}x^i$. Note that $\deg(x^{t_1-t_3-1}f_3(x)) = t_1 - 1$ but $\deg(f_1(x)\sum_{i=0}^{n-t_1-1} \beta_{13}^{(i)}x^i) \ge t_1$ and $\deg(f_3(x)\sum_{i=0}^{t_1-t_3-2} \beta_{31}^{(i)}x^i) \le t_1 - 2$. Hence, this gives a contradiction. □

Theorem 7. *Let n be a positive integer relatively prime to p and C be a cyclic code of length n over the ring $R_{u^2,v^2,w^2,p}$. If $C = \langle f_1(x) + uf_2(x), vf_3(x) + uvf_4(x), w(f_5(x) + uf_6(x)), w(vf_7(x) + uvf_8(x))\rangle$ with $t_i = \deg(f_i(x))$, $1 \le i \le 8$, and $t_7' = \min\{t_3, t_5\}$, then C has rank $n + t_1 + t_7' - t_3 - t_5 - t_7$. The minimal spanning set B of the code C is $B = \{f_1(x) + uf_2(x)$, $x(f_1(x) + uf_2(x))$, $\dots$, $x^{n-t_1-1}(f_1(x) + uf_2(x))$, $vf_3(x) + uvf_4(x)$, $x(vf_3(x) + uvf_4(x))$, $\dots$, $x^{t_1-t_3-1}(vf_3(x) + uvf_4(x))$, $w(f_5(x) + uf_6(x))$, $xw(f_5(x) + uf_6(x))$, $\dots$, $x^{t_1-t_5-1}w(f_5(x) + uf_6(x))$, $w(vf_7(x) + uvf_8(x))$, $xw(vf_7(x) + uvf_8(x))$, $\dots$, $x^{t_7'-t_7-1}w(vf_7(x) + uvf_8(x))\}$.*

Proof. The proof is similar to the above theorem. □

4. Minimum distance

Let n be a positive integer not relatively prime to p. Let C be a cyclic code of length n over $R_{u^2,v^2,w^2,p}$. From Eq. (2.9), we have $C_8 = \{f(x) \in \mathbb{F}_p[x] \mid uvwf(x) \in C\} = \langle f_8(x)\rangle$. Also, we know that C_8 is a cyclic code over $\mathbb{F}_p$.

Theorem 8. *Let n be a positive integer not relatively prime to p. If $C = \langle A_1, A_2, \ldots, A_8\rangle$ is a cyclic code of length n over $R_{u^2,v^2,w^2,p}$, then $w_H(C) = w_H(C_8)$.*

Proof. Let $M(x) = m_0(x)+um_1(x)+vm_2(x)+uvm_3(x)+wm_4(x)+uwm_5(x)+vwm_6(x) + uvwm_7(x) \in C$, where $m_0(x), m_1(x), \ldots, m_7(x) \in \mathbb{F}_p[x]$. We have $uvwM(x) = uvwm_0(x)$, $w_H(uvwM(x)) \le w_H(M(x))$ and $uvwC$ is subcode of C with $w_H(uvwC) \le w_H(C)$. Thus $w_H(uvwC) = w_H(C)$. Therefore, it is sufficient to focus on the subcode $uvwC$ in order to prove the theorem. Since $w_H(C_8) = w_H(uvwC)$, we get $w_H(C) = w_H(C_8)$. $\square$

Definition. Let $m = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1p + b_0$, $b_i \in \mathbb{F}_p, 0 \le i \le l-1$, be the p-adic expansion of m.

(1) If $b_{l-i} \neq 0$ for all $1 \le i \le q, q < l$, and $b_{l-i} = 0$ for all $i, q+1 \le i \le l$, then m is said to have a p-adic length q zero expansion.
(2) If $b_{l-i} \neq 0$ for all $1 \le i \le q, q < l$, $b_{l-q-1} = 0$ and $b_{l-i} \neq 0$ for some $i, q+2 \le i \le l$, then m is said to have p-adic length q non-zero expansion.
(3) If $b_{l-i} \neq 0$ for $1 \le i \le l$, then m is said to have a p-adic length l expansion or p-adic full expansion.

