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MULTIPLICATION OPERATORS ON BERGMAN SPACES OVER POLYDISKS ASSOCIATED WITH INTEGER MATRIX

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Abstract. This paper mainly considers a tuple of multiplication operators on Bergman spaces over polydisks which essentially arise from a matrix, their joint reducing subspaces and associated von Neumann algebras. It is shown that there is an interesting link of the non-triviality for such von Neumann algebras with the determinant of the matrix. A complete characterization of their abelian property is given under a more general setting.

1. Introduction

Let $\mathbb{D}$ denote the unit disk in the complex plane $\mathbb{C}$, and Ω denote a bounded domain in the complex space $\mathbb{C}^d$. The Bergman space $L_a^2(\Omega)$ is the Hilbert space consisting of all holomorphic functions over Ω which are square integrable with respect to the Lebesgue measure dA in $\mathbb{C}^d$.

For a bounded holomorphic function ϕ over Ω, let M_ϕ denote the multiplication operator with the symbol ϕ on $L_a^2(\Omega)$, given by

$$M_\phi f = \phi f,\ f \in L_a^2(\Omega).$$

In general, for a tuple $\Phi = \{\phi_j : 1 \le j \le n\}$, let $\mathcal{V}^*(\Phi, \Omega)$ denote the von Neumann algebra

$$\{M_{\phi_j}, M_{\phi_j}^* : 1 \le j \le n\}',$$

which consists of all bounded operators on $L_a^2(\Omega)$ commuting with both M_{ϕ_j} and $M_{\phi_j}^*$ for each j. Also one can define $\mathcal{V}^*(\Phi, \Omega)$ in a similar way if Φ is a family of bounded holomorphic functions on Ω. It is known that there is a close connection between orthogonal projections in $\mathcal{V}^*(\Phi, \Omega)$ and all joint reducing subspaces of $\{M_{\phi_j} : 1 \le j \le n\}$. Precisely, the range of an orthogonal projection in $\mathcal{V}^*(\Phi, \Omega)$ is exactly a joint reducing subspace of $\{M_{\phi_j} : 1 \le j \le n\}$, and vice versa. We say that $\mathcal{V}^*(\Phi, \Omega)$ is trivial if $\mathcal{V}^*(\Phi, \Omega) = \mathbb{C}I$. This is

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equivalent to that the only nonzero joint reducing subspace of $\{M_{\phi_j,\Omega} : 1 \leq j \leq n\}$ is the whole space $L_a^2(\Omega)$.

It has been a focus to study commutants and reducing subspaces of analytic multiplication operators on function spaces, specifically on the Bergman space. In single-variable case, a lot of remarkable work has been made on the Bergman space $L_a^2(\mathbb{D})$ [1–3, 5–10, 17–19]. It is worthy to note that things are different if the underlying domain moves from the unit disk $\mathbb{D}$ to polygons [13].

However, in multi-variable case not much has been done on this line. On the Bergman space $L_a^2(\mathbb{D}^2)$ Lu and Zhou [14] characterized all reducing subspaces of a class of multiplication operators defined by monomials. In [4] the authors of this paper completely describe the reducing subspaces of multiplication operators on $L_a^2(\mathbb{D}^2)$ by the symbol $z_1^k + z_2^l$ with $k, l \in \mathbb{Z}_+$. Techniques and computations in these papers are completely different. This indicates that the structure of reducing subspaces heavily depends on the symbols of multiplication operators. To investigate new symbols, it seems likely that one needs to develop new techniques. It is worthy to note that the work in [4] was later generalized in [21], and it is remarkable that Guo and Wang [11] study a wide class of operators in the setting of abstract operator theory, including the multiplication operators $M_{z_1^k+z_2^l}$ as a special example. In all of these papers, these authors consider a single multiplication operator. We also call the reader's attention to [20] which gives a distinct approach to the study of a tuple of multiplication operators.

