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A STRUCTURE OF NONCENTRAL IDEMPOTENTS

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Abstract. We focus on the structure of the set of noncentral idempotents whose role is similar to one of central idempotents. We introduce the concept of quasi-Abelian rings which unit-regular rings satisfy. We first observe that the class of quasi-Abelian rings is seated between Abelian and direct finiteness. It is proved that a regular ring is directly finite if and only if it is quasi-Abelian. It is also shown that quasi-Abelian property is not left-right symmetric, but left-right symmetric when a given ring has an involution. Quasi-Abelian property is shown to do not pass to polynomial rings, comparing with Abelian property passing to polynomial rings.

1. A concept related to noncentral idempotents

Throughout this paper all rings are associative with identity unless otherwise specified. Let R be a ring and denote 1 the identity of R. $I_e(R)$ is used to denote the set of all idempotents of R, and $I(R) = I_e(R)\backslash\{0, 1\}$. We use $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ to denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. For $n \geq 2$, denote the n by n full matrix ring over R by $\mathrm{Mat}_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. We use E_{ij} to denote the n by n matrix with (i, j)-entry 1 and zeros elsewhere. The polynomial ring with an indeterminate x over R is denoted by $R[x]$. Let $\mathbb{Z}$ ($\mathbb{Z}_n$) denote the ring of integers (modulo n). The preceding notations follow the literature.

A ring is usually called *Abelian* if every idempotent is central. A ring is usually called *reduced* if it has no nonzero nilpotent elements. It is easily checked that reduced rings are Abelian but not conversely.

Lemma 1.1. *Let R be a ring with $I(R)$ nonempty. Then the following conditions are equivalent*:

(1) *R is Abelian*;

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(2) *For every* $(e, r) \in I(R) \times R$, $er^m = r^m e$ *for all* $m \geq 1$;
(3) *For every* $(e, r) \in I(R) \times R$, $er^k = r^k e$ *for some* $k \geq 1$;
(4) *For every* $(e, r) \in I(R) \times R$, *there exists* $s \in R$ *such that* $er = se$.

Proof. (1)⇒(2), (2)⇒(3), and (1)⇒(4) are obvious.

(3)⇒(1): Let the condition (3) hold, and assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Set $\alpha = e + ea(1-e)$. By the condition (3), there exists $k \geq 1$ such that $e\alpha^k = \alpha^k e$. But $e\alpha^k = e\alpha = \alpha \neq 0 = \alpha e = \alpha^k e$, a contradiction. Thus R is Abelian.

(4)⇒(1): Let the condition (4) hold, and assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Set $\beta = ea(1-e)$. By the condition (4), there exists $s \in R$ such that $e\beta = se$. Then $0 \neq \beta = e\beta = se \in R(1-e) \cap Re = 0$, a contradiction. Thus R is Abelian. □

Lemma 1.1 leads us to the following definition.

Definition 1.2. A ring R is said to be *right* (resp., *left*) *quasi-Abelian* provided that either $I(R)$ is empty, or else for any $(e, a) \in I(R) \times R$ (resp., $(a, e) \in R \times I(R)$) there exists $(b, f) \in R \times I(R)$ (resp., $(f, b) \in I(R) \times R$) such that $ea = bf$ (resp., $ae = fb$). The ring R is said to be *quasi-Abelian* if it is both right and left quasi-Abelian.

Remark 1.3. Let R be a ring with $I(R)$ nonempty. Then the following can be easily obtained.

(1) A ring R is right quasi-Abelian if and only if for any $e \in I(R)$ and $r \in R$, there exists $f \in I(R)$ such that $ea \in Rf$.

(2) If R is a right quasi-Abelian ring, then for any $r \in R$ and $e \in I(R)$, there exist $r_1, r_2 \in R$ and $f_1, f_2 \in I(R)$ such that
(i) $er = r_1 f_1$ and $(1-e)r = r_2 f_2$;
(ii) $er = erf_1 = er_1 f_1$ and $(1-e)r = (1-e)rf_2 = (1-e)r_2 f_2$;
(iii) $r = er + (1-e)r = erf_1 + (1-e)rf_2$.

Abelian rings are clearly quasi-Abelian, but not conversely by the following.

Example 1.4. Consider the non-Abelian ring $R = U_2(\mathbb{Z}_2)$ and the fact that

$$I(R) = \left\{ E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then

$$E_1 R = E_2 R = \left\{ 0, E_1, E_2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, E_3 R = \{0, E_3\}, E_4 R = \{0, E_4\};$$

and

$$RE_1 = \{0, E_1\}, RE_2 = \{0, E_2\}, RE_3 = RE_4 = \left\{ 0, E_3, E_4, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}.$$

So every matrix in each $E_i R$ is contained in some RE_j, concluding that R is right quasi-Abelian. The left quasi-Abelian case is similarly shown.

The condition of quasi-Abelian is not left-right symmetric as the following shows.

Example 1.5. (1) Let K be a field and define a function $\sigma : K[t] \to K[t]$ by $\sigma(f(t)) = f(t^2)$, where $K[t]$ is a polynomial ring with an indeterminate t over K. Let R_0 be the skew polynomial ring $K[t][x;\sigma]$ with an indeterminate x over $K[t]$, only subject to $xf(t) = \sigma(f(t))x$ for all $f(t) \in K[t]$, where every polynomial of $K[t][x;\sigma]$ is expressed by $a_0 + a_1a + \cdots + a_nx^n$ with $a_i \in K[t]$. Next let R be the subring

$$\begin{pmatrix} K & R_0x \\ 0 & R_0 \end{pmatrix} = \begin{pmatrix} K & K[t][x;\sigma]x \\ 0 & K[t][x;\sigma] \end{pmatrix}$$

of $U_2(R_0)$.