Lemma 3. *Let C be a cyclic code over $R_{u^2,v^2,w^2,p}$ of length p^l, where l is a positive integer. Let $C = \langle f(x)\rangle$ where $f(x) = (x^{p^{l-1}} - 1)^b h(x)$, $1 \le b < p$. If $h(x)$ generates a cyclic code of length p^{l-1} and Hamming distance d, then the Hamming distance $d(C)$ of C is $(b+1)d$.*

Proof. For $c \in C$, we have $c = (x^{p^{l-1}} - 1)^b h(x)m(x)$ for some $m(x) \in \frac{R_{u^2,v^2,w^2,p}[x]}{(x^{p^l}-1)}$. Since $h(x)$ generates a cyclic code of length p^{l-1}, we have

$$\begin{aligned} w(c) &= w((x^{p^{l-1}} - 1)^b h(x)m(x)) \\ &= w(x^{p^{l-1}b} h(x)m(x)) + w({}^bC_1 x^{p^{l-1}(b-1)} h(x)m(x)) + \cdots \\ &\quad + w({}^bC_{b-1} x^{p^{l-1}} h(x)m(x)) + w(h(x)m(x)). \end{aligned}$$

Thus, $d(C) = (b+1)d$. $\square$

Theorem 9. *Let C be a cyclic code over $R_{u^2,v^2,w^2,p}$ of length p^l, where l is a positive integer. Then, $C = \langle A_1, A_2, \ldots, A_8\rangle$ where $f_1(x) = (x-1)^{t_1}, f_2(x) = (x-1)^{t_2}, \ldots, f_8(x) = (x-1)^{t_8}$ for some $t_1 > t_2, t_3 > t_4 > t_8 > 0$, $t_2 > t_6$, $t_3 > t_7$ and $t_1 > t_5 > t_6, t_7 > t_8 > 0$.*

(1) *If* $t_8 \leq p^{l-1}$, *then* $d(C) = 2$.

(2) *If* $t_8 > p^{l-1}$, *let* $t_8 = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1 p + b_0$ *be the p-adic expansion of* t_8 *and* $f_8(x) = (x-1)^{t_8} = (x^{p^{l-1}} - 1)^{b_{l-1}}(x^{p^{l-2}} - 1)^{b_{l-2}} \cdots (x^{p^1} - 1)^{b_1}(x^{p^0} - 1)^{b_0}$.

(a) *If* t_8 *has a p-adic length q zero expansion or full expansion* ($l = q$), *then* $d(C) = (b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$.

(b) *If* t_8 *has a p-adic length q non-zero expansion, then* $d(C) = 2(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$.

Proof. The first claim easily follows from Theorem 2. From Theorem 8, we see that $d(C) = d(C_8) = d(\langle (x-1)^{t_8} \rangle)$. Hence, we only need to determine the minimum weight of $C_8 = \langle (x-1)^{t_8} \rangle$.

(1) If $t_8 \leq p^{l-1}$, then $(x-1)^{t_8}(x-1)^{p^{l-1}-t_8} = (x-1)^{p^{l-1}} = (x^{p^{l-1}} - 1) \in C$. Thus, $d(C) = 2$.

(2) Let $t_8 > p^{l-1}$. (a) If t_8 has a p-adic length q zero expansion, we have $t_8 = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_{l-q}p^{l-q}$, and $f_8(x) = (x-1)^{t_8} = (x^{p^{l-1}} - 1)^{b_{l-1}}(x^{p^{l-2}} - 1)^{b_{l-2}} \cdots (x^{p^{l-q}} - 1)^{b_{l-q}}$. Let $h(x) = (x^{p^{l-q}} - 1)^{b_{l-q}}$. Then $h(x)$ generates a cyclic code of length p^{l-q+1} and minimum distance $(b_{l-q} + 1)$. By Lemma 3, the subcode generated by $(x^{p^{l-q+1}} - 1)^{b_{l-q+1}} h(x)$ has minimum distance $(b_{l-q+1} + 1)(b_{l-q} + 1)$. By induction on q, we can see that the code generated by $f_8(x)$ has minimum distance $(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$. Thus, $d(C) = (b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$.

