Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 361–369
https://doi.org/10.4134/CKMS.c150243
pISSN: 1225-1763 / eISSN: 2234-3024

THE OUTER-CONNECTED VERTEX EDGE DOMINATION NUMBER OF A TREE

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Abstract. For a given graph $G = (V, E)$, a set $D \subseteq V(G)$ is said to be an outer-connected vertex edge dominating set if D is a vertex edge dominating set and the graph $G \setminus D$ is connected. The outer-connected vertex edge domination number of a graph G, denoted by $\gamma_{ve}^{oc}(G)$, is the cardinality of a minimum outer connected vertex edge dominating set of G. We characterize trees T of order n with l leaves, s support vertices, for which $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$ and also characterize trees with equal domination number and outer-connected vertex edge domination number.

1. Introduction

Let $G = (V, E)$ be a simple graph. The degree of a vertex v, denoted by $d_G(v)$ or $\deg(v)$, is the cardinality of its open neighborhood. The open neighborhood of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the set $N_G[v] = N_G(v) \cup \{v\}$ is called its closed neighborhood. A vertex of degree one is a leaf, while the edge incident with a leaf is called an end edge. The vertex adjacent to a leaf is called a support vertex and a support vertex is strong (weak, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). The length of the shortest $u - v$ path in the connected graph G is the distance between u and v, denoted by $d(u, v)$ and $\max\{d(u, v) : u, v \in V(G)\}$ is the diameter of G denoted by diam(G). The path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. A complete graph on n vertices is denoted by K_n and a complete bipartite graph with partition set of cardinality m, n is denoted by $K_{m,n}$. The complement of a graph G denoted by G^c or $\overline{G}$ is defined as the graph on the same vertices such that two distinct vertices in G^c are adjacent if and only if they are not adjacent in G. Let T be a tree, and let v be a vertex of T. A vertex v is said

Received December 17, 2015; Revised November 10, 2017; Accepted November 30, 2017.
2010 *Mathematics Subject Classification.* 05C05, 05C69.
Key words and phrases. outer-connected domination, vertex edge domination, outer-connected vertex edge domination, tree.

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to be adjacent to a path P_n if there is a neighbor of v, say x, such that the subtree $T - vx$ containing x is a path P_n and x is a leaf in it.

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of a graph G if every vertex of $V(G) \setminus D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, see [3].

A vertex $v \in V(G)$ dominates an edge $e \in E(G)$ if v is incident with e or with an edge adjacent to e. A subset $D \subseteq V(G)$ is a vertex-edge dominating set, abbreviated VEDS, of a graph G if every edge of G is vertex-edge dominated by a vertex in D. The vertex-edge domination number of a graph G, denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex-edge dominating set of G. Vertex-edge domination in graphs was introduced in [6] and further studied in [1, 4, 5].

A subset D of $V(G)$ is an outer-connected dominating set, abbreviated OCD, of a graph G if D is a dominating set and the graph $G \setminus D$ is connected. The outer-connected domination number of a graph G, denoted by $\widetilde{\gamma}_c(G)$, is the cardinality of a minimum outer-connected dominating set of G. The outer-connected domination number of a graph was introduced in [2].

A subset D of $V(G)$ is an outer-connected vertex edge dominating set, abbreviated OCVEDS, of a graph G if D is a vertex edge dominating set of G and the graph $G \setminus D$ is connected. The outer-connected vertex edge domination number of a graph G, denoted by $\gamma_{ve}^{oc}(G)$, is the minimum cardinality of an outer-connected vertex edge dominating set of G. The problem OCVEDS has possible applications in computer networks. Consider a client-server architecture based model. Let D denote the set of servers and $V \setminus D$ be the set of clients. Two servers may or may not be related directly. But these servers have the unique property of being able to communicate not only with the clients who are directly linked to them, but also to the clients who are at a distance two from the servers with the main emphasis to the connectivity of the clients to the servers. Each client can communicate with another client in the group either directly or through another client. The smallest group of servers with this property is a minimum outer-connected vertex edge dominating set for the graph which represents the computer network.

