Bull. Korean Math. Soc. **55** (2018), No. 1, pp. 51–61
https://doi.org/10.4134/BKMS.b160844
pISSN: 1015-8634 / eISSN: 2234-3016

ON SOME SPECIAL DIFFERENCE EQUATIONS OF MALMQUIST TYPE

Jie Zhang

Abstract. In this article, we mainly use Nevanlinna theory to investigate some special difference equations of malmquist type such as $f^2 + (\Delta_c f)^2 = \beta^2$, $f^2+(\Delta_c f)^2 = R$, $f'^2+(\Delta_c f)^2 = R$ and $f^2+\big(f(z+c)\big)^2 = R$, where β is a nonzero small function of f and R is a nonzero rational function respectively. These discussions extend one related result due to C. C. Yang et al. in some sense.

1. Introduction and main results

In this article, we say that any function f is meromorphic always means it is analytic everywhere except at its poles in the whole complex plane $\mathbb{C}$, *i.e.*, if no pole occurs, then f is an entire function. We shall always assume that any reader is familiar with the standard notations and basic results of the Nevanlinna theory (see e.g., [4, 8, 9]). Let us denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, outside of a set E with finite logarithmic measure possibly, which is not necessary to be same at each occurrence. One meromorphic function a is said to be a small function with respect to f if and only if it satisfies $T(r, a) = S(r, f)$. For any meromorphic function f of finite order, we define its forward difference operator $\Delta_c f = f(z+c) - f(z)$, or we can denote by $\Delta f = f(z + c) - f(z)$ briefly, where c is a nonzero constant.

In 2004, Yang and Li [7] studied one certain nonlinear differential equation of malmquist type, and they provided successfully a necessary condition for such type of equation to have one admissible solution. We present their result as follows.

Theorem A (see [7]). *Let a_1, a_2 and a_3 be nonzero meromorphic functions. Then a necessary condition for the differential equation*

$$a_1 f^2 + a_2 f'^2 = a_3$$

Received October 20, 2016; Accepted February 13, 2017.
2010 *Mathematics Subject Classification.* Primary 30D35, 30D05, 34M10.
Key words and phrases. Nevanlinna theory, uniqueness, difference equation, differential equation.
This work was financially supported by the Fundamental Research Funds for the Central Universities (No. 2015QNA52).

to have a transcendental meromorphic solution satisfying $T(r,a_k)=S(r,f), k=1,2,3,$ *is* $a_1/a_3\equiv$ *constant.*

Our main purpose of this article is to consider some special difference counterparts of Theorem A utilizing the Nevanlinna theory, and now we describe our results (see Theorems 1.1–1.4) in detail as follows.

Theorem 1.1. *Let f be a transcendental entire function with finite order. Suppose that f is a solution of the difference equation*

$$f^2+(\Delta_c f)^2=\beta^2, \tag{1}$$

where β is a nonzero small function of f. Then after the transformation $f=\beta g$, the difference equation of f above reduces to a differential equation of g such that

$$g^2+(\alpha g')^2=1,$$

where α is a small meromorphic function of g, and α must have one zero at least.

However in Theorem 1.1, we fail to find out any such necessary condition as in Theorem A. Then for the case of difference equation (1) to admit a transcendental entire function of finite order, we propose one conjecture with regard to Theorem A as follows.

Conjecture 1. *Let f be a transcendental entire function with finite order. Suppose that f is a solution of the difference equation*

$$f^2+(\Delta_c f)^2=\beta^2,$$

where β is a small function of f. Then $\beta\equiv 0$.

Example 1. Let $f(z)=(1+i)^z$. Then it is a transcendental entire function with finite order of the difference equation $f^2+(\Delta_c f)^2=0$, where $c=1$.

This example shows the existence of possible solution to equation (1) exactly if $\beta\equiv 0$.

Remark 1. Suppose that β is a nonzero constant, for example, and we may assume $\beta=1$ without loss of generality. Then in a same way as in Theorem 1.3 later, we can get $2f=ae^{\alpha}+a^{-1}e^{-\alpha}$ and $2i\Delta_c f=ae^{\alpha}-a^{-1}e^{-\alpha}$, where a is a nonzero constant and α is a nonconstant polynomial, furthermore we can get $e^{\Delta\alpha}=1-i$ and $e^{-\Delta\alpha}=1+i$ by eliminating f and $\Delta_c f$, a contradiction. Thus the difference equation (1) admits no transcendental entire function with finite order if β is a nonzero constant in Theorem 1.1.