Let $E = \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) \in I(R)$. Then $a^2 = a, b^2 = b$ and $ac+cb = c$. $E \in I(R)$ implies that either $(a,b) = (1,0)$ (i.e., $E = \left(\begin{smallmatrix} 1 & c \\ 0 & 0 \end{smallmatrix}\right)$) or $(a,b) = (0,1)$ (i.e., $E = \left(\begin{smallmatrix} 0 & c \\ 0 & 1 \end{smallmatrix}\right)$). Thus

$$I(R) = \left\{ \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} \mid c, d \in K[t][x;\sigma]x \right\}.$$

We claim that R is right quasi-Abelian. To see that, let $E = \left(\begin{smallmatrix} 1 & c \\ 0 & 0 \end{smallmatrix}\right), F = \left(\begin{smallmatrix} 0 & d \\ 0 & 1 \end{smallmatrix}\right) \in I(R)$ and $A = \left(\begin{smallmatrix} u & v \\ 0 & w \end{smallmatrix}\right) \in R$. Then $EA = \left(\begin{smallmatrix} u & v+cw \\ 0 & 0 \end{smallmatrix}\right)$ and $FA = \left(\begin{smallmatrix} 0 & dw \\ 0 & w \end{smallmatrix}\right)$. Consider the following computation:

$$\begin{pmatrix} u & v+cw \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & u^{-1}(v+cw) \\ 0 & 0 \end{pmatrix} \text{ when } u \neq 0;$$

$$\begin{pmatrix} 0 & v+cw \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v+cw \\ 0 & 1 \end{pmatrix} \text{ when } u = 0;$$

and

$$\begin{pmatrix} 0 & dw \\ 0 & w \end{pmatrix} = \begin{pmatrix} u & dw \\ 0 & w \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

noting that

$$\begin{pmatrix} 1 & u^{-1}(v+cw) \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & v+cw \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in I(R).$$

Therefore R is right quasi-Abelian.

Next, assume that R is left quasi-Abelian. Let $B = \left(\begin{smallmatrix} 0 & tx \\ 0 & t \end{smallmatrix}\right) \in R$ and $G = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)$. Then $BG = \left(\begin{smallmatrix} 0 & tx \\ 0 & t \end{smallmatrix}\right)$. Assume that

$$BG = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \alpha c \\ 0 & c \end{pmatrix}$$

for some $\left(\begin{smallmatrix} 0 & \alpha \\ 0 & 1 \end{smallmatrix}\right) \in I(R)$ and $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in R$. Then $c = t$ and $\alpha c = tx$. Note that α is of the form $a_1x + \cdots + a_nx^n$ with $a_i \in K[t]$. This yields

$$tx = \alpha t = (a_1x + \cdots + a_nx^n)t = a_1t^2x + \cdots + a_nt^{2^n}x^n,$$

which is impossible. Therefore R is not left quasi-Abelian.

(2) Under the same condition of the preceding construction in (1), let

$$R = \begin{pmatrix} R_0 & xR_0 \\ 0 & K \end{pmatrix} = \begin{pmatrix} K[t][x;\sigma] & xK[t][x;\sigma] \\ 0 & K \end{pmatrix},$$

where $f(t)x = x\sigma(f(t))$ for all $f(t) \in K[t]$ and every polynomial of $K[t][x;\sigma]$ is expressed by $a_0 + xa_1 + \cdots + x^n a_n$ with $a_i \in K[t]$.

Then R is not right quasi-Abelian but left quasi-Abelian via a similar computation to (1).

Recall that an involution on a ring R is a function $* : R \to R$ which satisfies the properties that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, and $(x^*)^* = x$ for all $x, y \in R$. We get $0^* = 0$ because $0^* = (0+0)^* = 0^* + 0^*$, and $e^* = (ee)^* = e^*e^*$ implies that $e^* \in I_e(R)$ for all $e \in I_e(R)$. It is also easily shown that $u^* \in U(R)$ for all $u \in U(R)$, where $U(R)$ denotes the group of all units in R.

We see in the following a condition under which the quasi-Abelian property is left-right symmetric.

Proposition 1.6. *Let R be a ring with an involution $*$. Then the following conditions are equivalent*:

(1) *R is right quasi-Abelian*;

(2) *R is left quasi-Abelian.*

Proof. First note that $e^* \in I(R)$ for all $e \in I(R)$. Note that $e \neq 0$ and $e \neq 1$, entailing $1 - e \in I(R)$. If $e^* = 0$, then $e = (e^*)^* = 0^* = 0$, a contradiction. So $e^* \neq 0$. If $e^* = 1$, then $0 = (0^*)^* = ((e(1-e))^*)^* = ((1-e)^*e^*)^* = ((1-e)^*)^* = 1 - e$, a contradiction. So $e^* \neq 1$.

(1)⇔(2): Let R be right quasi-Abelian, and $(a, e) \in R \times I(R)$. Then there exists $(b, f) \in R \times I(R)$ such that $e^*a^* = bf$, noting $e^* \in I(R)$. Thus we have

$$ae = ((ae)^*)^* = (e^*a^*)^* = (bf)^* = f^*b^*.$$

But $f^* \in I(R)$, so R is left quasi-Abelian. The converse can be similarly proved. □

By Proposition 1.6, a ring with an involution is quasi-Abelian when it is right or left quasi-Abelian.

The condition "R is a ring with an involution $*$" in Proposition 1.6 is not superfluous as we see the ring R in Example 1.5(1) which is right quasi-Abelian but not left quasi-Abelian. In fact, this ring cannot have any involution by applying Proposition 1.6. We examine this fact in details in the following.

Remark 1.7. Let R be the ring R in Example 1.5(1). Then R is right quasi-Abelian but not left quasi-Abelian. So R is not able to have any involution by Proposition 1.6. We check this fact on a case-by-case computation. Assume on the contrary that R has an involution $*$. We first examine the basic properties of the involution in R as follows. Note that $N(R) = \left(\begin{smallmatrix} 0 & R_0x \\ 0 & 0 \end{smallmatrix}\right)$ and recall that $E^* \in I(R)$ for all $E \in I(R)$ by the proof of Proposition 1.6. Let $A, B \in R$.

(1) If $AB \neq 0$, then $0 \neq AB = ((AB)^*)^* = (B^*A^*)^*$ and so $B^*A^* \neq 0$ (otherwise, $B^*A^* = 0$ implies $AB = 0$). Especially, if $A \neq 0$, then $A^* \neq 0$, letting $B = 1_R$ where 1_R denotes the identity matrix in R.

(2) If $A^2 = 0$, then $0 = (A^2)^* = A^*A^*$. This implies that $A^* \in N(R)$ for all $A \in N(R)$.

(3) Since $1_R = (E_{11} + E_{22})^* = E_{11}^* + E_{22}^*$ and $E_{11}^*, E_{22}^* \in I(R)$, we have

$$E_{11}^* = \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}, E_{22}^* = \begin{pmatrix} 0 & -c \\ 0 & 1 \end{pmatrix} \text{ or } E_{11}^* = \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix}, E_{22}^* = \begin{pmatrix} 1 & -d \\ 0 & 0 \end{pmatrix},$$

where $c, d \in R_0x$.