We characterize trees T of order n with l leaves, s support vertices, for which $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$.

2. Preliminary result

Let K_n, C_n and P_n denote the complete graph, the cycle and the path of order n, respectively. By a star of order n we mean the bipartite graph $K_{1,n-1}$ for $n \geq 2$. In the first observation we present the OCVE domination number of standard graphs.

Observation 1:

(i) $\gamma_{ve}^{oc}(K_n) = 1$ for $n \geq 1$.

(ii) $\gamma_{ve}^{oc}(C_n) = \begin{cases} 1, & \text{if } n = 3, 4, \\ n-3, & \text{if } n \geq 5. \end{cases}$

(iii) $\gamma_{ve}^{oc}(P_n) = \begin{cases} 1, & \text{if } n = 2, 3, \\ 2, & \text{if } n = 4, 5, \\ n-3, & \text{if } n \geq 6. \end{cases}$

(iv) $\gamma_{ve}^{oc}(K_{m,n}) = 1$ for $(m, n) \geq 1$.

Observation 2: Every OCVEDS is a VEDS of a graph G. Thus we have $\gamma_{ve}(G) \leq \gamma_{ve}^{oc}(G)$.

Observation 3: Let D be a $\widetilde{\gamma}_c(G)$-set. Then D is a OCD set of G. That is D is dominating set and $G \setminus D$ is connected. Every dominating set is a VEDS of G. Hence, D is a OCVEDS of G. Thus, we get $\gamma_{ve}^{oc}(G) \leq \widetilde{\gamma}_c(G)$.

Observation 4: For any graph G, $1 \leq \gamma_{ve}^{oc}(G) \leq n-1$. The upper bound is attained for K_2.

We now improve the lower bound on the outer connected vertex-edge domination number of trees. First we show that if T is a nontrivial tree of order n with l leaves and s support vertices, then $\gamma_{ve}^{oc}(T)$ is bounded below by $(n - l + s + 1)/3$. For the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3 or P_5. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a P_3 by joining one of its leaves to a vertex of T_k adjacent to a path P_3.
- Operation $\mathcal{O}_3$: Attach a P_3 by joining one of its leaves to a support vertex of T_k.

We prove that for every tree T of the family $\mathcal{T}$ we have $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$.

Lemma 1. *If $T \in \mathcal{T}$, then $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_3$, then $(n - l + s + 1)/3 = (3 - 2 + 1 + 1)/3 = 1 = \gamma_{ve}^{oc}(T)$.

Let k be a positive integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let n' be the order of the tree T', l' the number of its leaves and s' the number of support vertices. Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. We have $n = n'+1$, $l = l' + 1$ and $s = s'$. It is straightforward to see that any $\gamma_{ve}^{oc}(T')$-set is a OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T')$. Obviously $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T)$. This implies that $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T')$. We now get $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') = (n' - l' + s' + 1)/3 = (n - 1 - l + 1 + s + 1)/3 = (n - l + s + 1)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We have $n = n'+3$, $l = l' + 1$ and $s = s' + 1$. We denote by x the leaf to which a path $v_1v_2v_3$ is attached. Let v_1 be adjacent to x. Let $u_1u_2u_3$ be another P_3 path adjacent to x. Let u_1 be adjacent to x. Let D' be a $\gamma_{ve}^{oc}(T')$-set. It is easy to see that $D' \cup \{v_3\}$ is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is easy to see that $D \setminus \{v_3\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. We now get $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') + 1 = ((n' - l' + s' + 1)/3) + 1 = (n - 3 - l + 1 + s - 1 + 1 + 3)/3 = (n - l + s + 1)/3$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We have $n = n'+3$, $l = l' + 1$ and $s = s' + 1$. We denote by x the support vertex to which P_3 is attached. Let $v_1v_2v_3$ be the attached path. Let v_1 be joined to x. Let y be a leaf adjacent to x. Let D' be any $\gamma_{ve}^{oc}(T')$-set. It is easy to see that $D' \cup \{v_3\}$ is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T') + 1$. Now let us observe that there exists a $\gamma_{ve}^{oc}(T)$-set that does not contain v_1 and v_2. Let D be such a set. To dominate the edges v_2v_1 and xv_1, we have $v_3, y \in D$. Observe that $D \setminus \{v_3\}$ is an OCVEDS of the tree T'. Therefore $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. We now conclude that $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') + 1$. We get $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') + 1 = ((n' - l' + s' + 1)/3) + 1 = (n - 3 - l + 1 + s - 1 + 1 + 3)/3 = (n - l + s + 1)/3$. $\square$