For the difference counterpart of Theorem A, people also have researched the topic on the finite order transcendental entire solutions of some special difference equations such as

$$f^2+P^2f^2(z+\eta)=Q,\quad f^2+P^2f^2(z+\eta)=Qe^{\alpha},$$

where P, Q, α are some polynomials and η is nonzero constant. For the detailed description of those results, please refer to [1, 6, 10] etc. Here we also present a result as follows, which may be considered as the general case that Q is a nonzero rational function in some sense.

Theorem 1.2. *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. If f is a solution of the differential-difference equation*

$$f'^2 + f(z+c)^2 = R, \tag{2}$$

where R is a nonzero rational function and c is a nonzero constant, then R is a nonzero constant and f is form of

$$f(z) = c_1 e^{iz} + c_2 e^{-iz}, c = k\pi,$$

where c_1, c_2 are two nonzero constants such that $4c_1c_2 = R$ and k is an integer.

Theorem 1.3. *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. Then f can not be a solution of the difference equation*

$$f^2 + \Delta_c f^2 = R, \tag{3}$$

where R is a nonzero rational function and c is a nonzero constant.

Theorem 1.4. *Let f be a transcendental meromorphic function with finitely many poles and $\sigma(f) < \infty$. If f is a solution of the differential-difference equation*

$$f'^2 + \Delta_c f^2 = R, \tag{4}$$

where R is a nonzero rational function and c is a nonzero constant, then R is a nonzero constant and f is of form

$$f(z) = c_1 e^{2iz} + c_2 e^{-2iz} + b, \quad c = k\pi + \pi/2,$$

where c_1, c_2 are two nonzero constants such that $16c_1c_2 = R$, b is a constant and k is an integer.

2. Some lemmas

To prove our results, we need some lemmas as follows.

Lemma 2.1. *If f is a nonconstant rational function and it satisfies the following differential-difference equation*

$$f' = k\Delta_c f,$$

where k and c are two nonzero constants, then $ck = 1$ and f is a polynomial of degree one.

Proof. Let us express our assumption $f' = k\Delta_c f$ in another form

$$f(z+c) = (1/k)f' + f. \tag{5}$$

First of all, we shall show that f is a polynomial. On the contrary, suppose that z_0 is a pole of f, then $z_0 + c$ also is a pole of f by comparing the order of pole z_0 on both sides of equation (5). By the cyclic utilization of this operation, we can obtain that $z_0 + 2c, z_0 + 3c, \dots, z_0 + nc, \dots$ are also poles of f, which is impossible for a nonconstant rational function. Therefore f is a polynomial.

If f is a polynomial of degree at least two, then we may set its Taylor expansion as follows

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \tag{6}$$

where $a_n, a_{n-1}, \dots, a_0$ are some constants such that $a_n \neq 0$, and n is a positive integer at least 2. From equation (6), we see

$$f'(z) = na_n z^{n-1} + (n-1)a_{n-1}z^{n-2} + \cdots + a_1 \tag{7}$$

and

$$\begin{aligned}\Delta_c f(z) &= a_n\big((z+c)^n - z^n\big) + a_{n-1}\big((z+c)^{n-1} - z^{n-1}\big) + \cdots + a_1 c\\ &= a_n\big(ncz^{n-1} + C_n^2 c^2 z^{n-2} + \cdots\big) + a_{n-1}\big((n-1)cz^{n-2} + \cdots\big)\\ &\quad + \cdots + a_1 c\\ &= a_n ncz^{n-1} + \big(a_n C_n^2 c^2 + a_{n-1}(n-1)c\big)z^{n-2} + \cdots + a_1 c.\end{aligned} \tag{8}$$

So from equations (5)-(8), we see

$$\begin{cases} a_n nck = na_n;\\ k\big(a_n C_n^2 c^2 + a_{n-1}(n-1)c\big) = (n-1)a_{n-1},\end{cases}$$

which means $\begin{cases} ck = 1;\\ a_n C_n^2 c = 0.\end{cases}$ But it is impossible.