Suppose that $E_{11}^* = \left(\begin{smallmatrix} 1 & c \\ 0 & 0 \end{smallmatrix}\right)$ (and so $E_{22}^* = \left(\begin{smallmatrix} 0 & -c \\ 0 & 1 \end{smallmatrix}\right)$). For $0 \neq f \in R_0x$,

$$0 \neq \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \in N(R), \text{ and so we let } \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & f' \\ 0 & 0 \end{pmatrix},$$

where $f' \neq 0$ by (1) and (2). Hence

$$0 \neq \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}^* = \left(E_{11}\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}\right)^* = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}^* E_{11}^* = \begin{pmatrix} 0 & f' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix} = 0.$$

This induces a contradiction. Thus we conclude that

$$E_{11}^* = \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} \text{ and } E_{22}^* = \begin{pmatrix} 1 & -d \\ 0 & 0 \end{pmatrix}.$$

(4) Let $A = \left(\begin{smallmatrix} a & 0 \\ 0 & f \end{smallmatrix}\right) \in R$ with $0 \neq a, f$. Then $A^* = \left(\begin{smallmatrix} a' & g \\ 0 & f' \end{smallmatrix}\right)$, where $a' \neq 0$ and $f' \neq 0$. Indeed, if $A^* = \left(\begin{smallmatrix} 0 & g \\ 0 & f' \end{smallmatrix}\right) \in R$ with $0 \neq f'$, then

$$0 \neq (E_{12}A)^* = A^*E_{12}^* = \begin{pmatrix} 0 & g \\ 0 & f' \end{pmatrix}\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = 0 \text{ where } 0 \neq h \in R_0x$$

by (1) and (2), noting that $0 \neq E_{12} \in N(R)$. This is a contradiction. Next, if $A^* = \left(\begin{smallmatrix} a' & g \\ 0 & 0 \end{smallmatrix}\right) \in R$ with $0 \neq a'$, then $0 \neq (AE_{12})^* = E_{12}^*A^* = 0$. This is also a contradiction.

Therefore we have $A^* = \left(\begin{smallmatrix} a' & g \\ 0 & f' \end{smallmatrix}\right)$ with $a' \neq 0$ and $f' \neq 0$.

(5) Let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right) \in R$. Then

$$A^* = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}^* = \left(E_{11} + \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right)^* = E_{11}^* + \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}^* = \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}^*.$$

By (4), we must have $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} k & e \\ 0 & f' \end{smallmatrix}\right)$ for some $0 \neq k \in K$, $e \in K[t][x;\sigma]x$ and $f' \in K[t][x;\sigma]$. Here if $f' \neq 0$, then

$$0 \neq E_{12}^*\begin{pmatrix} k & e \\ 0 & f' \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} E_{12}\right)^* = 0,$$

a contradiction. Thus $\left(\begin{smallmatrix} 0 & 0 \\ 0 & x \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} k & e \\ 0 & 0 \end{smallmatrix}\right)$. It then follows that

$$A^* = \begin{pmatrix} 0 & d \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} k & e \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} k & d+e \\ 0 & 1 \end{pmatrix},$$

entailing $A^* \in U(R)$. This forces $A \in U(R)$, a contradiction.

Following the literature, a ring R is called *directly finite* (or *Dedekind finite*) if $ab = 1$ implies $ba = 1$ for $a, b \in R$. It is easily checked that Abelian rings are directly finite. The class of directly finite rings contains rings that satisfy either the ascending or the descending chain condition for principal right ideals generated by idempotents by [4, Theorem 1]. So left or right Artinian rings are directly finite. This implies that there exist many directly finite rings which are non-Abelian. Moreover we have the following.

Example 1.8. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ and consider

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \ E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R.$$

Then $E \in I(R)$ and $EA = A$. Assume that there exists

$$\left(B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, F = \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right) \in R \times I(R)$$

such that $A = BF$. Then $ad = 2$ and $d^2 = d$, forcing $a = 2$ and $d = 1$. But $F \in I(R)$ and so $f = 0$. This yields

$$\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = A = BF = \begin{pmatrix} 2 & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2e \\ 0 & 0 \end{pmatrix},$$

a contradiction. Thus R is not right quasi-Abelian. But R is directly finite by [3, Proposition 2.7(1)] since $N_*(R) = N^*(R) = N(R)$.

Following von Neumann [6], a ring R (possibly without identity) is said to be *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Such a ring is also called *von Neumann regular* by Goodearl [2]. It is shown that R is regular if and only if every principal right (left) ideal of R is generated by an idempotent in [2, Theorem 1.1]. This result implies that every regular ring R is semiprimitive (i.e., $J(R) = 0$).

Following Ehrlich [1], a ring R is called *unit-regular* if for each $a \in R$ there exists a unit $u \in R$ such that $a = aua$. Every unit-regular ring is directly finite by [2, Proposition 5.2], and clearly regular. But there exists a directly finite regular ring but not unit-regular by [2, Example 5.10]. We see that there exists a regular ring which is not directly finite by Example 1.10. So if a regular ring is directly finite, then it is right quasi-Abelian as we see in the following.

Theorem 1.9. (1) *A right or left quasi-Abelian ring is directly finite.*

(2) *Directly finite regular rings are quasi-Abelian.*

(3) *Unit-regular rings are quasi-Abelian.*

(4) *Semisimple Artinian rings are quasi-Abelian.*

Proof. (1) Let R be a right quasi-Abelian ring and suppose that $ab = 1$ for $a, b \in R$. Assume on the contrary that $ba \neq 1$. Then $ba \in I(R)$. Since R is quasi-Abelian, there exists $(c, e) \in R \times I(R)$ such that $(ba)b = ce$. This yields

$$b(1-e) = (bab)(1-e) = ce(1-e) = 0.$$

So we have $1 - e = ab(1-e) = a0 = 0$ because $ab = 1$. Thus $e = 1$, a contradiction because $e \in I(R)$. Thus R is directly finite. The left case can be similarly proved.

