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DISCOUNT BARRIER OPTION PRICING WITH A STOCHASTIC INTEREST RATE: MELLIN TRANSFORM TECHNIQUES AND METHOD OF IMAGES

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Abstract. In finance, barrier options are options contracts with a payoff that depends on whether the price of the underlying asset hits a predetermined barrier level during the option's lifetime. Based on exotic options and random fluctuations of interest rates in the marketplace, we consider discount barrier options with a stochastic interest rate driven by the Hull-White process. This paper derives the closed-form solutions of the discount barrier option and the discount double barrier option using Mellin transform methods and the PDE (partial differential equation) method of images.

1. Introduction

The pricing of many path-dependent options models in the marketplace has created interesting mathematical challenges. Barrier options, a type of path-dependent option, represent a derivative contract that is activated or nullified when the underlying asset price reaches a predetermined level. Barrier options are popular because they provide the insurance value of an option when acting as a hedge but are cheaper than the vanilla option. However, problems remain in managing option pricing models and generalizing the Black-Scholes model. A closed-form solution of barrier options based on the Black-Scholes framework was first suggested by Robert Merton [10]. Rubinstein and Reiner [12] derived the pricing formulas for all eight barrier types.

In real situations, random changes in interest rates over time have a significant influence on option prices. From this perspective, many studies have examined European option pricing with a stochastic interest rate. Amin and Jarrow [1] derived the closed-form formula of European option prices with a Gaussian Ornstein-Uhlenbeck process using probabilistic approaches. Additionally, Kim, Yoon, and Yu [9] investigated a multiscale stochastic volatility model with a Hull-White stochastic interest rate and demonstrated that the

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performance of the data fitting of the multiscale stochastic volatility model with stochastic interest rates outperformed that of the constant interest rate described in Jean-Pierre Fouque et al. [5]. Yoon, Lee, and Kim [15] studied an option pricing formula on the stochastic elasticity of a variance model with stochastic interest rates. Particularly, the authors found that the combined structure of the model enhances the convexity of implied surface as time-to-maturity shortens.

Based on the importance of the stochastic interest rate on option prices, we discuss the pricing of discount barrier options under the Gaussian Hull-White interest rate model. Bernard et al. [2] derived an exact pricing formula for discount barrier options with a Hull-White interest rate using probabilistic approaches. We utilize Mellin transform methods, one of the integral transform methods, and the method of images to find the closed-form solution of the discount barrier option with a stochastic interest rate.

The Mellin transform technique, which is defined by the multiplicative version of the two-sided Laplace transform, assists the resolution of computation complexity in comparison with other transform approaches. Yoon [13] studied a closed-form solution for European options under a stochastic interest rate using Mellin transforms. Yoon and Kim [14] exploited double Mellin transforms to find the exact form solution for European vulnerable options under a Hull-White stochastic interest rate and a constant interest rate. Additionally, Jeon, Yoon, and Kang [8] considered the pricing of path-dependent options using double Mellin transforms to investigate an explicit (closed)-form pricing formula of path-dependent options, and Jeon et al. [7] derived the explicit solution of time-dependent coefficients Black-Scholes partial differential equation by exploiting the techniques of the double Mellin transform to find a closed-form formula of vulnerable geometric Asian options.

The method of images was first proposed by Peter Buchen [3] and is closely related to the reflection principle of the expectations solution. The author found the closed solution of barrier option price more easily than the established method. By using the method of images, we transform the partial differential equation (PDE) with a restricted domain (a boundary condition and a final condition) into the PDE with an unrestricted domain (a final condition) and can solve the PDE problem more effectively.

This paper is organized as follows. Section 2 shows the use of the Mellin transform approach in deriving the pricing formula of a European call option with a Hull-White interest rate. Section 3 considers the pricing of discount barrier options with the Hull-white interest rate, and we obtain the closed solution of the discount barrier option using the method of images and the Mellin transform. Section 4 shows an explicit analytic solution for the discount double barrier option, and Section 5 presents concluding remarks.

2. European call option pricing with a Hull-White interest rate

The pricing formula of European call options with the Hull-White stochastic interest rate has been derived by Yoon [13]. However, in this section, we investigate the closed-form solution of the European option price using other methods. Particularly, by changing variables, we reduce the given PDE as the form of the simplest heat equation, and using the Mellin transform method we solve the reduced heat equation more easily. These approaches help us to determine the closed-form solution of discount European barrier options with a stochastic interest rate in Section 3.