Motivated by the above work, we focus on a tuple of multiplication operators, rather than a single operator. Precisely, for an index $\beta = (\beta_1, \ldots, \beta_d)$ in $\mathbb{Z}_+^d$, define

$$z^\beta = \prod_{j=1}^{d} z_j^{\beta_j}, \ z = (z_1, \ldots, z_d) \in \mathbb{C}^d.$$

Let A denote a $d \times d$ $\mathbb{Z}_+$-entry matrix, and let $\alpha^1, \ldots, \alpha^d$ be d row vectors from A. Write

$$\Phi_A(z) = (z^{\alpha^1}, \ldots, z^{\alpha^d}), \ z \in \mathbb{D}^d.$$

The following shows that the structure of $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ has close connection with properties of integer matrix.

Theorem 1.1. *Suppose Φ_A is defined as above. Then $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is trivial if and only if A is a unimodular matrix.*

Example 1.2. Let

$$A_1 = \begin{pmatrix} 4 & 3 & 0 \\ 5 & 4 & k \\ 0 & 0 & 1 \end{pmatrix}, (k \in \mathbb{Z}_+), \quad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We have $\det A_1 = \det A_2 = 1$. Note that $\Phi_{A_1} = (z_1^4 z_2^3, z_1^5 z_2^4 z_3^k, z_3)$, and $\Phi_{A_2} = (z_1, z_1 z_2, z_2 z_3)$, and by Theorem 1.1 we get that both $\mathcal{V}^*(\Phi_{A_1}, \mathbb{D}^3)$ and $\mathcal{V}^*(\Phi_{A_2}, \mathbb{D}^3)$ are trivial.

But $\det A_3 = 2 \neq \pm 1$, and $\Phi_{A_3} = (z_1 z_3, z_1 z_2, z_2 z_3)$. Theorem 1.1 shows that $\mathcal{V}^*(\Phi_{A_3}, \mathbb{D}^3) \neq \mathbb{C}I$. Note that A_3 and A_2 have almost same entries.

The following is of interest.

Theorem 1.3. *Suppose that Φ is a family of monomials in d-variables. Then $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ is not abelian if and only if there exist two different integers s_1, s_2 in $\{1, \ldots, d\}$ and $t_1, t_2 \in \mathbb{Z}_+$ such that each monomial in Φ is a function in $z_{s_1}^{t_1} z_{s_2}^{t_2}$ and variables $z_s (s \notin \{s_1, s_2\})$.*

We have the following corollary.

Corollary 1.4. *Suppose Φ_A and A are defined as in Theorem 1.1. If* $\det A \neq 0$, *then $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is abelian.*

Some comments for Theorem 1.3 are in order. If each monomial in Φ does not depend on z_2, then each can be written as a function of $z_1 (= z_1^1 z_2^0)$, $z_3, \ldots, z_d$. In this case, by Theorem 1.3 $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ is not abelian.

Here are more examples.

Example 1.5. Write

$$p_1(z) = z_1 z_2^3, \quad p_2(z) = z_1^2 z_2^6 z_3$$

and

$$p_3(z) = z_4^2 z_5, \quad p_4(z) = z_3 z_4.$$

Since each of $p_1, \ldots, p_4$ can be written as a function of $z_1 z_2^3, z_3, z_4$ and z_5, by Theorem 1.3 $\mathcal{V}^*(p_1, p_2, p_3, p_4, \mathbb{D}^5)$ is not abelian.

For $d \geq 3$, let $p(z) = z_1 \cdots z_d$ and

$$q(z) = z_1 z_2^2 \cdots z_d^d.$$

Applying Theorem 1.3 we have that $\mathcal{V}^*(p, q, \mathbb{D}^d)$ is abelian.

Let p and q be monomials defined in Example 1.5, and define

$$\Phi = (q, p, p^2, \ldots, p^{d-1}).$$

Letting A be the corresponding matrix, we have $\Phi = \Phi_A$ and $\det A = 0$. Since

$$\mathcal{V}^*(\Phi_A, \mathbb{D}^d) = \mathcal{V}^*(p, q, \mathbb{D}^d),$$

$\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is abelian. Then we have the following corollary of Theorem 1.3.