We now give a lower bound on the outer connected vertex edge domination number of a tree together with the characterization of the extremal trees.

Theorem 2. *If T is a nontrivial tree of order $n \geq 3$ with l leaves and s support vertices, then $\gamma_{ve}^{oc}(T) \geq (n - l + s + 1)/3$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $diam(T) = 2$, then T is a star. If P is P_3, then $n = 3$, $l = 2$ and $s = 1$. Consequently, $(n - l + s + 1)/3 = (3 - 2 + 1 + 1)/3 = 1 = \gamma_{ve}^{oc}(T)$. If T is a star other than P_3, we obtain T from P_3 by finite number of operation $\mathcal{O}_1$ on the support vertex. Thus $T \in \mathcal{T}$.

Now assume that $diam(T) \geq 3$. Thus the order n of the tree T is at least four. We obtain the result by induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s$. Obviously $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T')$. We get $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') \geq (n' - l' + s' + 1)/3 = (n - 1 - l + 1 + s + 1)/3 = (n - l + s + 1)/3$. If $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$, then obviously $\gamma_{ve}^{oc}(T') = (n' - l' + s' + 1)/3$. By the inductive hypothesis, we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $diam(T) \geq 4$, then let w be the parent of u. If $diam(T) \geq 5$, then let d be the parent of w. If $diam(T) \geq 6$, then let e be the parent of d. By T_x, we denote the sub-tree induced by a vertex x and its descendants in the rooted tree T.

Assume that some child of u, say x, is a leaf. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edge tv, the vertex $t \in D$. It is easy to see that $D \setminus \{t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 1 \geq ((n' - l' + s' + 1)/3) + 1 = ((n - 2 - l + 1 + s - 1 + 1)/3) + 1 = (n - l + s + 2)/3 > (n - l + s + 1)/3$.

Now assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is obvious that $t \in D$. It is clear that $D \setminus \{t\}$ is an OCVEDS of the tree T'. We now get $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 1 \geq ((n' - l' + s' + 1)/3) + 1 \geq (n - 2 - l + 1 + s - 1 + 1 + 3)/3 > (n - l + s + 1)/3$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. First assume that some child of w, say x, such that the distance of w to the most distant vertex of T_x is three. It suffices to consider only the possibilities when T_x is $P_3 = xyz$. Let $T' = T - T_x$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is obvious that $z \in D$. It is easy to see that $D \setminus \{z\}$ is an OCVEDS of the tree T'. We now get $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 1 \geq ((n' - l' + s' + 1)/3) + 1 = ((n-3-l+1+s-1+1)/3)+1 = (n-l+s+1)/3$. If $\gamma_{ve}^{oc}(T) = (n-l+s+1)/3$, then obviously $\gamma_{ve}^{oc}(T') = (n' - l' + s' + 1)/3$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$.

Assume that some child of w, say x, such that the distance of w to the most distant vertex of T_x is two. It suffices to consider the possibilities when $T_x = P_2 = xy$. Let $T' = T - T_x$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edge xy, the vertex $y \in D$. It is obvious that $D \setminus \{y\}$ is an OCVEDS of the tree T'. We now get $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 1 \geq ((n' - l' + s' + 1)/3) + 1 = ((n - 2 - l + 1 + s - 1 + 1)/3) + 1 > (n - l + s + 1)/3$.