If X_t is the value of the underlying asset (stock) of the option with a drift rate of stock μ_t and the volatility of the underlying asset σ, then the dynamics of X_t are described by the SDE $dX_t = \mu_t X_t dt + \sigma X_t dW_t$, where W_t is the standard Brownian motion. Under an equivalent martingale measure, the above model is changed into the following SDEs

$$dX_t = r_t X_t dt + \sigma X_t dW_t^*, \quad dr_t = (b(t) - ar_t)dt + \check{\sigma} dW_t^{(r)}, \tag{2.1}$$

where W_t^* is the standard Brownian motion satisfying the following relation $W_t^* = W_t + \int_0^t \frac{\mu_s - r_s}{\sigma}$, and the correlation of W_t^* and $W_t^{(r)}$ is given by

$$d\langle W_t^*, W_t^{(r)}\rangle_t = \rho_{xr} dt,$$

where $-1 \le \rho_{xr} \le 1$. Additionally, r_t represents the Hull-White interest rate with a mean reversion rate of interest, a, volatility of interest rate, $\check{\sigma}$, and average direction of interest rate movement, $b(t)$. Under the risk-neutral probability measure, the no-arbitrage price of a European call option with a payoff function $h(x) = (x-K)^+$ is expressed by

$$P(t,x,r) = E^* \left\{ \exp\left(-\int_t^T r_{t^*} dt^* \right) (X_T - K)^+ \,|\, X_t = x, r_t = r \right\}.$$

Using the Feynman-Kac formula (cf. Bernt Oksendal [11]), $P(t,x,r)$ is the solution that satisfies the following PDE problem

(2.2)

$$\hat{\mathcal{L}} P(t,x,r) = 0, \quad P(T,x,r) = h(x) = (x-K)^+,$$

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - I) + \rho_{xr}\sigma\check{\sigma} x \frac{\partial^2}{\partial x \partial r} + (b(t) - ar)\frac{\partial}{\partial r} + \frac{1}{2}\check{\sigma}^2 \frac{\partial^2}{\partial r^2},$$

where $P(T,x,r) = h(x)$ is the final condition, and I is the identity operator.

To derive the closed-form formula of $P(t,x,r)$ in (2.2), we review the bond price formula. Based on the Hull-White interest rate model for the short rate, r, given by $dr_t = (b(t) - ar_t)dt + \check{\sigma} dW_t^{(r)}$, the no-arbitrage price at time t of a zero-coupon bond maturing at time T, defined by $B(t,r;T)$, is given by

$$B(t,r;T) = E^* \left\{ \exp\left(-\int_t^T r_{t^*} dt^* \right) | \, r_t = r \right\},$$

where the terminal condition $B(T,r;T)=1$.

Then, using the Feynman-Kac formula, we obtain the following PDE

$$\frac{\partial B}{\partial t}+\frac{1}{2}\check{\sigma}^2\frac{\partial^2 B}{\partial r^2}+(b(t)-ar)\frac{\partial B}{\partial r}-rB=0,\ B(T,r;T)=1, \tag{2.3}$$

and the solution of the PDE has the form $B(T-\tau,r;T)=A(\tau)e^{-D(\tau)r}$, where $\tau=T-t$, $A(\tau)$ and $D(\tau)$ are expressed by

(2.4)

$$A(\tau)=\exp\left(\frac{\check{\sigma}^2}{2a^2}\left(\tau+\frac{2}{a}(e^{-a\tau}-1)-\frac{1}{2a}(e^{-2a\tau}-1)\right)-\int_0^\tau b(T-\tau^*)D(\tau^*)d\tau^*\right),$$

$$D(\tau)=\frac{1-e^{-a\tau}}{a}.$$

Next, to solve the PDE (2.2), we use the Mellin transform technique as in Hassan and Adem [6]. For a locally Lebesgue integrable function $f(s),\ s\in\mathbb{R}^+$, the Mellin transform $\mathcal{M}(f(s),w),\ w\in\mathbb{C}$ is defined as

$$\mathcal{M}(f(s),w):=\hat{f}(w)=\int_0^\infty f(s)s^{w-1}ds,$$

and if $a<\mathrm{Re}(w)<b$ and c such that $a<c<b$ exists, the inverse of the Mellin transform is given by

$$f(s)=\mathcal{M}^{-1}(\hat{f}(w))=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{f}(w)x^{-w}dw.$$

2.1. Transformation of PDE

By changing variables $U(t,v)=\frac{P(t,x,r)}{B(t,r;T)}$ and $v=\frac{x}{B(t,r;T)}$ in (2.2), we obtain the following relations