(2) Let R be a directly finite regular ring and $(e, a) \in I(R) \times R$. It suffices to consider the case of $ea \neq 0$. Since R is regular, $eabea = ea$ for some $b \in R$. Here $(bea)^2 = bea$. Here $0 \neq ea = eabea$ implies $bea \neq 0$. Next if $bea = 1$, then $eab = 1$ because R is directly finite, and so we obtain

$$0 = (1-e)eab = 1 - e \neq 0,$$

a contradiction. So $bea \in I(R)$. We have now $ea = (ea)(bea)$ with $bea \in I(R)$, implying that R is right quasi-Abelian. The left case can be similarly proved.

(3) Let R be a unit-regular ring. Then R is directly finite by [2, Proposition 5.2]. Thus R is quasi-Abelian by (2).

(4) is an immediate consequence of (2), or obtained from (3) because semi-simple Artinian rings are unit-regular by [2, Theorem 4.1]. □

Observe that the converse of Theorem 1.9(1) does not hold by Example 1.8. In fact, the ring R is not left quasi-Abelian either, considering an involution $*$ by defining

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^* = \begin{pmatrix} c & b \\ 0 & a \end{pmatrix}.$$

The class of regular rings and the class of right quasi-Abelian rings do not imply each other by the following example and Example 1.4, noting that $E_{12}AE_{12} = 0$ for any $A \in U_2(\mathbb{Z}_2)$.

Example 1.10. Let R be the ring of column finite infinite matrices over a field. Then R is regular. But it is not directly finite since $CA = 1$ and $AC = E \neq 1$. Thus R is not right quasi-Abelian by Theorem 1.9(1).

By Example 1.8 and Theorem 1.9(1), we can now say that the concept of right (left) quasi-Abelian ring is a ring property that is between Abelian property and direct finiteness.

Corollary 1.11. *For a regular ring R the following conditions are equivalent*:

(1) *R is directly finite*;
(2) *R is right quasi-Abelian*;
(3) *R is left quasi-Abelian.*

Proof. The proof is done by help (1) and (2) of Theorem 1.9. □

By Corollary 1.11, a regular ring is quasi-Abelian when it is right or left quasi-Abelian.

Recall that Abelian regular rings are reduced by [2, Theorem 3.2]. So, based on Theorem 1.9(2), one may ask whether directly finite regular rings are Abelian. However the conclusion "quasi-Abelian" of Theorem 1.9(2) cannot be replaced by the condition "Abelian" by the following example.

Example 1.12. Let D be a division ring. Consider $R = \mathrm{Mat}_n(D)$ for $n \geq 2$. Then R is a directly finite regular ring by [2, Theorem 1.7 and Proposition 5.5], but not Abelian. Note that R is quasi-Abelian by Theorem 1.9(2).

Considering Theorem 1.9(3), one may ask whether $\mathrm{Mat}_2(R)$ is quasi-Abelian when R is a quasi-Abelian ring. But there exists a domain (hence a quasi-Abelian ring) R such that $\mathrm{Mat}_2(R)$ is not directly finite by [7, Theorem 1.0], and so it is not quasi-Abelian by Theorem 1.9(1).

2. Matrices and polynomials over quasi-Abelian rings

In this section we study more properties of quasi-Abelian rings in relation to various kinds of ring extensions which have roles in ring theory. It is well known that $U_2(R)$ over any ring R is not Abelian. But we consider the case of $U_2(R)$ being quasi-Abelian as follows, comparing with Example 1.12.

Theorem 2.1. *Let R be a domain. Then the following conditions are equivalent:*
(1) *R is a division ring;*
(2) *$U_n(R)$ is a right quasi-Abelian ring for all $n \geq 2$;*
(3) *$U_n(R)$ is a left quasi-Abelian ring for all $n \geq 2$.*

Proof. (1)⇒(2). Let $E = U_2(R)$ and suppose that R is a division ring. Since R is a division ring, we have

$$I(E) = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha, \beta \in R \right\}.$$

Consider the product $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in E$. Then

$$\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & b+\alpha c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & a^{-1}(b+\alpha c) \\ 0 & 0 \end{pmatrix} \text{ when } a \neq 0$$

and

$$\begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & b+\alpha c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b+\alpha c \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ when } a = 0,$$

noting that $\left(\begin{smallmatrix} 1 & a^{-1}(b+\alpha c) \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(E)$.

Next consider the product $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ for $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in E$. Then

$$\begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \beta c \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & \beta c \\ 0 & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

noting that $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(E)$.

Therefore E is a right quasi-Abelian ring.

Let $E' = U_3(R)$. Since R is a division ring, we have

$$I(E') = \left\{ \begin{pmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha_3 & \alpha_3\alpha_4 \\ 0 & 1 & \alpha_4 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_6 \\ 0 & 0 & 1 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 0 & \alpha_7 \\ 0 & 1 & \alpha_8 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \alpha_9 & -\alpha_9\alpha_{10} \\ 0 & 0 & \alpha_{10} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha_{11} & \alpha_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \alpha_i \in R \right\}.$$

Let $a = (a_{ij}) \in E'$. Recall that E is right quasi-Abelian by the argument above. We use this fact freely.

(i) Consider $e = \left(\begin{smallmatrix} 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I(E')$. Then $e_0 = \left(\begin{smallmatrix} 1 & \alpha_1 \\ 0 & 0 \end{smallmatrix}\right) \in I(E)$, and so there exist $f_0 = \left(\begin{smallmatrix} f_1 & f_2 \\ 0 & f_3 \end{smallmatrix}\right) \in I(E)$ and $b_0 = \left(\begin{smallmatrix} b_1 & b_2 \\ 0 & b_3 \end{smallmatrix}\right) \in E$ such that $e_0a_0 = b_0f_0$, where $a_0 = \left(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{smallmatrix}\right) \in E$. Consider

$$b = \begin{pmatrix} b_1 & b_2 & a_{13} + \alpha_1 a_{23} + \alpha_2 a_{33} \\ 0 & b_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} f_1 & f_2 & 0 \\ 0 & f_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in E'.$$

Here $f_0 \in I(E)$ implies exactly one of f_1 and f_3 is zero, and so $f \in I(E')$. Moreover we get $ea = bf$.

(ii) Consider $e = \left(\begin{smallmatrix} 0 & \alpha_3 & \alpha_3\alpha_4 \\ 0 & 1 & \alpha_4 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I(E')$. Then $ea = (b_{ij})$ with $b_{11} = 0$. Let $b = (b_{ij})$ and $f = E_{22} + E_{33} \in E'$. Then $f \in I(E')$ and $ea = bf$.