Corollary 1.6. *For $d \geq 3$, there is a singular matrix $A \in M_d(\mathbb{Z}_+)$ such that $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is abelian.*

This paper is arranged as follows. Section 2 presents the proof of Theorem 1.1. Also, one will see that $\mathcal{V}^*(\Phi_A, \mathbb{D}^2)$ is abelian if and only if $\det A \neq 0$. In Section 3 we mainly characterize the abelian property of $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ for a family of monomials, and give the proof of Theorem 1.3.

2. Joint reducing subspaces and unimodular matrice

In this section, we mainly present the proof of Theorem 1.1, which says that for $A \in M_n(\mathbb{Z}_+)$, the tuple M_{Φ_A} has no nontrivial joint reducing subspaces if and only if A is a unimodular matrix.

Proof of Theorem 1.1. To see the "if" part, assume $\det A = 1$ without loss of generality. Since A is a unimodular matrix, its elementary row transformations involves two classes [15, Theorem 22.5]:

(i) rows switching;
(ii) multiplying all entries of a row by an integer constant and then adding to another.

Note that all involved elementary matrices are of determinant 1 or -1. After finitely many times of elementary row transformations, A becomes an upper-triangular matrix $\widetilde{A}$ with integer entries:

$$\begin{pmatrix} p_1 & * & * & * & * \\ 0 & p_2 & \cdots & * & * \\ 0 & 0 & \ddots & * & * \\ \vdots & 0 & \cdots & p_{d-1} & * \\ 0 & 0 & \cdots & 0 & p_d \end{pmatrix}.$$

Since either $\det \widetilde{A} = \det A$ or $\det \widetilde{A} = -\det A$, we have

$$p_1 \cdots p_d = \pm 1.$$

Since each $p_j (1 \leq j \leq d)$ is an integer,

$$p_j = \pm 1, 1 \leq j \leq d.$$

Then by finitely many elementary row transformations as (ii), A becomes a diagonal matrix whose diagonal entries are 1 or -1. Immediately, for each integer $j (1 \leq j \leq d)$, there are integers $\{n_j^k : 1 \leq j \leq d\}$ such that

$$\sum_{1 \leq k \leq d} n_j^k \alpha^k = \mathbf{e}_j^T. \tag{2.1}$$

Given $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ with $z_1 \cdots z_d \neq 0$, write

$$\Phi_A(w) = \Phi_A(z). \tag{2.2}$$

Then the only solution w for (2.2) is given by

$$w_s = \prod_{k=1}^{d} w^{n_s^k \alpha^k} = \prod_{k=1}^{d} z^{n_s^k \alpha^k} = z_s, 1 \leq s \leq d.$$

Then there is an open ball V in $\mathbb{D}^d$ such that

$$\Phi_A^{-1}(\Phi_A(V)) = V,$$

and following the proof of [8, Theorem 2.2] we have that each operator in $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ must be a scalar multiple of the identity operator. Therefore, $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is trivial.

To see the inverse direction, assume $\det A \neq \pm 1$. Before continuing, we observe that $\det A = \pm 1$ if and only if there are integers $\{n_j^k : 1 \leq j \leq d\}$ such that (2.1) holds. For this, it suffices to prove that if (2.1) holds, then

$$\det A = \pm 1.$$

In fact, note that

$$I = \begin{pmatrix} \mathbf{e}_1^T \\ \vdots \\ \mathbf{e}_d^T \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^d n_1^k \alpha^k \\ \vdots \\ \sum_{k=1}^d n_d^k \alpha^k \end{pmatrix}.$$

By the property of determinant, we have

$$1 = \sum_{k_1=1}^{d} n_1^{k_1} \det \begin{pmatrix} \alpha^{k_1} \\ \vdots \\ \sum_{k=1}^d n_d^k \alpha^k \end{pmatrix} = \sum_{k_1,\dots,k_d=1}^{d} n_1^{k_1} \cdots n_d^{k_d} \det \begin{pmatrix} \alpha^{k_1} \\ \vdots \\ \alpha^{k_d} \end{pmatrix}.$$

Since for such integers $k_1, \dots, k_d$,

$$\det \begin{pmatrix} \alpha^{k_1} \\ \vdots \\ \alpha^{k_d} \end{pmatrix} = \pm \det \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^d \end{pmatrix} \quad \text{or} \quad 0.$$

Therefore, 1 equals to an integer multiple of

$$\det \begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^d \end{pmatrix},$$

forcing $\det A = \pm 1$, as desired.