Thus f is a polynomial of degree one, and then we can obtain $ck = 1$ easily. □

Lemma 2.2 (see [8])**.** *Let f be a nonconstant meromorphic function in the complex plane and*

$$R(f) = \frac{p(f)}{q(f)},$$

where $p(f) = \sum_{k=0}^{p} a_k f^k$ $(a_p \not\equiv 0)$ and $q(f) = \sum_{j=0}^{q} b_j f^j$ $(b_q \not\equiv 0)$ are two mutually prime polynomials in f. If the coefficients a_k, b_j are small functions of f, then

$$T\big(r, R(f)\big) = \max\{p,q\}T(r,f) + S(r,f).$$

Lemma 2.3 (see [8])**.** *Suppose that $f_1, f_2, \dots, f_n$ $(n \geq 2)$ are meromorphic functions and $g_1, g_2, \dots, g_n$ are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$;

(ii) *$g_j - g_k$ are not constants for $1 \le j < k \le n$;*
(iii) *For $1 \le j \le n$, $1 \le h < k \le n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}(r \to \infty, r \notin E)$.*

Then $f_j \equiv 0$ $(j = 1, 2, \ldots, n)$.

Lemma 2.4 (see [5])**.** *Let f be a transcendental meromorphic solution of finite order ρ of a difference equation of the form*

$$H(z, f)P(z, f) = Q(z, f),$$

where $H(z, f), P(z, f), Q(z, f)$ are difference polynomials in f such that the total degree of $H(z, f)$ in f and its shifts is n and that the corresponding total degree of $Q(z, f)$ is at most n. If $H(z, f)$ just contains one term of maximal total degree, then for any $\varepsilon > 0$, holds

$$m\big(r, P(z, f)\big) = O(r^{\rho-1+\varepsilon}) + S(r, f)$$

possibly outside of an exceptional set of finite logarithmic measure.

3. The proofs of main theorems

3.1. The proof of Theorem 1.1

Proof. On the one hand, by differentiating both sides of equation (1), we get

$$ff' + \Delta_c f \cdot \Delta_c f' = \beta\beta'. \tag{9}$$

Multiplying by $\Delta_c f$ on the both sides of equation (9), together with equation (1), leads to

$$ff'\Delta_c f + \Delta_c f'(\beta^2 - f^2) = \beta\beta'\Delta_c f, \tag{10}$$

namely

$$f(\Delta_c f \cdot f' - \Delta_c f' \cdot f) = \beta\beta'\Delta_c f - \beta^2\Delta_c f'. \tag{11}$$

Set

$$\varphi = \Delta_c f \cdot f' - \Delta_c f' \cdot f. \tag{12}$$

Then it follows from equations (11)-(12) that

$$f\varphi = \beta\beta'\Delta_c f - \beta^2\Delta_c f'. \tag{13}$$

By applying Lemma 2.4 to equation (13), we see

$$T(r, \varphi) = m(r, \varphi) = S(r, f),$$

which is to say that φ is a small entire function of f.

If $\varphi \equiv 0$, then it follows from equation (13) that $\Delta_c f = c_0\beta$, where c_0 is a nonzero constant. By substituting $\Delta_c f = c_0\beta$ into equation (1), we see

$$f^2 = (1 - {c_0}^2)\beta^2,$$

which is impossible. Therefore we may assume $\varphi \not\equiv 0$ in what follows.

On the other hand, by eliminating β, β' on the right sides of equations (1) and (9), we get

$$\beta'(f^2 + (\Delta_c f)^2) - \beta(ff' + \Delta_c f \cdot \Delta_c f') = 0, \tag{14}$$

which leads to

(15)

$$f(\beta' f - \beta f') = \beta \Delta_c f \cdot \Delta_c f' - \beta'(\Delta_c f)^2 = \Delta_c f(\beta \Delta_c f' - \beta' \Delta_c f) = -\frac{\Delta_c f f \varphi}{\beta}.$$

Therefore it follows from equation (15) that

$$\Delta_c f = \frac{\beta}{\varphi}(\beta f' - \beta' f). \tag{16}$$

Set $f = g\beta$ and $\alpha = \beta^2/\varphi$. Then it follows from equation (16) that

$$g' = \frac{\beta f' - \beta' f}{\beta^2} = \frac{\varphi \Delta_c f}{\beta^3}, \tag{17}$$

namely $\Delta_c f = g'\beta^3/\varphi$. Substituting this into equation (1) leads to

$$g^2 + (\alpha g')^2 = 1. \tag{18}$$

Lastly, if $\alpha \neq 0$, then we see obviously that g is an entire function by comparing the order of possible pole on both sides of equation (18), and then we can rewrite equation (18) as the following form