(2.5)

$$\frac{\partial P}{\partial t}=U\frac{\partial B}{\partial t}+B\frac{\partial U}{\partial t}-v\frac{\partial U}{\partial v}\frac{\partial B}{\partial t},\ \frac{\partial P}{\partial r}=U\frac{\partial B}{\partial r}-v\frac{\partial U}{\partial v}\frac{\partial B}{\partial r},$$

$$\frac{\partial P}{\partial x}=\frac{\partial U}{\partial v},\ \frac{\partial^2 P}{\partial x^2}=\frac{1}{B}\frac{\partial^2 U}{\partial v^2},\ \frac{\partial^2 P}{\partial r^2}=U\frac{\partial^2 B}{\partial^2 r}-v\frac{\partial U}{\partial v}\frac{\partial^2 B}{\partial r^2}-\frac{v^2}{B}\frac{\partial^2 U}{\partial v^2}\left(\frac{\partial B}{\partial r}\right)^2,$$

$$\frac{\partial P}{\partial r\partial x}=-\frac{v}{B}\frac{\partial^2 U}{\partial v^2}\frac{\partial B}{\partial r}.$$

Then, plugging (2.5) into PDE (2.2) leads to

$$\begin{aligned}\frac{\partial U}{\partial t}&+\frac{1}{2}\left[\sigma^2\frac{x^2}{B^2}-2\sigma\check{\sigma}\rho_{xr}\frac{v^2}{B}\frac{\partial B}{\partial r}+\frac{1}{2}\check{\sigma}^2v^2\left(\frac{1}{B}\frac{\partial B}{\partial r}\right)^2\right]\frac{\partial^2 U}{\partial v^2}\\&+\frac{v}{B}\left[\frac{\partial B}{\partial t}+\frac{1}{2}\check{\sigma}^2\frac{\partial^2 B}{\partial r^2}+(b(t)-ar)\frac{\partial B}{\partial r}-rB\right]\frac{\partial U}{\partial v}\\&+\frac{v}{B}\left[\frac{\partial B}{\partial t}+\frac{1}{2}\check{\sigma}^2\frac{\partial^2 B}{\partial r^2}+(b(t)-ar)\frac{\partial B}{\partial r}-rB\right]U=0,\end{aligned} \tag{2.6}$$

and by (2.3), the above equation can be reduced to the following PDE:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\hat{\sigma}^2(t)v^2\frac{\partial^2 U}{\partial v^2} = 0, \quad U(T,v) = h(v) = (v-K)^+, \tag{2.7}$$

where $\hat{\sigma}^2(t) = \sigma^2 + 2\rho\sigma\check{\sigma}X(t) + \check{\sigma}^2X^2(t)$ and $X(t) = -\frac{1}{B}\frac{\partial B}{\partial r}$.

2.2. Derivation of the option pricing formula with a stochastic interest rate

This subsection derives the formula of European call options with a Hull-white interest rate using the Mellin transform. Because the payoff function h described in (2.7) is not bounded, the Mellin transform does not exist. Therefore, we define h as $\lim_{n\to\infty} h_n(v) = h(v)$ by a bounded sequence h_n such that each h_n is given by

$$h_n(v) = \begin{cases} v-K, & \text{if } K < v < n, \\ 0, & \text{if } v < K. \end{cases}$$

If we define a sequential function $U_n(t,v)$ satisfying the following PDE

$$\frac{\partial U_n}{\partial t} + \frac{1}{2}\hat{\sigma}^2(t)v^2\frac{\partial^2 U_n}{\partial v^2} = 0, \quad U_n(T,v) = h_n(v) \tag{2.8}$$

on domain $\{(t,v) : 0 \le t < T, 0 \le v < \infty\}$, then the limit $U(t,v) = \lim_{n\to\infty} U_n(t,v)$ has to be the solution of the PDE (2.7).

To solve the PDE (2.8), if we define $\hat{u}_n(t,v^*)$ as the Mellin transform of $U_n(t,v)$, then the inverse of the Mellin transform is expressed by

$$U_n(t,v) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{u}_n(t,v^*)v^{-v^*}dv^*. \tag{2.9}$$

Hence, plugging (2.9) into the PDE (2.8) yields

$$U_n(t,v) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{h}_n(v^*)e^{\alpha(t)(v^{*2}+v^*)}v^{-v^*}dv^*, \tag{2.10}$$

where $\alpha(t) = \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^*$ and $\hat{h}_n(v^*)$ is the Mellin transform of $h_n(v)$.