(iii) Consider $e = \left(\begin{smallmatrix} 0 & 0 & \alpha_5 \\ 0 & 0 & \alpha_6 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in I(E')$. Then $ea = (b_{ij})$ with $b_{11} = 0$. Let $b = (b_{ij})$ and $f = E_{22} + E_{33} \in E'$. Then $f \in I(E')$ and $ea = bf$.

(iv) Consider $e = \left(\begin{smallmatrix} 1 & 0 & \alpha_7 \\ 0 & 1 & \alpha_8 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in I(E')$. Then $ea = \left(\begin{smallmatrix} a_{11} & a_{12} & a_{13}+\alpha_7 a_{33} \\ 0 & a_{22} & a_{23}+\alpha_8 a_{33} \\ 0 & 0 & 0 \end{smallmatrix}\right)$.

(iv)-1-1. Assume that $a_{11} \neq 0$ and $a_{22} \neq 0$. Then we let

$$b = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$
$$f = \begin{pmatrix} 1 & 0 & a_{11}^{-1}[(a_{13} + \alpha_7 a_{33}) - a_{12}a_{22}^{-1}(a_{23} + \alpha_8 a_{33})] \\ 0 & 1 & a_{22}^{-1}(a_{23} + \alpha_8 a_{33}) \\ 0 & 0 & 0 \end{pmatrix} \in E'.$$

Then $f \in I(E')$ and $ea = bf$.

(iv)-1-2. Assume that $a_{11} \neq 0$ and $a_{22} = 0$. Then we let

$$b = \begin{pmatrix} a_{11} & 0 & a_{13} + \alpha_7 a_{33} \\ 0 & 0 & a_{23} + \alpha_8 a_{33} \\ 0 & 0 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 1 & a_{11}^{-1}a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in E'.$$

Then $f \in I(E')$ and $ea = bf$.

(iv)-2. Assume that $a_{11}=0$. Then we let

$$b=\begin{pmatrix}0 & a_{12} & a_{13}+\alpha_7 a_{33}\\ 0 & a_{22} & a_{23}+\alpha_8 a_{33}\\ 0 & 0 & 0\end{pmatrix} \text{ and } f=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\in E'.$$

Then $f\in I(E')$ and $ea=bf$.

(v) Consider $e=\left(\begin{smallmatrix}1 & \alpha_9 & -\alpha_9\alpha_{10}\\ 0 & 0 & \alpha_{10}\\ 0 & 0 & 1\end{smallmatrix}\right)\in I(E')$. Then

$$ea=\begin{pmatrix}a_{11} & a_{12}+\alpha_9 a_{22} & a_{13}+\alpha_9 a_{23}-\alpha_9\alpha_{10}a_{33}\\ 0 & 0 & \alpha_{10}a_{33}\\ 0 & 0 & a_{33}\end{pmatrix}.$$

(v)-1. If $a_{11}\neq 0$, then we let

$$b=\begin{pmatrix}a_{11} & 0 & a_{13}+\alpha_9 a_{23}-\alpha_9\alpha_{10}a_{33}\\ 0 & 0 & \alpha_{10}a_{33}\\ 0 & 0 & a_{33}\end{pmatrix} \text{ and}$$

$$f=\begin{pmatrix}1 & a_{11}^{-1}(a_{12}+\alpha_9 a_{22}) & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\in E'.$$

Then $f\in I(E')$ and $ea=bf$.

(v)-2. If $a_{11}=0$, then we let

$$b=\begin{pmatrix}0 & a_{12}+\alpha_9 a_{22} & a_{13}+\alpha_9 a_{23}-\alpha_9\alpha_{10}a_{33}\\ 0 & 0 & \alpha_{10}a_{33}\\ 0 & 0 & a_{33}\end{pmatrix} \text{ and } f=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}\in E'.$$

Then $f\in I(E')$ and $ea=bf$.

(vi) Consider $e=\left(\begin{smallmatrix}0 & \alpha_{11} & \alpha_{12}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{smallmatrix}\right)\in I(E')$. Then $ea=(b_{ij})$ with $b_{11}=0$. Let $b=(b_{ij})$ and $f=E_{22}+E_{33}\in E'$. Then $f\in I(E')$ and $ea=bf$.

Therefore $E'=U_3(R)$ is a right quasi-Abelian ring by the computation of the preceding six cases.

We next summarize the computation of $U_3(R)$ via four cases. Let $e=(e_{ij})\in I(U_3(R))$, $e_0=\left(\begin{smallmatrix}e_{11} & e_{12}\\ 0 & e_{22}\end{smallmatrix}\right)$, $a_0=\left(\begin{smallmatrix}a_{11} & a_{12}\\ 0 & a_{22}\end{smallmatrix}\right)$, and $b=ea=(b_{ij})\in U_3(R)$. Then $b_0=e_0a_0=\left(\begin{smallmatrix}b_{11} & b_{12}\\ 0 & b_{22}\end{smallmatrix}\right)$.

(I) Suppose that $e_{11}=0$ or $a_{11}=0$. This is the cases of (i) when $a_{11}=0$, (ii), (iii), (iv)-2, (v)-2, and (vi) of $U_3(R)$. Then $b_{11}=0$, and so $ea=(b_{ij})(E_{22}+E_{33})$. Note $E_{22}+E_{33}\in I(U_3(R))$.

(II) Suppose that $e_{11}=1$, $\operatorname{rank}(e_0)=1$ (i.e., $e_{22}=0$), and $a_{11}\neq 0$. This is the cases of (i) when $a_{11}\neq 0$ and (v)-1 of $U_3(R)$. Then there exist $c_0=(c_{st})\in U_2(R)$ and $f_0=(f_{st})\in I(U_2(R))$ such that $e_0a_0=c_0f_0$. Letting

$$c=\begin{pmatrix}c_{11} & c_{12} & \sum_{i=1}^{3}e_{1i}a_{i3}\\ 0 & c_{22} & \sum_{i=2}^{3}e_{2i}a_{i3}\\ 0 & 0 & e_{33}a_{33}\end{pmatrix} \text{ and } f=\begin{pmatrix}f_{11} & f_{12} & 0\\ 0 & f_{22} & 0\\ 0 & 0 & 1\end{pmatrix},$$

we have $ea = cf$. Note that $f_0 \in I(U_2(R))$ implies $f \in I(U_3(R))$.