$$g^2 + (\alpha g')^2 = (g + i\alpha g')(g - i\alpha g') = 1,$$

which means $g + i\alpha g'$ and $g - i\alpha g'$ both do not have any pole or zero in a similar way. Recalling g is of finite order, we can set

$$g + i\alpha g' = e^{\gamma}$$

and

$$g - i\alpha g' = e^{-\gamma}$$

respectively, where γ is a nonconstant polynomial. Therefore we obtain $g = \cos p$, where $p = i\gamma$ is also a polynomial. Substituting $g = \cos p$ and $f = \beta g$ into equation (1) leads to

$$\begin{aligned} \frac{\beta(z+c)}{\beta} = \frac{\cos p \pm \sin p}{\cos(p + \Delta p)} &= \frac{\cos p \pm \sin p}{\cos p \cos \Delta p - \sin p \sin \Delta p} \\ &= \frac{\frac{1 \mp i}{2} e^{2ip} + \frac{1 \pm i}{2}}{\frac{\cos \Delta p + i \sin \Delta p}{2} e^{2ip} + \frac{\cos \Delta p - i \sin \Delta p}{2}}. \end{aligned}$$

We notice that the coefficients $\cos \Delta p, \sin \Delta p, \cos \Delta p \pm i \sin \Delta p (\not\equiv 0)$ are just small functions of e^p. Thus by applying Lemma 2.2 to the equation above, we see

$$T(r, e^{2ip}) = T\left(r, \frac{\beta(z+c)}{\beta}\right) + S(r, g) = S(r, g),$$

which is impossible. Thus α must have one zero at least. □

3.2. The proof of Theorem 1.2

Proof. Suppose that equation (2) admits a transcendental meromorphic solution f with finitely many poles and $\sigma(f) < \infty$, then we rewrite equation (2) as the following form

$$\big(f' + if(z+c)\big)\big(f' - if(z+c)\big) = R. \tag{19}$$

Since f just has finitely many poles and R is a nonzero rational function, then equation (19) implies that $f' + if(z+c)$ and $f' - if(z+c)$ both have just finitely many poles and zeros. Then equation (19) is equal to equation set

$$\begin{cases} f' + if(z+c) = R_1 e^{p}; \\ f' - if(z+c) = R_2 e^{-p}, \end{cases} \tag{20}$$

where R_1, R_2 are two nonzero rational functions such that $R_1R_2 = R$ and p is a nonconstant polynomial. By solving $f, f(z+c)$ from equation set (20), we obtain

$$\begin{cases} f' = \dfrac{R_1e^p + R_2e^{-p}}{2}; \\ f(z+c) = \dfrac{R_1e^p - R_2e^{-p}}{2i}. \end{cases} \tag{21}$$

By eliminating f from equation set (21), we see

$$\{iR_1(z+c)e^{\Delta p} - (R_1' + R_1p')\}e^p + \{iR_2(z+c)e^{-\Delta p} + (R_2' - R_2p')\}e^{-p} = 0. \tag{22}$$

Since p is a nonconstant polynomial and $e^{\Delta p}$ is just a small function of e^p, then by applying Lemma 2.2 to equation (22), we see

$$\begin{cases} iR_1(z+c)e^{\Delta p} = R_1' + R_1p'; \\ iR_2(z+c)e^{-\Delta p} = -(R_2' - R_2p'). \end{cases} \tag{23}$$

Thus equation set (23) means $e^{\Delta p}$ is a rational function, that is equal to say p is polynomial of degree one. Set $p(z) = az + b$, where $a(\neq 0), b$ are two constants. But equation set (23) shows

$$ie^{ac} \leftarrow ie^{ac}\frac{R_1(z+c)}{R_1} = \frac{R_1'}{R_1} + a \to a$$

and

$$ie^{-ac} \leftarrow ie^{-ac}\frac{R_2(z+c)}{R_2} = -\frac{R_2'}{R_2} + a \to a,$$

as $z \to \infty$, which leads to $\begin{cases} ie^{ac} = a; \\ ie^{-ac} = a, \end{cases}$ namely $\begin{cases} e^{ac} = 1; \\ a = i, \end{cases}$ or $\begin{cases} e^{ac} = -1; \\ a = -i. \end{cases}$

So we can obtain $c = k\pi, k \in \mathbb{Z}$.