To compute (2.10), we consider

$$R(t,v) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\alpha(t)(v^{*2}+v^*)}v^{-v^*}dv^*. \tag{2.11}$$

Because $\alpha(t) = \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^* \ge 0$, by Lemma 1 described in Yoon [13], we obtain

$$R(t,v) = \frac{1}{2\sqrt{\pi\alpha(t)}}\, e^{-\frac{1}{4}\alpha(t)}v^{\frac{1}{2}}e^{-\frac{(\ln v)^2}{4\alpha(t)}}. \tag{2.12}$$

Remark 2.1. The Mellin convolution of f and g is given by the inverse Mellin transform of $\hat{f}(v^*)\,\hat{g}(v^*)$ as follows:

$$f(v) * g(v) = M_{v^*}^{-1}\left[\hat{f}(v^*)\hat{g}(v^*); v\right] = \int_0^\infty u^{-1} f\left(\frac{v}{u}\right) g(u) du,$$

where $f(v) * g(v)$ is the symbol of the Mellin convolution of f and g, and $M_{v^*}^{-1}$ is the symbol of the inverse Mellin transform.

Because $e^{\alpha(t)(v^{*2}+v^*)}$ and $\hat{h}_n(v^*)$ are the Mellin transforms of $R(t,v)$ and $h(v)$, respectively, the Mellin convolution property mentioned above yields

$$\begin{aligned} U_n(t,v) &= \int_0^n h_n(u) R(t, \frac{v}{u}) \frac{1}{u} du \\ &= \int_K^n \frac{1}{2\sqrt{\pi\alpha(t)}} e^{-\frac{\alpha(t)}{4}} \left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{u} e^{-\frac{1}{4\alpha(t)}(\ln\frac{v}{u})^2} (v-K) du. \end{aligned}$$

By taking the limit $n \to \infty$ on both sides, we have

$$U(t,v) = \int_K^\infty \frac{1}{2\sqrt{\pi\alpha(t)}} e^{-\frac{\alpha(t)}{4}} \left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{u} e^{-\frac{1}{4\alpha(t)}(\ln\frac{v}{u})^2} (v-K) du. \tag{2.13}$$

Theorem 2.1. *Under the given condition $P(T,x,r) = (X_T - K)^+$, the formula of the European call option with a Hull-White interest rate is expressed by*

(2.14)

$$P(t,x,r) = x\Phi\left(d_1(t, \frac{x}{KB(t,r;T)})\right) - KB(t,r;T)\Phi\left(d_2(t, \frac{x}{KB(t,r;T)})\right),$$

$$d_1(t,\xi_1) := \frac{\ln\xi_1 + \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^*}{\sqrt{\int_t^T \hat{\sigma}^2(t^*)dt^*}}, \quad d_2(t,\xi_1) := d_1(t,\xi_1) - \sqrt{\int_t^T \hat{\sigma}^2(t^*)dt^*},$$

where $B(T-\tau, r; T)$ is the price of the zero-coupon bond mentioned in (2.3) *to* (2.4), $\hat{\sigma}(t) = \sqrt{\sigma^2 + 2\rho_{sr}\sigma\check{\sigma}M(t) + \check{\sigma}^2 M^2(t)}$ *with $M(t) = -\frac{1}{B}\frac{\partial B}{\partial r}$ and Φ is the normal cumulative distribution function defined by $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-\frac{z^2}{2}} dz$.*

Proof. The direct calculation of (2.13) leads to

$$\begin{aligned} U(t,v) &= \int_K^\infty \frac{1}{2\sqrt{\pi\alpha(t)}} e^{-\frac{\alpha(t)}{4}} \left(\frac{v}{u}\right)^{\frac{1}{2}} \frac{1}{u} e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2} (v-K) du \\ &= \frac{1}{2\sqrt{\pi\alpha(t)}} \int_K^\infty e^{-\frac{1}{4\alpha(t)}(\ln\frac{v}{u}-\alpha(t))^2} (v-K) \frac{1}{u} du \\ &= \frac{v}{2\sqrt{\pi\alpha(t)}} \int_K^\infty e^{-\frac{1}{4\alpha(t)}(\ln\frac{v}{u}+\alpha(t))^2} \frac{1}{u} du \\ &\quad - \frac{K}{2\sqrt{\pi\alpha(t)}} \int_K^\infty e^{-\frac{1}{4\alpha(t)}(\ln\frac{v}{u}-\alpha(t))^2} \frac{1}{u} du \\ &= v\,\Phi(d_1(t, \frac{v}{K})) - K\Phi(d_2(t, \frac{v}{K})), \end{aligned} \tag{2.15}$$

where

$$\alpha(t) = \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^*,\ d_1(t,\xi_1) = \frac{\ln\xi_1 + \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^*}{\sqrt{\int_t^T \hat{\sigma}^2(t^*)dt^*}},$$

$$d_2(t,\xi_1) = \frac{\ln\xi_1 - \frac{1}{2}\int_t^T \hat{\sigma}^2(t^*)dt^*}{\sqrt{\int_t^T \hat{\sigma}^2(t^*)dt^*}} \text{ and } \Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^z e^{-\frac{z^2}{2}}dz.$$