(III) Suppose that $\text{rank}(e_0) = 2$ (i.e., $e_0 = E_{11} + E_{22}$), and $a_{11} \neq 0$, $\text{rank}(a_0) = 1$ (i.e., $a_{22} = 0$). Then $e_{33} = 0$. This is the case of (iv)-1-2 of $U_3(R)$. Then $e_0a_0 = E_{11}(e_0a_0)$ and $E_{11} \in I(U_2(R))$. So there exist $c_0 = (c_{st}) \in U_2(R)$ and $f_0 = (f_{st}) \in I(U_2(R))$ such that $e_0a_0 = c_0f_0$. Letting

$$c = \begin{pmatrix} c_{11} & c_{12} & \sum_{i=1}^{3} e_{1i}a_{i3} \\ 0 & c_{22} & \sum_{i=2}^{3} e_{2i}a_{i3} \\ 0 & 0 & e_{33}a_{33} \end{pmatrix} \text{ and } f = \begin{pmatrix} f'_{11} & f'_{12} & 0 \\ 0 & f'_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have $ea = cf$. Note that $f'_0 \in I(U_2(R))$ implies $f \in I(U_3(R))$.

(IV) Suppose that $\text{rank}(e_0) = 2$ (i.e., $e_0 = E_{11} + E_{22}$), and $\text{rank}(a_0) = 2$ (i.e., $a_{ii} \neq 0$ for all $i = 1, 2$). Then $e_{33} = 0$. Note $a_{ii} = b_{ii}$ for all $i = 1, 2$. This is the case of (iv)-1-1 of $U_3(R)$. So letting

$$c = \begin{pmatrix} a_{11} & b_{12} & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } f = \begin{pmatrix} 1 & 0 & a_{11}^{-1}[\sum_{i=1}^{3} e_{1i}a_{i3} - a_{12}a_{22}^{-1}(\sum_{i=2}^{3} e_{2i}a_{i3})] \\ 0 & 1 & a_{22}^{-1}(\sum_{i=2}^{3} e_{2i}a_{i3}) \\ 0 & 0 & 0 \end{pmatrix},$$

we have $f \in I(U_3(R))$ and $ea = cf$.

We now extend the argument of $U_3(R)$ to the general situation $U_n(R)$ for $n \geq 4$. We will proceed by induction on n, based on the affirmative results for the cases of $n = 2, 3$ as above.

Let $E'' = U_n(R)$ for $n \geq 4$ and $e = (e_{ij}) \in I(E'')$, $a = (a_{ij}) \in E''$. Write $b = ea = (b_{ij})$. Set $e_0 = (e'_{st})$, $a_0 = (a'_{st})$, $b_0 = (b'_{st}) \in U_{n-1}(R)$ such that $e'_{st} = e_{st}$, $a'_{st} = a_{st}$, $b'_{st} = b_{st}$ for all $1 \leq s, t \leq n-1$. Then $e_0 \in I_e(U_{n-1}(R))$ and $e_0a_0 = b_0$.

`Case 1.` Suppose that $e_{11} = 0$ or $a_{11} = 0$. Then $b_{11} = 0$, and so $ea = b(E_{22} + E_{33} + \cdots + E_{nn})$. Note $E_{22} + E_{33} + \cdots + E_{nn} \in I(E'')$.

`Case 2.` Suppose that $e_{11} = 1$, $\text{rank}(e_0) < n-1$ (i.e., $e_{jj} = 0$ for some $2 \leq j \leq n-1$), and $a_{11} \neq 0$. Then $e_0 \in I(U_{n-1}(R))$. So there exist $f_0 = (f'_{st}) \in I(U_{n-1}(R))$ and $c_0 = (c'_{hk}) \in U_{n-1}(R)$ such that $e_0a_0 = c_0f_0$. Let $c = (c_{ij}) \in E''$ with $c_{ij} = c'_{ij}$ for $1 \leq i, j \leq n-1$ and $c_{in} = b_{in}$ for $i = 1, \ldots, n$; and $f = (f_{ij}) \in E''$ with $f_{ij} = f'_{st}$ for all $1 \leq i, j \leq n$ and $f_{1n} = f_{2n} = \cdots = f_{(n-1)n} = 0$, $f_{nn} = 1$. Then we have $ea = cf$. Note that $f_0 \in I(U_{n-1}(R))$ implies $f \in I(E''))$.

`Case 3.` Suppose that $\text{rank}(e_0) = n-1$ (i.e., $e_0 = E_{11} + E_{22} + \cdots + E_{(n-1)(n-1)}$), and $a_{11} \neq 0$, $\text{rank}(a_0) < n-1$. Then $a_{hh} = 0$ for some $2 \leq h \leq n-1$. Say that k is the largest integer such that $a_{kk} = 0$. Set $a'_0 = (a'_{st}) \in U_k(R)$ with $a'_{st} = a_{st}$ for all $1 \leq s, t \leq k$, and $e'_0 = E_{11} + \cdots + E_{(k-1)(k-1)} \in U_k(R)$. Then $e'_0 \in I(U_k(R))$ and $(E_{11} + \cdots + E_{(k-1)(k-1)} + E_{kk})a'_0 = e'_0[(E_{11} + \cdots + E_{(k-1)(k-1)} + E_{kk})a'_0]$. So there exist $f'_0 = (f'_{st}) \in I(U_k(R))$ and $c'_0 = (c'_{st}) \in U_k(R)$ such that $e'_0a'_0 = c'_0f'_0$. Let $c = (c_{ij}) \in E''$ with $c_{ij} = c'_{ij}$ for $1 \leq i, j \leq k$, and $c_{lm} = b_{lm}$ for $k+1 \leq l, m \leq n$; and $f = (f_{ij}) \in E''$ with $f_{ij} = f'_{st}$ for all $1 \leq i, j \leq k$, and $f_{ss} = 1$ for all $s = k+1, \ldots, n$, $f_{st} = 0$ for all $k+1 \le s,t \le n$ with $s \ne t$. Then $ea = cf$. Note that $f'_0 \in I(U_k(R))$ implies $f \in I(E'')$.