Since $\det A \neq \pm 1$, at least one of the equations (2.1) does not hold. Therefore,

$$\mathbb{Z}\alpha^1 + \cdots + \mathbb{Z}\alpha^d \neq \mathbb{Z}^d,$$

and thus

$$[\mathbb{Z}\alpha^1 + \cdots + \mathbb{Z}\alpha^d] \cap \mathbb{Z}_+^d \neq \mathbb{Z}_+^d.$$

Write

$$M = \overline{span\{z^\beta : \beta \in [\mathbb{Z}\alpha^1 + \cdots + \mathbb{Z}\alpha^d] \cap \mathbb{Z}_+^d\}}.$$

It is easy to check that M is a nonzero joint reducing subspace for the tuple M_{Φ_A} and $M \neq L_a^2(\mathbb{D}^d)$. This implies that $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is nontrivial. In conclusion, $\mathcal{V}^*(\Phi_A, \mathbb{D}^d)$ is trivial if and only if $\det A \neq \pm 1$. $\square$

3. Abelian property of $\mathcal{V}^*(\Phi, \mathbb{D}^d)$

In this section, we will study the von Neumann algebra $\mathcal{V}^*(\Phi, \mathbb{D}^d)$, where Φ is a family of monomials.

As follows, we will prove Theorem 1.3, that is restated as follows.

Theorem 3.1. *Suppose that Φ is a family of monomials in d-variables. Then $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ is not abelian if and only if there exist two different integers s_1, s_2 in $\{1, \ldots, d\}$ and $t_1, t_2 \in \mathbb{Z}_+$ such that each monomial in Φ is a function in $z_{s_1}^{t_1} z_{s_2}^{t_2}$, and variables $z_s (s \notin \{s_1, s_2\})$.*

Proof. To prove "if" part, assume that there exist two different integers s_1, s_2 in $\{1, \ldots, d\}$ and $t_1, t_2 \in \mathbb{Z}_+$ such that each monomial is a function in $z_{s_1}^{t_1} z_{s_2}^{t_2}$, and variables $z_s (s \notin \{s_1, s_2\})$. Then it is direct to check that $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ contains $\mathcal{V}^*(z_{s_1}^{t_1} z_{s_2}^{t_2}, z_s (s \in \{1, \ldots, d\} \setminus \{s_1, s_2\}), \mathbb{D}^d)$, and the latter is $*$-isomorphic to $\mathcal{V}^*(z_1^{t_1} z_2^{t_2}, \mathbb{D}^2)$. Since $\mathcal{V}^*(z_1^{t_1} z_2^{t_2}, \mathbb{D}^2)$ is not abelian [14, 16], $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ is not abelian.

Recall that

$$\{e_n(\lambda) = \sqrt{n+1}\lambda^n, n = 0, 1, \ldots\}$$

is an orthonormal basis of $L_a^2(\mathbb{D})$. By direct computations we have

$$M_{\lambda^k} e_n = \sqrt{\frac{n+1}{k+n+1}} e_{n+k}, n \geq 0,$$

and

$$M_{\lambda^k}^* e_n = \sqrt{\frac{n-k+1}{n+1}} e_{n-k}, n \geq k.$$

Thus,

$$M_{\lambda^k}^* M_{\lambda^k} e_n = \frac{n+1}{k+n+1} e_n, n \geq 0. \tag{3.3}$$

Note that $\{z^\alpha : \alpha \in \mathbb{Z}_+^d\}$ is an orthogonal basis of $L_a^2(\mathbb{D}^d)$. By (3.3) we have

$$M_{z^\beta}^* M_{z^\beta} z^\alpha = \prod_{1 \leq i \leq d} \frac{\alpha_i + 1}{\beta_i + \alpha_i + 1} z^\alpha. \tag{3.4}$$