Because $U(t,v) = \frac{P(t,x,r)}{B(t,r;T)}$ and $v = \frac{x}{B(t,r;T)}$, we obtain the closed-form solution of $P(t,x,r)$

(2.16)

$$P(t,x,r) = x\Phi\left(d_1(t, \frac{x}{KB(t,r;T)})\right) - KB(t,r;T)\Phi\left(d_2(t, \frac{x}{KB(t,r;T)})\right). \quad \square$$

3. Discount barrier options with a Hull-White interest rate

This section derives the closed-form solution of discount barrier options with a stochastic interest rate driven by the Hull-White process. First, we set up the PDE on the discount barrier option model and find a unique solution of the PDE on the extended domain using Mellin transform methods and the image solution using the PDE method of images described in Peter Buchen [3]. Combining the unique solution and the image solution, we derive the pricing formula of discount barrier options with a Hull-White interest rate.

3.1. PDE formulation

This subsection considers a European up-and-out call option with a Hull-White interest rate process r_t and a discount stochastic barrier $\beta(t) = H \cdot B(t,r;T)$ where H is a constant and $B(t,r;T)$ is the price of a zero coupon at time t. Under the risk-neutral measure P^*, the price of the up-and-out call with the discount barrier $\beta(t)$ is given by

$$P(t,x,r) = E^*\left\{\exp\left(-\int_t^T r_t^* dt^*\right)\tilde{h}(X_T)|X_t = x, r_t = r\right\},$$

and the payoff function $\tilde{h}$ is expressed by

$$\tilde{h}(X_T) = (X_T - K)^+ 1_{\{\max_{0\le\gamma\le T}(X_\gamma - \beta(\gamma))<0\}}.$$

Using a similar method and the Feynman-Kac formula, the solution of $P(t,x,r)$ satisfies the following PDE

(3.1)

$$\hat{\mathcal{L}}P(t,x,r)=0,\ P(T,x,r)=h(x)=(x-K)^+,\ x<H,\ P\left(t,H\cdot B(t,r;T),r\right)=0,$$

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - I) + \rho_{xr}\sigma\check{\sigma}x\frac{\partial^2}{\partial x\partial r} + (b(t) - ar)\frac{\partial}{\partial r} + \frac{1}{2}\check{\sigma}^2\frac{\partial^2}{\partial r^2}$$

with the domain $\{(t,x) : 0 \le t \le T,\ 0 \le x < H \cdot B(t,r;T)\}$.

Additionally, with the change of variables $U(t,v)=\frac{P(t,x,r)}{B(t,r;T)}$ and $v=\frac{x}{B(t,r;T)}$ in Section 2.1. We obtain the following reduced PDE:

$$\frac{\partial U}{\partial t}+\frac{1}{2}\hat{\sigma}^2(t)v^2\frac{\partial^2 U}{\partial v^2}=0, U(T,v)=h(v)=(v-K)^+, v<H, U(t,H0)=0 \tag{3.2}$$

with the domain $\{(t,v) : 0\le t\le T,\ 0\le v<H\}$, $\hat{\sigma}^2(t)=\sigma^2+2\rho\sigma\check{\sigma}M(t)+\check{\sigma}^2M^2(t)$, and $M(t)=-\frac{1}{B}\frac{\partial B}{\partial r}$.

3.2. Derivation of the discount barrier option pricing formula with a stochastic interest rate: Image function approach

Remark 3.1. If $q(t,x)$ is a differentiable function with respect to (t,x), then the image function of $q(t,x)$ denoted by $q^*(t,x)=\mathcal{I}_Bq(t,x)$ to the barrier B has the four following properties:

(1) ${\mathcal{I}_B}^2=I$, where I is the identity operator.
(2) If $\mathcal{L}\,q(t,x)=0$, then $\mathcal{L}\,\mathcal{I}_B[q(t,x)]=0$.
(3) When $x=B$, $\mathcal{I}_B[q]=q$, that is, $(I-\mathcal{I}_B)[q]=0$.
(4) If $x>B$ $(x<B)$ is the active domain of $q(t,x)$, then $x<B$ $(x>B)$ is the active domain of $q^*(t,x)$,

where $\mathcal{I}_B$ is the image operator with respect to the barrier level B, and $\mathcal{L}$ is a parabolic differential operator.