If $e^{ac} = 1$ and $a = i$, then equation set (23) reduces to

$$\begin{cases} i\Delta R_1 = R_1'; \\ i\Delta R_2 = -R_2'. \end{cases} \tag{24}$$

If R_1 is a nonconstant rational function, then from Lemma 2.1 and $i\Delta R_1 = R_1'$ presented in equation (24), we see $c = -i$. However $e^{ac} = 1$ means $1 = ic = ac = 2k\pi i$, $k \in \mathbb{Z}$, which is a contradiction. So R_1 is a nonzero constant, and we can obtain that R_2 also is a nonzero constant in a same way.

If $e^{ac} = -1$ and $a = -i$, then equation set (23) reduces to

$$\begin{cases} i\Delta R_1 = -R_1'; \\ i\Delta R_2 = R_2'. \end{cases} \tag{25}$$

If R_1 is a nonconstant rational function, then from Lemma 2.1 and $i\Delta R_1 = -R_1'$ presented in equation (25), we see $c = i$. However $e^{ac} = -1$ means $-1 = ic = -ac = 2k\pi i + \pi i$, $k \in \mathbb{Z}$, which is a contradiction. So R_1 is a nonzero constant, and we can obtain that R_2 also is a nonzero constant in a same way.

Therefore we see that R_1, R_2 are just two nonzero constants, and then from the second equation in equation set (21), we see that f is of form

$$f(z) = c_1 e^{iz} + c_2 e^{-iz}.$$

By substituting this form into equation (2), we see $c_1 c_2 = 4R$ based on a simple calculation. □

3.3. The proof of Theorem 1.3

Proof. Suppose that equation (3) admits a transcendental meromorphic solution f with finitely many poles and $\sigma(f) < \infty$, then we reset equation (3) as the following form

$$\big(f + i\Delta_c f\big)\big(f - i\Delta_c f\big) = R. \tag{26}$$

Since f just has finitely many poles and R is a nonzero rational function, then equation (26) implies that $f + i\Delta_c f$ and $f - i\Delta_c f$ both have just finitely many poles and zeros. Then equation (26) is equal to equation set

$$\begin{cases} f + i\Delta_c f = R_1 e^p; \\ f - i\Delta_c f = R_2 e^{-p}, \end{cases} \tag{27}$$

where R_1, R_2 are two nonzero rational functions such that $R_1 R_2 = R$ and p is a nonconstant polynomial. By solving $f', f(z + c)$ from equation set (27), we obtain

$$\begin{cases} f = \dfrac{R_1 e^p + R_2 e^{-p}}{2}; \\ \Delta_c f = \dfrac{R_1 e^p - R_2 e^{-p}}{2i}. \end{cases} \tag{28}$$

By eliminating f from equation set (28), we see

$$\{i(R_1(z+c)e^{\Delta p} - R_1) - R_1\}e^p + \{i(R_2(z+c)e^{-\Delta p} - R_2) + R_2\}e^{-p} = 0. \tag{29}$$

Then by applying Lemma 2.2 to equation (29), we see

$$\begin{cases} i(R_1(z+c)e^{\Delta p} - R_1) = R_1; \\ i(R_2(z+c)e^{-\Delta p} - R_2) = -R_2. \end{cases} \tag{30}$$

Thus equation set (30) means $e^{\Delta p}$ is a rational function, that is to say p is polynomial with degree one. Set $p(z) = az + b$, where $a(\neq 0), b$ are two constants. But equation set (30) shows

$$i(e^{ac} - 1) \leftarrow i(e^{ac}\frac{R_1(z+c)}{R_1} - 1) = 1$$

and

$$i(e^{-ac} - 1) \leftarrow i(e^{-ac}\frac{R_2(z+c)}{R_2} - 1) = -1$$

as $z \to \infty$, which leads to $e^{ac} = 1 - i$ and $e^{-ac} = 1 + i$ respectively, a contradiction. □

3.4. The proof of Theorem 1.4

Proof. Suppose that equation (4) admits a transcendental meromorphic solution f with finitely many poles and $\sigma(f) < \infty$, then we reset equation (4) as the following form

$$\big(f' + i\Delta_c f\big)\big(f' - i\Delta_c f\big) = R. \tag{31}$$

Since f just has finitely many poles and R is a rational function, then equation (31) implies that $f' + i\Delta_c f$ and $f' - i\Delta_c f$ both have just finitely many poles and zeros. Then equation (31) is equal to equation set

$$\begin{cases} f' + i\Delta_c f = R_1 e^p; \\ f' - i\Delta_c f = R_2 e^{-p}, \end{cases} \tag{32}$$

where R_1, R_2 are two nonzero rational functions such that $R_1 R_2 = R$ and p is a nonconstant polynomial. By solving $f', f(z+c)$ from equation set (32), we obtain

$$\begin{cases} f' = \dfrac{R_1 e^p + R_2 e^{-p}}{2}; \\ \Delta_c f = \dfrac{R_1 e^p - R_2 e^{-p}}{2i}. \end{cases} \tag{33}$$