Next we will prove the "only if" part. For this, assume that the latter condition on Φ fails, and we must prove that $\mathcal{V}^*(\Phi, \mathbb{D}^d)$ is abelian. Note that Φ contains at least two monomials. Pick monomials $q_1, \ldots, q_t$ and p from Φ, and write

$$p(z) = z^\beta \quad \text{and} \quad q_1(z) = z^{\gamma^1}, \ldots, q_t(z) = z^{\gamma^t},$$

where β and $\gamma^1, \ldots, \gamma^t$ are in $\mathbb{Z}_+^d$. Since Φ does not satisfy the condition in Theorem 3.1, for each $i \in \{1, \ldots, d\}$ there is a monomial in Φ depending on z_i and by multiplying finite of such monomials we get z^β. Hence for simplicity we may assume that $z^\beta \in \Phi$ and

$$\beta_1 \cdots \beta_d \neq 0.$$

By reasonable choice of $\gamma^1, \ldots, \gamma^t$, one can show that for fixed i with $1 \leq i \leq d$,

$$\frac{1}{\beta_i}(\gamma_i^1, \ldots, \gamma_i^t) = \frac{1}{\beta_l}(\gamma_l^1, \ldots, \gamma_l^t) \tag{3.5}$$

only if $i = l$. To see this, we pick finite members from Φ

$$z^{\gamma^1}, \ldots, z^{\gamma^t}$$

such that not all of them can be written as functions of $z_{s_1}^{t_1} z_{s_2}^{t_2}$, and variables $z_s (s \notin \{s_1, s_2\})$ where $s_1 < s_2, t_1, t_2$ are arbitrary. Assume (3.5) holds, and then the matrix

$$\begin{pmatrix} \gamma_i^1, & \ldots, & \gamma_i^t \\ \gamma_l^1, & \ldots, & \gamma_l^t \end{pmatrix}$$

is of rank 1. That is, the integer-entry vectors $(\gamma_i^1, \gamma_l^1), \ldots, (\gamma_i^t, \gamma_l^t)$ are constant tuples of a vector. Then one can show that there is a nonzero vector $\mathbf{v} = (v_1, v_2)$ in $\mathbb{Z}_+^2$ such that all of $(\gamma_i^1, \gamma_l^1), \ldots, (\gamma_i^t, \gamma_l^t)$ can be written as an integer multiple of $\mathbf{v}$. If $i \neq l$, this would gives that

$$z^{\gamma^1}, \ldots, z^{\gamma^t}$$

are functions of $z_i^{v_1} z_l^{v_2}$ and other variables $z_s (s \neq i, l)$, which is a contradiction. Thus $i = l$.

Letting j and $k_1, \ldots, k_t$ be integers in $\mathbb{Z}_+$, we write $\mathbf{k} = (k_1, \ldots, k_t)$,

$$\mathbf{k} \cdot \gamma = \sum_{m=1}^{t} k_m \gamma^m,$$

and let

$$Q(z) = z^{j\beta + \mathbf{k} \cdot \gamma}.$$

By (3.4)

$$M_Q^* M_Q z^\alpha = \omega(j, \mathbf{k}, \alpha) z^\alpha,$$

where

$$\omega(j, \mathbf{k}, \alpha) = \prod_{1 \leq i \leq d} \frac{\alpha_i + 1}{j\beta_i + (\mathbf{k} \cdot \gamma)_i + \alpha_i + 1} = \prod_{1 \leq i \leq d} \frac{\alpha_i + 1}{j\beta_i + \sum_m k_m \gamma_i^{(m)} + \alpha_i + 1}.$$

Our idea is to study the spectrum projection of $M_Q^* M_Q$. Let $P_\alpha(j, \mathbf{k})$ denote the orthogonal projection onto the closed linear span of

$$\{z^I : \omega(j, \mathbf{k}, \alpha) = \omega(j, \mathbf{k}, I), I \in \mathbb{Z}_+^d\}$$

and P_α be the infimum of $P_\alpha(j, \mathbf{k})$; that is, P_α equals the orthogonal projection onto the closed linear span of

$$\{z^I : \omega(j, \mathbf{k}, \alpha) = \omega(j, \mathbf{k}, I), j \in \mathbb{Z}_+, \mathbf{k} \in \mathbb{Z}_+^t\}.$$

By special choice of $\gamma^1, \ldots, \gamma^t$ in Φ, we will show that P_α exactly equals the orthogonal projection onto $\mathbb{C}z^\alpha$. Note that by spectral theorem, $P_\alpha(j, \mathbf{k})$ lies in $\mathcal{W}^*(\Phi, \mathbb{D}^d)$, and so does P_α.