Then, for a differentiable function $\tilde{q}(t,x)$, to derive the solution of the following PDE

$$\mathcal{L}\tilde{q}(t,x)=0;\ \tilde{q}(t,B)=0,\ x>B\ ,\ t<T,\ \ \tilde{q}(T,x)=g(x), \tag{3.3}$$

where $g(x)$ is the payoff function of $\tilde{q}(t,x)$, we consider the related PDE with unrestricted domain with respect to x

$$\mathcal{L}q(t,x)=0;\ \ q(T,x)=g(x)\mathbf{1}_{\{x>B\}},\ x>0\ ,\ t<T.$$

Then, the solution of the PDE (3.3) is described by $\tilde{q}(t,x)=q(t,x)-q^*(t,x)$.

In (3.2), letting $W(t,v)=U(t,Hv)$, the PDE (3.2) is transformed into

$$\frac{\partial W}{\partial t}+\frac{1}{2}\hat{\sigma}(t)v^2\frac{\partial^2 W}{\partial v^2}=0, W(T,v)=h(Hv)=(Hv-K)^+, v<1, W(t,1)=0 \tag{3.4}$$

in the region $\{(t,v) : 0\le t\le T,\ 0\le v<1\}$.

From Remark 3.1, to solve the problem (3.4), we consider $\bar{W}(t,v)$ satisfying the following PDE:

$$\frac{\partial \bar{W}}{\partial t}+\frac{1}{2}\hat{\sigma}(t)v^2\frac{\partial^2 \bar{W}}{\partial v^2}=0,\ \bar{W}(T,v)=h(Hv)\,1_{\{v<1\}} \tag{3.5}$$

with the domain $\{(t,v) : 0 \le t \le T,\ 0 \le v < \infty\}$. Then, the Mellin transform approach and the Mellin convolution property mentioned before yield

$$\bar{W}(t,v) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{h}(v^*)e^{\alpha(t)(v^{*2}+v^*)}dv^* \ \text{ and} \tag{3.6}$$
$$\bar{W}(t,v) = \int_0^\infty h(Hu)\, 1_{\{u<1\}} R(t,\frac{v}{u})\frac{1}{u}du,$$

respectively, where $\hat{h}$ is the Mellin transform of $h(Hv)1_{\{v<1\}}$ and $R(t,v)$ is denoted by (2.13). If we let $Q(t,v^*) = \alpha(t)\,(v^{*2}+v^*)$, then $Q(t,v^*) = Q(t,-1-v^*)$ and this relation yields

$$\begin{aligned} R(t,v) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{Q(t,v^*)}v^{-v^*}dv^* \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{Q(t,-1-v^*)}v^{1+v^*}dv^* = v\,R(t,\frac{1}{v}). \end{aligned} \tag{3.7}$$

From (3.6), we obtain

$$v\bar{W}(t,\frac{1}{v}) = \int_0^\infty h(\frac{H}{u})\, 1_{\{u>1\}} R(t,\frac{v}{u})\frac{1}{u}du. \tag{3.8}$$

Theorem 3.1. *Let $\bar{W}^*(t,v) = v\,\bar{W}(t,\frac{1}{v})$. Then $\bar{W}^*(t,v)$ is an image function of $\bar{W}(t,v)$ satisfying $\hat{L}\,\bar{W}^*(t,v) = 0$, where $\hat{L} = \frac{\partial}{\partial t} + \frac{1}{2}\hat{\sigma}(t)v^2\frac{\partial^2}{\partial v^2}$. Moreover, the solution of* (3.2) *is given by*

$$U(t,v) = \bar{U}(t,v) - \bar{U}^*(t,v) = \bar{U}(t,v) - \frac{v}{H}\bar{U}(t,\frac{H^2}{v}), \tag{3.9}$$

where $\bar{U}(t,v)$ is a solution of $\hat{L}\,\bar{U}(t,v) = 0$ with terminal condition $\bar{U}(T,v) = h(v)1_{\{v<H\}}$, and $\bar{U}^*(t,v) = \frac{v}{H}\bar{U}(t,\frac{H^2}{v})$ is the image function of $\bar{U}(t,v)$.