Assume that some child of w, say x, is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is obvious that $t, x \in D$. It is easy to see that $D \setminus \{t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 1 \geq ((n' - l' + s' + 1)/3) + 1 = ((n - 3 - l + 1 + s - 1 + 1)/3) + 1 = (n - l + s + 1)/3$. If $\gamma_{ve}^{oc}(T) = (n - l + s + 1)/3$, then obviously $\gamma_{ve}^{oc}(T') = (n' - l' + s' + 1)/3$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{T}$.

Now assume that $d_T(w) = 2$. Let $d_T(d) = 1$. Then $T = P_5$. We have $(n - l + s + 1)/3 = (5 - 2 + 2 + 1)/3 = 2 = \gamma_{ve}^{oc}(T)$. Now assume that some child of d is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l$ and $s' = s - 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is obvious that $t \in D$. To dominate the edge wu, the vertex $v \in D$. It is clear that $D \setminus \{v, t\}$ is an OCVEDS of the tree T'. We now get $\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 2 \geq ((n' - l' + s' + 1)/3) + 2 \geq ((n - 3 - l + s - 1 + 1)/3) + 2 > (n - l + s + 1)/3$.

Now assume that no child of d is a leaf. Let $T' = T - T_u$. We have $n' = n - 3$, $l' = l$ and $s' = s$. Let D be a $\gamma_{ve}^{oc}(T)$-set. It is easy to see that $v, t \in D$. It is obvious that $D \setminus \{v, t\}$ is an OCVEDS of the tree T'. We now get

$\gamma_{ve}^{oc}(T) \geq \gamma_{ve}^{oc}(T') + 2 \geq ((n' - l' + s' + 1)/3) + 2 = ((n - 3 - l + s + 1)/3) + 2 > (n - l + s + 1)/3.$ □

3. Domination and outer-connected vertex edge domination

We begin this section with a lemma which will be useful.

Lemma 3. *Let T be a tree with $|V(T)| \geq 2$. There exists a minimum dominating set D of T such that D contains every support vertex.*

Proof. Let D be a minimum dominating set of T. Assume that D does not contain a support vertex u. Let x be the leaf adjacent to u. To dominate x, the vertex $x \in D$ as u is not in D. The set $(D \setminus \{x\}) \cup \{u\}$ is a minimum dominating set. Every support vertex is in D. If v is a vertex adjacent to a support vertex but not a support vertex is in D, then $D \setminus \{v\}$ is a dominating set, a contradiction. □

We now characterize the trees attaining the equality of domination number and outer connected vertex-edge domination number. For this purpose, we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a path P_3 or P_4. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_k.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertex to a vertex of T_k adjacent to a path P_2.
- Operation $\mathcal{O}_3$: Attach a path P_2 by joining one of its vertex to a support vertex of T_k.
- Operation $\mathcal{O}_4$: Attach a path P_3 by joining one of its leaf to a support vertex of T_k.

We prove that for every tree in the family $\mathcal{F}$, the domination number is equal to the outer connected vertex edge domination number.

Lemma 4. *If $T \in \mathcal{F}$, then $\gamma(T) = \gamma_{ve}^{oc}(T)$.*

Proof. We use the induction on the number k of operations performed to construct the tree T. If $T = T_1 = P_3$, then $\gamma(T) = 1 = \gamma_{ve}^{oc}(T)$. If $T = P_4$, then $\gamma(T) = 2 = \gamma_{ve}^{oc}(T)$. Let k be a positive integer. Assume the result is true for every tree $T' = T_k$ of the family $\mathcal{F}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. It is straightforward to see that any $\gamma(T')$-set is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T')$. Obviously $\gamma(T') \leq \gamma(T)$. This implies that $\gamma(T) = \gamma(T')$. It is also easy to obtain $\gamma_{ve}^{oc}(T') = \gamma_{ve}^{oc}(T)$. We now get $\gamma(T) = \gamma(T') = \gamma_{ve}^{oc}(T') = \gamma_{ve}^{oc}(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. We denote by x the vertex to which a path $P_2 = u_1u_2$ is attached. Let u_1 be joined to x. Let v_1v_2 be a path different from u_1u_2 adjacent to x. Let v_1 be joined to x. Let