To continue, we study a necessary condition for

$$\omega(j,\mathbf{k},\alpha)=\omega(j,\mathbf{k},\alpha'),$$

where $\alpha,\alpha'\in\mathbb{Z}_+^d$. Precisely, define

$$h(\zeta_1)=\omega(j,\zeta_1,k_2,\dots,k_t,\alpha)-\omega(j,\zeta_1,k_2,\dots,k_t,\alpha')$$

which is a bounded holomorphic function on the right half plane

$$\{\zeta_1\in\mathbb{C}: Re\,\zeta_1>0\}.$$

Since $h(k)=0$ for $k=1,2,\dots$, h is identically zero [12]. That is,

$$\prod_{1\le i\le d}\frac{\alpha_i+1}{j\beta_i+\zeta_1\gamma_i^1+\sum_{m\ge 2}k_m\gamma_i^{(m)}+\alpha_i+1}$$
$$=\prod_{1\le i\le d}\frac{\alpha_i'+1}{j\beta_i+\zeta_1\gamma_i^1+\sum_{m\ge 2}k_m\gamma_i^{(m)}+\alpha_i'+1}.$$

Both sides extend throughout the complex plane except for finitely many points. By similar reasoning, we have

$$\prod_{1\le i\le d}\frac{\alpha_i+1}{\zeta_0\beta_i+\sum_{1\le m\le t}\zeta_m\gamma_i^{(m)}+\alpha_i+1}=\prod_{1\le i\le d}\frac{\alpha_i'+1}{\zeta_0\beta_i+\sum_{1\le m\le t}\zeta_m\gamma_i^{(m)}+\alpha_i'+1},$$

where $\zeta_0,\dots,\zeta_t$ are complex variables. Hence, as a finite sequence

$$\{\frac{1}{\beta_i}(\sum_{1\le m\le t}\zeta_m\gamma_i^{(m)}+\alpha_i+1):1\le i\le d\}$$

is a permutation of

$$\{\frac{1}{\beta_i}(\sum_{1\le m\le t}\zeta_m\gamma_i^{(m)}+\alpha_i'+1):1\le i\le d\}.$$

This immediately gives that the sequence of vectors in $\mathbb{C}^{d+1}$

$$\{\frac{1}{\beta_i}(\gamma_i^1,\dots,\gamma_i^t,\alpha_i+1):1\le i\le d\}$$

is a permutation of

$$\{\frac{1}{\beta_i}(\gamma_i^1,\dots,\gamma_i^t,\alpha_i'+1):1\le i\le d\}.$$

For fixed i with $1\le i\le d$, the equation (3.5)

$$\frac{1}{\beta_i}(\gamma_i^1,\dots,\gamma_i^t)=\frac{1}{\beta_l}(\gamma_l^1,\dots,\gamma_l^t)$$

holds only if $i=l$. This immediately gives $\alpha_i=\alpha_i'$, and by arbitrariness of i, we have $\alpha=\alpha'$, forcing P_α to be the orthogonal projection onto $\mathbb{C}z^\alpha$.

Since $\mathcal{W}^*(\Phi,\mathbb{D}^d)$ contains all those projections P_α, it follows that each operator T in $\mathcal{V}^*(\Phi,\mathbb{D}^d)$ commutes with all these projections, and hence T is diagonal with respect to $\{z^\alpha:\alpha\in\mathbb{Z}_+^d\}$. Therefore, we deduce that $\mathcal{V}^*(\Phi,\mathbb{D}^d)$ is abelian to finish the proof. $\square$

For the case of dimension $d = 2$, we have the following characterization for abelian property of $\mathcal{V}^*(\Phi_A, \mathbb{D}^2)$.

Corollary 3.2. *Suppose* $A \in M_2(\mathbb{Z}_+)$. *Then* $\mathcal{V}^*(\Phi_A, \mathbb{D}^2)$ *is abelian if and only if* $\det A \neq 0$.

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