Proof. As described previously, using the convolution property of the Mellin transform, $\bar{W}(t,v)$ in (3.6) is a solution of (3.5). Therefore, from the relationship between (3.5)(PDE problem) and (3.6)(Mellin convolution), $\bar{W}^*(t,v) = v\bar{W}(t,\frac{1}{v})$ given by (3.8) is a solution of the PDE $\hat{L}\,\bar{W}^*(t,v) = 0$ with the terminal condition $\bar{W}^*(T,v) = v\,h(\frac{H}{v})\,1_{\{v>1\}}$. Additionally, to prove that $\bar{W}^*(t,v)$ is an image function of $W(t,v)$, let us define $W(t,v) = \bar{W}(t,v) - \bar{W}^*(t,v)$.

Then,

$$\hat{L}\,\bar{W}(t,v) = 0 \text{ and } \hat{L}\,\bar{W}^*(t,v) = 0,\ W(t,1) = \bar{W}(t,1) - \bar{W}^*(t,1) = 0, \tag{3.10}$$

and

$$\begin{aligned} W(T,v) &= \bar{W}(T,v) - \bar{W}^*(T,v) \\ &= h(Hv)\mathbf{1}_{\{v<1\}} - h(\frac{H}{v})v\mathbf{1}_{\{v>1\}} = \begin{cases} h(Hv) & \text{if } v<1 \\ -vh(\frac{H}{v}) & \text{if } v>1 \end{cases} \end{aligned} \tag{3.11}$$

are satisfied. Hence, from the properties of the image function mentioned in Peter Buchen [4] and Remark 3.1, $\bar{W}^*(t,v)$ is the image function of $\bar{W}(t,v)$.

Additionally, from $W(t,v) = \bar{W}(t,v) - \bar{W}^*(t,v)$ and (3.10), we have $\hat{L}\,W(t,v) = 0$ and by combining (3.10) and (3.11), $W(t,v)$ is the solution of the PDE (3.4), which is expressed by

$$W(t,v) = \bar{W}(t,v) - v\bar{W}(t,\frac{1}{v}). \tag{3.12}$$

Finally, by replacing v with $\frac{v}{H}$ in (3.12), we obtain

$$W(t,\frac{v}{H}) = \bar{W}(t,\frac{v}{H}) - \frac{v}{H}\bar{W}(t,\frac{H}{v}) \tag{3.13}$$

and from $W(t,v) = U(t,Hv)$, $U(t,v)$ is given by

$$U(t,v) = \bar{U}(t,v) - \bar{U}^*(t,v) = \bar{U}(t,v) - \frac{v}{H}\bar{U}(t,\frac{H^2}{v}). \tag{3.14}$$

□

Theorem 3.2. *The price of the discount barrier call option with a Hull-White interest rate defined in* (3.1)*, is described by*

$$\begin{aligned} P(t,x,r) = {} & x\Phi\left(d_1(t,\frac{x}{KB})\right) - KB\Phi\left(d_2(t,\frac{x}{KB})\right) \\ & - x\Phi\left(d_1(t,\frac{x}{HB})\right) + KB\Phi\left(d_2(t,\frac{x}{HB})\right) \\ & - HB\Phi\left(d_1(t,\frac{H^2B}{Kx})\right) + \frac{xK}{H}\Phi\left(d_2(t,\frac{H^2B}{xK})\right) \\ & + HB\Phi\left(d_1(t,\frac{HB}{x})\right) - \frac{xK}{H}\Phi\left(d_2(t,\frac{HB}{x})\right), \end{aligned} \tag{3.15}$$

where d_1 and d_2 are defined by Theorem 2.1, and $B = B(t,r;T)$ is the price of the zero-coupon bond denoted by (2.3) *to* (2.4)*.*

Proof. In Theorem 3.1, the $\bar{U}(t,v)$ is the solution satisfying

$$\hat{\mathcal{L}}\bar{U} = 0, \quad \bar{U}(T,v) = h(v)\mathbf{1}_{\{v<H\}} = (v-K)^+\mathbf{1}_{\{v<H\}}, \tag{3.16}$$

and, using the same procedure as in Section 2,

$$\begin{aligned} \bar{U}(t,v) = {} & \int_K^H \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du \\ = {} & \int_K^\infty \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du \\ & - \int_H^\infty \frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du \\ = {} & v\,\Phi\left(d_1(t,\frac{v}{K})\right) - K\Phi\left(d_2(t,\frac{v}{K})\right) \\ & - \left(v\,\Phi\left(d_1(t,\frac{v}{H})\right) - K\Phi\left(d_2(t,\frac{v}{H})\right)\right). \end{aligned} \tag{3.17}$$