D' be any $\gamma_{ve}^{oc}(T')$-set. It is easy to see that $D' \cup \{u_2\}$ is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edge xv_1 and v_1v_2, the vertex $v_2 \in D$. To dominate the edges xu_1 and u_1u_2, the vertex $u_2 \in D$. It is easy to see that $D \setminus \{u_2\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. This implies that $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') + 1$. Let D' be a $\gamma(T')$-set. It is easy to observe that $D' \cup \{u_1\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma(T)$-set. To dominate v_2 and u_2, the vertices v_1 and u_1 is in D. It is easy to see that $D \setminus \{u_1\}$ is a DS of the tree T'. Thus $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. We now get $\gamma(T) = \gamma(T') + 1 = \gamma_{ve}^{oc}(T') + 1 = \gamma_{ve}^{oc}(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. We denote by x the support vertex to which a path $P_2 = u_1u_2$ is attached. Let u_1 be joined to x. Let y be a leaf adjacent to x. Let D' be any $\gamma(T')$-set. It is easy to see that $D' \cup \{u_1\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma(T)$-set. Clearly $x, u_1 \in D$. It is easy to observe that $D \setminus \{u_1\}$ is a DS of the tree T'. Thus $\gamma(T') \leq \gamma(T) - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. Let D' be a $\gamma_{ve}^{oc}(T')$-set. It is obvious that $D' \cup \{u_2\}$ is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges xu_1 and u_1u_2, the vertex $u_2 \in D$. To dominate the edge xy, the edge $y \in D$. It is clear that $D \setminus \{u_2\}$is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. This implies that $\gamma_{ve}^{oc}(T') = \gamma_{ve}^{oc}(T) - 1$. We now get $\gamma(T) = \gamma(T') + 1 = \gamma_{ve}^{oc}(T') + 1 = \gamma_{ve}^{oc}(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_4$. We denote by x the support vertex to which a path $P_3 = u_1u_2u_3$ is attached. Let u_1 be joined to x. Let y be a leaf adjacent to x. Let D' be a $\gamma(T')$-set. It is obvious that $D' \cup \{u_2\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma(T)$-set. To dominate y and u_3, the vertices $x, u_2 \in D$. It is easy to see that $D \setminus \{u_2\}$ is a DS of the tree T'. Thus $\gamma(T) \leq \gamma(T') - 1$. This implies that $\gamma(T) = \gamma(T') + 1$. Let D' be a $\gamma_{ve}^{oc}(T')$-set. It is clear that $D' \cup \{u_3\}$ is an OCVEDS of the tree T. Thus $\gamma_{ve}^{oc}(T) \leq \gamma_{ve}^{oc}(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges xy, u_1u_2 and u_2u_3, the vertices $y, u_3 \in D$. It is clear that $D \setminus \{u_3\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. This implies that $\gamma_{ve}^{oc}(T) = \gamma_{ve}^{oc}(T') + 1$. We now get $\gamma(T) = \gamma(T') + 1 = \gamma_{ve}^{oc}(T') + 1 = \gamma_{ve}^{oc}(T)$. $\square$

We now give the lower bound on the outer connected vertex-edge domination number of a tree in terms of domination number, together with the characterization of extremal trees.

Theorem 5. *Let T be a tree. If $\gamma(T) \leq \gamma_{ve}^{oc}(T)$ with equality if and only if $T = P_2$ or $T \in \mathcal{F}$.*

Proof. If $diam(T) = 1$, then $T = P_2 \in \mathcal{F}$. If $diam(T) = 2$, then T is a star. If $T = P_3$, then $T \in \mathcal{F}$. If T is a star different from P_3, then it can be obtained from P_3 by an appropriate number of operations $\mathcal{O}_1$. Thus $T \in \mathcal{F}$. Now assume that $diam(T) \geq 3$. Thus the order n of the tree is at least four. The result we obtain by the induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$.