By eliminating f from equation set (33), we see

$$\{i(R_1(z+c)e^{\Delta p} - R_1) - R_1 p' - R_1'\}e^p + \{i(R_2(z+c)e^{-\Delta p} - R_2) + R_2' - R_2 p'\}e^{-p} = 0. \tag{34}$$

By applying Lemma 2.2 to equation (34), we see

$$\begin{cases} i(R_1(z+c)e^{\Delta p} - R_1) = R_1 p' + R_1'; \\ i(R_2(z+c)e^{-\Delta p} - R_2) = R_2 p' - R_2'. \end{cases} \tag{35}$$

Thus equation set (35) means $e^{\Delta p}$ is a rational function, that is to say p is polynomial with degree one. Set $p(z) = az + b$, where $a(\neq 0), b$ are two constants. Then equation set (35) shows

$$i(e^{ac} - 1) \leftarrow i(e^{ac}\frac{R_1(z+c)}{R_1} - 1) = a + \frac{R_1'}{R_1} \to a$$

and

$$i(e^{-ac} - 1) \leftarrow i(e^{-ac}\frac{R_2(z+c)}{R_2} - 1) = a - \frac{R_2'}{R_2} \to a,$$

as $z \to \infty$, which leads to $e^{ac} = -1$ and $a = -2i$. Then equation set (35) reduces to

$$\begin{cases} -i\Delta R_1 = R_1'; \\ i\Delta R_2 = R_2'. \end{cases} \tag{36}$$

Then from Lemma 2.1 and $-i\Delta R_1 = R_1'$ presented in equation (36), we see $c = i$, however $e^{ac} = e^2 = -1$, which is a contradiction. So R_1 is a nonzero constant, and we can obtain that R_2 also is a nonzero constant in a same way. There we prove R_1, R_2 are nonzero constants and $a = -2i, c = k\pi + \pi/2, k \in \mathbb{Z}$.

Form the first equation of set (33), we see that f has form

$$f(z) = c_1e^{2iz} + c_2e^{-2iz} + b,$$

where c_1, c_2 are two nonzero constants and b is a constant. Substituting it into equation (4), we see $c_1c_2 = 16R$ by a simple calculation. □

Acknowledgments. The author would like to thank the referee for his/her comments and suggestions.

References

[1] W. Chen, P. C. Hu, and Y. Y. Zhang, *On solutions to some nonlinear difference and differential equations*, J. Korean Math. Soc. **53** (2016), no. 4, 835–846.

[2] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujian J. **16** (2008), no. 1, 105–129.

[3] ______, *On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions*, Trans. Amer. Math. Soc. **361** (2009), no. 7, 3767–3791.

[4] W. K. Hayman, *Meromorphic Functions*, Clarendon Press, Oxford, 1964.

[5] I. Laine and C. C. Yang, *Clunie theorem for difference and q-difference polynomials*, J. London Math. Soc. **76** (2007), no. 3, 556–566.

[6] K. Liu and L. Z. Yang, *On entire solutions of some differential-difference equations*, Comput. Methods Funct. Theory. **13** (2013), no. 3, 433–447.

[7] C. C. Yang and P. Li, *On the transcendental solutions of a certain type of nonlinear differential equations*, Arch. Math. (Basel) **82** (2004), no. 5, 442–448.

[8] C. C. Yang and H. X. Yi, *Uniqueness theory of meromorphic functions*, Science Press, Beijing, Second Printed in 2006.

[9] L. Yang, *Value Distribution Theory*, Springer-Verlag & Science Press, Berlin, 1993.

[10] J. Zhang, *Existence of entire solution of some certain type difference equation*, Houston J. Math. **39** (2013), no. 2, 625–635.

Jie Zhang
School of Mathematics
China University of Mining and Technology
Xuzhou 221116, P. R. China
Email address: zhangjie1981@cumt.edu.cn