`Case 4.` Suppose that $\mathrm{rank}(e_0) = n-1$ (i.e., $e_0 = E_{11} + E_{22} + \cdots + E_{(n-1)(n-1)}$), and $\mathrm{rank}(a_0) = n-1$ (i.e., $a_{ii} \ne 0$ for all $i = 1, \ldots, n-1$). Then $e_{nn} = 0$ and $a_{ii} = b_{ii}$ for all $i = 1, \ldots, n-1$. Let $c = (c_{ij}) \in E''$ with $c_{ij} = b_{ij}$ for $1 \le i,j \le n-1$, and $c_{in} = 0$ for $i = 1, \ldots, n$; and $f = (f_{ij}) \in E''$ such that $f_{ii} = 1$ for $i = 1, \ldots, n-1$, $f_{nn} = 0$, and $f_{st} = 0$ for all $1 \le s,t \le n-1$ with $s \ne t$; and

$$\begin{aligned}
f_{(n-1)n} &= a_{(n-1)(n-1)}^{-1} b_{(n-1)n}, \\
f_{(n-2)n} &= a_{(n-2)(n-2)}^{-1} [b_{(n-2)n} - b_{(n-2)(n-1)} f_{(n-1)n}], \\
f_{(n-3)n} &= a_{(n-3)(n-3)}^{-1} [b_{(n-3)n} - (b_{(n-3)(n-2)} f_{(n-2)n} + b_{(n-3)(n-1)} f_{(n-1)n})], \\
&\vdots \\
f_{1n} &= a_{11}^{-1} [b_{1n} - (\sum_{k=2}^{n-1} b_{1k} f_{kn})].
\end{aligned}$$

Then $ea = cf$. Note that $f_{nn} = 0$ implies $f \in I(E'')$.

Therefore $E'' = U_n(R)$ for $n \ge 4$ is right quasi-Abelian by Cases 1, 2, 3, and 4.

(2)⇒(1). Let $E = U_2(R)$ be right quasi-Abelian, and assume on the contrary that R is not a division ring. Say that a is a nonzero nonunit in R. Since R is a domain, we also have

$$I(E) = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha, \beta \in R \right\},$$

as in the case of R being a division ring.

Consider the product $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & 1+a \\ 0 & 0 \end{smallmatrix}\right)$ and let

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 1+a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 1+a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & c \\ 0 & d \end{pmatrix}\begin{pmatrix} s & t \\ 0 & u \end{pmatrix}$$

for some $\left(\begin{smallmatrix} b & c \\ 0 & d \end{smallmatrix}\right) \in E$ and $\left(\begin{smallmatrix} s & t \\ 0 & u \end{smallmatrix}\right) \in I(E)$. Then s must be 1, entailing $b = a$ and $t \ne 0$, $u = 0$. Thus we have

$$\begin{pmatrix} a & 1+a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & at \\ 0 & 0 \end{pmatrix}.$$

This yields $1 + a = at$ and $a(t-1) = 1$. This implies that a is a unit, a contradiction to the choice of a. Therefore R is a division ring.

The proof of (1)⇔(3) is similar to the proof (1)⇔(2). □

Consider the subring $U_2(\mathbb{Z})$ of $U_2(\mathbb{Q})$, where $\mathbb{Q}$ is the field of rational numbers. Then $U_2(\mathbb{Z})$ is not quasi-Abelian by Theorem 2.1(2), in spite of $U_2(\mathbb{Q})$ being quasi-Abelian by Theorem 2.1(1). This implies that the class of quasi-Abelian rings is not closed under subrings.

We next consider homomorphic images of right quasi-Abelian rings. First we can say that the class of right quasi-Abelian rings is not closed under homomorphic images. Indeed, by [2, Example 5.11], there exists a directly finite regular ring (hence quasi-Abelian by Theorem 1.9(2)) R with a right and left primitive ideal P such that R/P is not directly finite (hence not right quasi-Abelian by Theorem 1.9(1)).

We will find a condition under which homomorphic images preserve the right quasi-Abelian property. An ideal I of a ring R is usually said to be *idempotent-lifting* if idempotents in R/I can be lifted to R. It is well-known that nil ideals are idempotent-lifting.

Theorem 2.2. (1) *Let R be a right quasi-Abelian ring and I be a proper ideal of R. Suppose that $I \cap I(R) = \emptyset$ and I is idempotent-lifting. Then R/I is a right quasi-Abelian ring.*

(2) *A direct product of rings is right quasi-Abelian if and only if each individual ring is.*

Proof. (1) Let $\epsilon + I \in I(R/I)$ and $a + I \in R/I$ for $\epsilon, a \in R$. Since I is idempotent-lifting, there exists $e \in I_e(R)$ such that $\epsilon + I = e + I$. If $e = 1$ or $e = 0$, then $\epsilon + I \notin I(R/I)$ and so $e \in I(R)$. Since R is right quasi-Abelian, there exist $f \in I(R)$ and $b \in R$ such that $ea = bf$. Here $f \in I(R)$ implies $f + I \neq 0$; and since $I \cap I(R) = \emptyset$, $f + I \neq 1 + I$ (otherwise, $0 \neq 1 - f \in I$, contrary to $I \cap I(R) = \emptyset$). Thus $f + I \in I(R/I)$ such that $(\epsilon + I)(a + I) = (e + I)(a + I) = ea + I = bf + I = (b + I)(f + I)$, showing that R/I is right quasi-Abelian.

(2) For given a family $\{R_\gamma \mid \gamma \in \Gamma\}$ of rings, we denote the direct product of R_γ's by $R = \prod_{\gamma\in\Gamma} R_\gamma$. Let R_γ be right quasi-Abelian for all γ. Consider $(e_\gamma) \in I(R)$ and $(r_\gamma) \in R$. Then we have the following two cases:

(i) There exists a proper nonempty subset Γ_1 of Γ such that $e_\alpha \in I(R_\alpha)$ for all $\alpha \in \Gamma_1$ and $e_\beta \in \{0_{R_\beta}, 1_{R_\beta}\}$ for all $\beta \in \Gamma\backslash\Gamma_1$; and

(ii) there exists a proper nonempty subset Γ_2 of Γ such that $e_{\alpha'} = 0_{R_{\alpha'}}$ for all $\alpha' \in \Gamma_2$ and $e_{\beta'} = 1_{R_{\beta'}}$ for all $\beta' \in \Gamma\backslash\Gamma_2$.