(b) If t_8 has a p-adic length q non-zero expansion, we have

$$t_8 = b_{l-1}p^{l-1} + b_{l-2}p^{l-2} + \cdots + b_1 p + b_0, b_{l-q-1} = 0.$$

Let $r = b_{l-q-2}p^{l-q-2} + b_{l-q-3}p^{l-q-3} + \cdots + b_1 p + b_0$ and $h(x) = (x-1)^r = (x^{p^{l-q-2}} - 1)^{b_{l-q-2}}(x^{p^{l-q-3}} - 1)^{b_{l-q-3}} \cdots (x^{p^1} - 1)^{b_1}(x^{p^0} - 1)^{b_0}$. Since $r < p^{l-q-1}$, we have $p^{l-q-1} = r + j$ for some non-zero j. Thus, $(x-1)^{p^{l-q-1}-j} h(x) = (x^{p^{l-q-1}} - 1) \in C$. Hence, the subcode generated by $h(x)$ has minimum distance 2. By Lemma 3, the subcode generated by $(x^{p^{l-q}} - 1)^{b_{l-q}} h(x)$ has minimum distance $2(b_{l-q} + 1)$. By induction on q, we can see that the code generated by $f_8(x)$ has minimum distance $2(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$. Thus, $d(C) = 2(b_{l-1} + 1)(b_{l-2} + 1) \cdots (b_{l-q} + 1)$. □

Let C be a cyclic code of length n over the ring $R_{u^2,v^2,w^2,p}$. Let d_L be the Lee distance of a code C. By Theorem 1 of [9], we have

$$\left[\frac{d_L - 1}{8} \right] \leq n - \log_{p^8} |C|.$$

If C is a free code, then we get

$$\left[\frac{d_L - 1}{8} \right] \leq n - \mathrm{Rank}(C).$$

5. The Gray map

Let w_L and w_H denote the Lee weight and the Hamming weight respectively. We define the Lee weight as follows:

$$w_L(\alpha) = w_H(\phi_L(\alpha)) \text{ for all } \alpha \in R_{u^2,v^2,w^2,p},$$

where the Gray map $\phi_L : R_{u^2,v^2,w^2,p} \to \mathbb{F}_p^8$ is defined as follows:

$$\begin{aligned}&\phi_L(\alpha_1 + u\alpha_2 + v\alpha_3 + uv\alpha_4 + w\alpha_5 + uw\alpha_6 + vw\alpha_7 + uvw\alpha_8)\\ =\ &(\alpha_8, \alpha_6 + \alpha_8, \alpha_7 + \alpha_8, \alpha_4 + \alpha_8, \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_4 + \alpha_6 + \alpha_8,\\ &\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8).\end{aligned}$$

The Gray map naturally extend to $R^n_{u^2,v^2,w^2,p}$ as distance preserving isometry $\phi_L : (R^n_{u^2,v^2,w^2,p},$ Lee weight$) \to (\mathbb{F}_p^{8n},$ Hamming weight$)$ as follows

$$\phi_L(\alpha_1, \alpha_2, \ldots, \alpha_n) \to (\phi_L(\alpha_1), \phi_L(\alpha_2), \ldots, \phi_L(\alpha_n)),\ \forall\ a_i \in R_{u^2,v^2,w^2,p}.$$

By linearity of the map ϕ_L we obtain the following theorem.

Theorem 10. *If C is a linear code over $R_{u^2,v^2,w^2,p}$ of length n, size p^k and minimum lee weight d, then $\phi_L(C)$ is a p-ary linear code with parameters $[8n, k, d]$.*

5.1. Gray images of cyclic codes over $R_{u^2,v^2,w^2,p}$

Definition. Let T be the cyclic shift such that $T(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$. Then a linear code C is an l-quasicyclic code of length n if it is invariant under T^l.

Lemma 4. *Let T be the cyclic shift. Then*

$$\phi \circ T = T^8 \circ \phi_L.$$

Proof. Let $\widetilde{r} = (r_0, r_1, \ldots, r_{n-1}) \in R^n_{u^2,v^2,w^2,p}$ and ϕ_L be the Gray map defined in 2.2. Then

$$(\phi \circ T) = (\widetilde{r}) = (\phi_L(r_{n-1}), \phi(r_0), \ldots, \phi_L(r_{n-2})).$$

We also know that

$$\phi_L(r_0, r_1, \ldots, r_{n-1}) = (\phi_L(r_0), \phi_L(r_1), \ldots, \phi_L(r_{n-1})),$$

where each $\phi_L(r_i)$ is of length 8. Therefore, if we apply the cyclic shift eight times, the whole of $\phi_L(r_{n-1})$ will shift from the end to the beginning, which means we will get

$$(T^8 \circ \phi_L)(\widetilde{r}) = (\phi_L(r_{n-1}), \phi_L(r_0), \ldots, \phi_L(r_{n-2})).$$

Hence, we get the result. □

Theorem 11. *Let C be a cyclic code of length n over the ring $R_{u^2,v^2,w^2,p}$. Then $\phi_L(C)$ is a 8-quasicyclic binary linear code of length $8n$ over $\mathbb{F}_p$.*