First assume that some support vertex of T, say x, is strong. Let y be a leaf adjacent to x. Let $T' = T - y$. We have $\gamma(T) \leq \gamma(T')$. Obviously $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T)$. We get $\gamma(T) \leq \gamma(T') \leq \gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T)$. If $\gamma(T) = \gamma_{ve}^{oc}(T)$, then obviously $\gamma(T') = \gamma_{ve}^{oc}(T')$. By the inductive hypothesis, we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{F}$. Henceforth, we can assume that every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r, v be the parent of t, and u be the parent of v in the rooted tree. If $diam(T) \geq 4$, then let w be the parent of u. If $diam(T) \geq 5$, then let d be the parent of w. If $diam(T) \geq 6$, then let e be the parent of d. By T_x we denote by subtree induced by a vertex x and its descendants in the rooted tree.

Assume that among the children of u there is a support vertex other than v. Let $T' = T - T_v$. Let D' be a $\gamma(T')$-set. It is clear that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges vt and uv, the vertex $t \in D$. It is obvious that $D \setminus \{t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. We now get $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_{ve}^{oc}(T') + 1 \leq \gamma_{ve}^{oc}(T)$. If $\gamma(T) = \gamma_{ve}^{oc}(T)$, then obviously $\gamma(T') = \gamma_{ve}^{oc}(T')$. By the inductive hypothesis, we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{F}$.

Assume that some child of u, say x, is a leaf. Let $T' = T - T_v$. Let D' be a $\gamma(T')$-set. It is obvious that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges vt and uv, the vertex $t \in D$. It is clear that $D \setminus \{t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. We now get $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_{ve}^{oc}(T') + 1 \leq \gamma_{ve}^{oc}(T)$. If $\gamma(T) = \gamma_{ve}^{oc}(T)$, then obviously $\gamma(T') = \gamma_{ve}^{oc}(T')$. By the inductive hypothesis, we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by operation $\mathcal{O}_3$. Thus $T \in \mathcal{F}$.

Now assume that $d_T(u) = 2$. Assume that $d_T(w) \geq 3$. Assume that no child of w is a leaf. Let $T' = T - T_u$. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges vt, vu and wu, the vertices $v, t \in D$. It is clear that $D \setminus \{v, t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 2$. We now get $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_{ve}^{oc}(T') + 1 \leq \gamma_{ve}^{oc}(T) - 1 < \gamma_{ve}^{oc}(T)$.

Assume that some child of w, say x is a leaf. Let $T' = T - T_u$. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges vt, vu and wu, the vertices $x, t \in D$. It is clear that $D \setminus \{t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 1$. We now get $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_{ve}^{oc}(T') + 1 \leq \gamma_{ve}^{oc}(T)$. If $\gamma(T) = \gamma_{ve}^{oc}(T)$, then obviously $\gamma(T') = \gamma_{ve}^{oc}(T')$. By the inductive hypothesis, we have $T' \in \mathcal{F}$. The tree T can be obtained from T' by operation $\mathcal{O}_4$. Thus $T \in \mathcal{F}$.

Now assume $d_T(w) = 2$. Let $d_T(d) \geq 2$. Let $T' = T - T_u$. Let D' be a $\gamma(T')$-set. It is easy to see that $D' \cup \{v\}$ is a DS of the tree T. Thus $\gamma(T) \leq \gamma(T') + 1$. Let D be a $\gamma_{ve}^{oc}(T)$-set. To dominate the edges vt, vu and wu, the vertices $v, t \in D$. It is clear that $D \setminus \{v, t\}$ is an OCVEDS of the tree T'. Thus $\gamma_{ve}^{oc}(T') \leq \gamma_{ve}^{oc}(T) - 2$. We now get $\gamma(T) \leq \gamma(T') + 1 \leq \gamma_{ve}^{oc}(T') + 1 \leq \gamma_{ve}^{oc}(T) - 1 < \gamma_{ve}^{oc}(T)$. □

Acknowledgement. Authors thank the anonymous referee(s) for their valuable suggestions and constructive comments which improved the manuscript. The authors express their gratitude to National Board of Higher Mathematics, Mumbai for the financial support and to DST FIST for the support to the Mathematics department, SASTRA University.

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