Consider the case of (i). Since every R_γ is right quasi-Abelian, there exists $g_\alpha \in I(R_\alpha)$ and $t_\alpha \in R_\alpha$ such that $e_\alpha r_\alpha = t_\alpha g_\alpha$ for all $\alpha \in \Gamma_1$. Set $(f_\gamma) \in I_e(R)$ and $(s_\gamma) \in R$ be such that

$$f_\alpha = g_\alpha \text{ for all } \alpha \in \Gamma_1 \text{ and } f_\beta = e_\beta \text{ for all } \beta \in \Gamma\backslash\Gamma_1;$$

and

$$s_\alpha = t_\alpha \text{ for all } \alpha \in \Gamma_1 \text{ and } s_\beta = r_\beta \text{ for all } \beta \in \Gamma\backslash\Gamma_1.$$

Then $(e_\gamma)(r_\gamma) = (s_\gamma)(f_\gamma)$ because $e_\beta r_\beta = s_\beta r_\beta$ for all $\beta \in \Gamma\backslash\Gamma_1$, noting that e_β is central in R_β. Moreover (f_β) is obviously contained in $I(R)$.

Consider the case of (ii). Here (e_γ) is central in R, and $(e_\gamma)(r_\gamma) = (r_\gamma)(e_\gamma)$ follows.

Conversely, let R be right quasi-Abelian and suppose that $e \in I(R_\alpha)$ and $r \in R_\alpha$ for $\alpha \in \Gamma$. If $er = 0$, then we are done. So assume that $er \neq 0$.

Consider $(e_\gamma) \in I_e(R)$ and $(r_\gamma) \in R$ such that

$$e_\alpha = e \text{ and } e_\beta = 1_\beta \text{ for all } \beta \in \Gamma\backslash\{\alpha\};$$

and

$$r_\alpha = r \text{ and } r_\beta = 1_\beta \text{ for all } \beta \in \Gamma\backslash\{\alpha\}.$$

Then $(e_\gamma) \in I(R)$ because $e \in I(R_\alpha)$. Since R is right quasi-Abelian, there exist $(f_\gamma) \in I(R)$ and $(s_\gamma) \in R$ such that $(e_\gamma)(r_\gamma) = (s_\gamma)(f_\gamma)$. Clearly $f_\gamma \in I(R_\gamma)$. But since $1_\beta = e_\beta r_\beta = s_\beta f_\beta$ for all β, we must get $f_\beta = 1_\beta$ (otherwise, $f_\beta \notin I_e(R_\beta)$). So $f_\alpha \neq 1_\alpha$ for (f_γ) to be in $I(R)$. Moreover $f_\alpha \neq 0_\alpha$ because $0 \neq er = e_\alpha r_\alpha = s_\alpha f_\alpha$. Consequently $f_\alpha \in I(R_\alpha)$, and therefore R_α is right quasi-Abelian. □

From Theorem 2.2, we can obtain an information for upper triangular matrix rings to be right quasi-Abelian.

Proposition 2.3. (1) *Let R be a ring and $n \geq 2$. If $U_n(R)$ is a right quasi-Abelian ring, then so is R.*

(2) *Let R, S be rings and ${}_RM_S$ be an R-S-bimodule. If $\left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ is a right quasi-Abelian ring, then R and S are both right quasi-Abelian.*

Proof. (1) Suppose that $U_n(R)$ is a right quasi-Abelian ring. Consider the nilpotent ideal $I = \{A \in U_n(R) \mid \text{each diagonal entry of } A \text{ is zero}\}$ of $U_n(R)$. Then I is clearly idempotent-lifting and $I \cap I(U_n(R)) = \emptyset$. Moreover, $U_n(R)/I \cong \oplus_{i=1}^n R_i$ where $R_i = R$ for all i. So R is right quasi-Abelian by Theorem 2.2. The proof of (2) is similar. □

Recall in [5, Lemma 8] that a ring R is Abelian if and only if $R[x]$ is Abelian and that if R is Abelian, then every idempotent of $R[x]$ is in R. We extend this fact a little in the following.

Proposition 2.4. *For a ring R the following conditions are equivalent*:

(1) *R is Abelian*;

(2) *$R[x]$ is Abelian*;

(3) *Every idempotent of $R[x]$ is in R.*

Proof. It suffices to prove (3) implying (1), by help of [5, Lemma 8]. Let the condition (3) hold. Assume on the contrary that there exist $e^2 = e, a \in R$ such that $ea(1-e) \neq 0$. Consider $f(x) = e + ea(1-e)x \in R[x]$. Then $f(x)^2 = f(x)$, but $f(x)$ is not contained in R, a contradiction to the condition (3). Thus R is Abelian. □

Based on [5, Lemma 8], one may conjecture that the right quasi-Abelian property can go up to polynomials. But there exists a quasi-Abelian ring over which the polynomial ring is not quasi-Abelian as we see in the following.

Proposition 2.5. (1) *The right quasi-Abelian property does not go up to polynomial rings.*

(2) *For a ring R, if $R[x]$ is a right quasi-Abelian ring, then so is R.*

Proof. (1) Consider the ring $R = U_2(K)$ over a field K. Then R is quasi-Abelian by Theorem 2.1(1). On the other hand, $K[x]$ is a domain but not a division ring, and so $U_2(K[x])$ is not right quasi-Abelian by Theorem 2.1(2). This concludes that $R[x]$ is not right quasi-Abelian, since $R[x] = U_2(K)[x] \cong U_2(K[x])$.

(2) Assume that $R[x]$ is right quasi-Abelian. Let $a \in R$ and $e \in I(R)$. Since $R[x]$ is right quasi-Abelian, there exist $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ and $f(x) = \sum_{i=0}^{m} f_i x^i \in I(R[x])$ such that $ea = g(x)f(x)$ and $f_0 \in I(R)$. Then $ea = b_0 f_0$ and so R is right quasi-Abelian. □

We end this note by raising the following.

Question. Let R be a ring. If $R[x]$ is a right quasi-Abelian ring, then is R an Abelian ring?

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References

[1] G. Ehrlich, *Unit-regular rings*, Portugal. Math. **27** (1968), 209–212.

[2] K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.

[3] S. U. Hwang, Y. C. Jeon, and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), no. 1, 186–199.

[4] N. Jacobson, *Some remarks on one-sided inverses*, Proc. Amer. Math. Soc. **1** (1950), 352–355.

[5] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), no. 2, 477–488.

[6] J. V. Neumann, *On regular rings*, Proceedngs of the National Academy of Sci. **22** (1936), 707–713.

[7] J. C. Shepherdson, *Inverses and zero-divisors in matrix ring*, Proc. London Math. Soc. **1** (1951), 71–85.

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