Proof. Let C be a cyclic code over $R_{u^2,v^2,w^2,p}$. Then we know that $T(C) = C$. Now applying ϕ_L to both sides we get

$$\phi_L(T(C)) = \phi_L(C).$$

But, by Lemma 4, we know that $\phi \circ T = T^8 \circ \phi_L$. Therefore,

$$\phi_L(C) = \phi_L(T(C)) = (\phi_L \circ T)(C) = T^8(\phi_L(C))$$

which implies that $\phi_L(C)$ is invariant under T^8. Hence, $\phi_L(C)$ is a 8-quasicyclic code. □

6. Examples

Example 1. Cyclic codes of length 4 over the ring $R_{u^2,v^2,w^2,2}$. We have

$$x^4 - 1 = (x-1)^4 \text{ over } \mathbb{F}_2.$$

Let $g = x - 1$, some of the non zero cyclic codes of length 4 over the ring $R_{u^2,v^2,w^2,2}$ with generator polynomials, rank and minimum distance are given in Table 1.

TABLE 1. Non zero cyclic codes of length 4 over $R_{u^2,v^2,w^2,2}$.

Non-zero generator polynomials	Rank	$d(C)$
$\langle vwg^2 + (c_0 + c_1x)uvw\rangle$	2	2
$\langle vwg + c_0uvw\rangle$	3	2
$\langle uwg^3 + c_1vwg^3 + c_0uvwg^2\rangle$	1	4
$\langle uwg^3 + c_0uvw(c_2 + c_3x), vwg^3 + (c_0 + c_1x)uvw\rangle$	3	4
$\langle wg^3 + c_2uwg^2 + c_3vwg^2 + c_4uvw,\ uvwg^2\rangle$	2	2
$\langle wg + c_1uw + c_2vw\rangle$	3	2
$\langle uvg^3 + c_1uwg^3 + c_2vwg + c_3uvwg, vwg^2 + c_0uvwg\rangle$	3	2
$\langle vg^3 + c_1uvg + c_1vwg, uvg^2 + c_0wg,\ wg^2, uw, vw\rangle$	8	1
$\langle g^2 + v + u + c_1w, vg + u, ug + c_1w, uv + c_1w, wg, uw, vw\rangle$	8	1

Example 2. Cyclic codes of length 3 over the ring $R_{u^2,v^2,w^2,3}$. We have

$$x^3 - 1 = (x-1)^3 \text{ over } \mathbb{F}_3.$$

Let $g = x - 1$, some of the non zero cyclic codes of length 3 over the ring $R_{u^2,v^2,w^2,3}$ with generator polynomials, rank and minimum distance are given in Table below:

Example 3. Cyclic codes of length 5 over the ring $R_{u^2,v^2,w^2,5}$. We have

$$x^5 - 1 = (x-1)^5 \text{ over } \mathbb{F}_5.$$

Let $g = x - 1$, some of the non zero cyclic codes of length 5 over the ring $R_{u^2,v^2,w^2,5}$ with generator polynomials, rank and minimum distance are given in Table below:

TABLE 2. Non zero cyclic codes of length 3 over $R_{u^2,v^2,w^2,3}$.

Non-zero generator polynomials	Rank	$d(C)$
$\langle uwg^2+vwg, vwg^2\rangle$	3	3
$\langle uwg+c_2vw+c_1uvw, vwg+c_0uvw\rangle$	4	2
$\langle wg+c_1uw+c_2vw, uwg+c_0vw\rangle$	3	2
$\langle uvg+c_2uw+c_1vw, wg+c_0uw\rangle$	5	1
$\langle vg^2+c_4uvg+c_3w+c_2uw, uvg+wg^2+c_1uw, uwg^2+c_0vw, vwg, uvw\rangle$	5	1
$\langle ug+c_3v+c_2wg+c_1vw, vg+c_1wg+c_1vw, uv+c_0wg$ $+c_1vw+c_0vw, wg^2+c_0vw+c_0uvw\rangle$	8	3
$\langle g^2+c_3u+c_0uvw, ug+c_2w+c_0uvw, v+c_1w+c_0uvw, wg+c_0uvw,$ $uwg+c_0uvw\rangle$	6	1

TABLE 3. Non zero cyclic codes of length 5 over $R_{u^2,v^2,w^2,5}$.

Non-zero generator polynomials	Rank	$d(C)$
$\langle vwg^4+uvwg^3\rangle$	1	5
$\langle uwg^4+vwg^3+uvwg, vwg^4+uvwg^2, uvwg^3\rangle$	3	4
$\langle uwg^3+c_4vwg+uvw(c_2+c_3x), vwg^2+(c_0+c_1x)uvw\rangle$	5	3
$\langle wg^4+uw(c_6+c_7x)+vwc_8+uvwc_9, uwg^2+vw(c_4+c_5x)+$ $uvw(c_2+c_3x), vwg^2+uvw(c_0+c_1x)\rangle$	5	3
$\langle uvg^4+wc'_2g^3+uw(c'_3+c'_4x)+vw(c'_0+c'_1x)+uvw, wg^4+c_9uwg+$ $vw(c_6+c_7x)+c_8uvw, uwg^2+vw(c_3+c_4x+c_5x^2)+uvw(c_1+c_2x),$ $vwg^2+uvw(c_0+c_1x)\rangle$	6	3
$\langle uvg^3+c'_4wg^2+c'_3uw+c'_2vw, wg^4+uwg+c'_1vw+c'_0uvw,$ $uwg^3+vw(c_3+c_4x)+c_5uvw, vwg^3+(c_0+c_1x)uvw, uvwg^2\rangle$	6	3
$\langle vg^4+c'_0uvg+w(c_1+c'_2x)+c'_3uw+c'_4vw, uvg^2+c'_7wg^2+c'_8uw$ $+c'_9vw+c'_5uvw, wg^4+c_4uwg^2+c_5vw+c_6uvw, uwg^3+vw$ $+uvwg, vwg+(c_0+c_1x)uvw\rangle$	8	2

References

[1] T. Abualrub and I. Siap, *Cyclic codes over the rings $\mathbb{Z}_2+u\mathbb{Z}_2$ and $\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2$*, Des. Codes Cryptogr. **42** (2007), no. 3, 273–287.

[2] M. Al-Ashker and M. Hamoudeh, *Cyclic codes over $\mathbb{Z}_2+u\mathbb{Z}_2+u^2\mathbb{Z}_2+\cdots+u^{k-1}\mathbb{Z}_2$*, Turkish J. Math. **35** (2011), no. 4, 737–749.

[3] A. Bonnecaze and P. Udaya, *Cyclic codes and self-dual codes over $\mathbb{F}_2+u\mathbb{F}_2$*, IEEE Trans. Inform. Theory **45** (1999), no. 4, 1250–1255.

[4] H. Q. Dinh, *Constacylic codes of length p^s over $\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$*, J. Algebra **324** (2010), no. 5, 940–950.

[5] H. Q. Dinh and S. Lopez-Permouth, *Cyclic and negacyclic codes over finite chain rings*, IEEE Trans. Inform. Theory **50** (2004), no. 8, 1728–1744.

[6] S. T. Dougherty, S. Karadeniz, and B. Yildiz, *Cyclic codes over R_k*, Des. Codes Cryptogr. **63** (2012), no. 1, 113–126.

[7] S. T. Dougherty and K. Shiromoto, *Maximum distance codes over rings of order 4*, IEEE Trans. Inform. Theory **47** (2001), no. 1, 400–404.

[8] P. K. Kewat, B. Ghosh, and S. Pattanayak, *Cyclic codes over the ring $\mathbb{Z}_p[u,v]/\langle u^2,v^2,uv-vu\rangle$*, Finite Fields Appl. **34** (2015), 161–175.

[9] K. Shiromoto, *Singleton bounds for codes over finite rings*, J. Algebraic Combin. **12** (2000), no. 1, 95–99.

[10] A. K. Singh and P. K. Kewat, *Cyclic codes over* $\mathbb{Z}_p[u]/\langle u^k \rangle$, Des. Codes Cryptogr. **74** (2015), no. 1, 1–13.

[11] R. Sobhani and M. Molakarimi, *Some results on cyclic codes over the ring* $R_{2,m}$, Turkish J. Math. **37** (2013), no. 6, 1061–1074.

[12] B. Yildiz and S. Karadeniz, *Cyclic codes over* $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, Des. Codes Cryptogr. **58** (2011), no. 3, 221–234.

Pramod Kumar Kewat
Department of Applied Mathematics
Indian Institute of Technology (ISM)
Dhanbad 826 004, India
Email address: kewat.pk.am@ismdhanbad.ac.in

Sarika Kushwaha
Department of Applied Mathematics
Indian Institute of Technology (ISM)
Dhanbad 826 004, India
Email address: sarika120189@gmail.com