By (3.9) in Theorem 3.1, we obtain $U(t,v) = \bar{U}(t,v) - \frac{v}{H}\bar{U}(t, \frac{H^2}{v})$. Therefore, $U(t,v)$ yields

$$
\begin{aligned}
U(t,v) = {} & v\,\Phi\left(d_1(t, \frac{v}{K})\right) - K\Phi\left(d_2(t, \frac{v}{K})\right) \\
& - \left(v\,\Phi\left(d_1(t, \frac{v}{H})\right) - K\Phi\left(d_2(t, \frac{v}{H})\right)\right) \\
& - H\,\Phi\left(d_1(t, \frac{H^2}{Kv})\right) + \frac{vK}{H}\Phi\left(d_2(t, \frac{H^2}{Kv})\right) \\
& + \left(H\,\Phi\left(d_1(t, \frac{H}{v})\right) - \frac{vK}{H}\Phi\left(d_2(t, \frac{H}{v})\right)\right).
\end{aligned} \tag{3.18}
$$

Because $U(t,v) = \frac{P(t,x,r)}{B(t,r;T)}$, $v = \frac{x}{B(t,r;T)}$,

$$
\begin{aligned}
P(t,x,r) = {} & x\Phi\left(d_1(t, \frac{x}{KB})\right) - KB\Phi\left(d_2(t, \frac{x}{KB})\right) \\
& - x\Phi\left(d_1(t, \frac{x}{HB})\right) + KB\Phi\left(d_2(t, \frac{x}{HB})\right) \\
& - HB\Phi\left(d_1(t, \frac{H^2B}{Kx})\right) + \frac{xK}{H}\Phi\left(d_2(t, \frac{H^2B}{xK})\right) \\
& + HB\Phi\left(d_1(t, \frac{HB}{x})\right) - \frac{xK}{H}\Phi\left(d_2(t, \frac{HB}{x})\right).
\end{aligned} \tag{3.19}
$$

□

4. Discount double barrier options with a Hull-White interest rate

This section investigates the price of discount double barrier options with a Hull-White interest rate utilizing consecutive chains of image operators described by Bucheon and Konstandatos [5] and Mellin transform techniques.

4.1. PDE formulation

We consider the arbitrage-free pricing of double knock-out discount barrier call options. A double barrier option has two barriers that contain an upstream barrier at $H \cdot B(t,r;T)$ and a downstream barrier at $L \cdot B(t,r;T)$. If the underlying asset X_t reaches one of the two barriers at any time before the expiration, then the invalidity of the option contract is aroused. Then, we obtain the price of double knock-out discount barrier options as follows:

$$P(t,x,r) = E^* \left\{ \exp\left(-\int_t^T r_t^* dt^*\right) \hat{h}(X_T) | X_t = x, r_t = r \right\},$$

where the payoff function h is given by

$$\hat{h}(X_T) = (X_T - K)^+ \, 1_{\{\{\max_{0\le\gamma\le T}(X_\gamma - H\cdot B(\gamma))<0\}\cap\{\min_{0\le\gamma\le T}(X_\gamma - L\cdot B(\gamma))>0\}\}}.$$

Using a similar method and the Feynman-Kac formula, the solution of $P(t,x,r)$ satisfies the following PDE

(4.1)

$$\begin{aligned}
&\hat{\mathcal{L}}P(t,x,r)=0,\\
&P(T,x,r)=h(x)=(x-K)^{+}\quad L<x<H,\\
&P\left(t,H\cdot B(t,r;T),r\right)=P\left(t,L\cdot B(t,r;T),r\right)=0,\\
&\hat{\mathcal{L}}=\frac{\partial}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2}{\partial x^2}+r(x\frac{\partial}{\partial x}-I)+\rho_{xr}\sigma\check{\sigma}x\frac{\partial^2}{\partial x\partial r}+(b(t)-ar)\frac{\partial}{\partial r}+\frac{1}{2}\check{\sigma}^2\frac{\partial^2}{\partial r^2}
\end{aligned}$$

with the domain $\{(t,x)\,:\,0\leq t\leq T,\,L\cdot B(t,r;T)<x<H\cdot B(t,r;T)\}$.

By changing variables $Q(t,v)=\frac{P(t,x,r)}{B(t,r;T)}$ and $v=\frac{x}{B(t,r;T)}$ described previously, PDE (4.1) leads to the following PDE:

$$\begin{aligned}
&\frac{\partial Q}{\partial t}+\frac{1}{2}\hat{\sigma}^2(t)v^2\frac{\partial^2 Q}{\partial v^2}=0,\;\; Q(T,v)=h(v)=(v-K)^{+},\\
&L<v<H,\;\; Q(t,H)=Q(t,L)=0
\end{aligned}\tag{4.2}$$

in the region $\{(t,v)\,:\,0\leq t\leq T,\,L<x<H\}$.

Then, we obtain the solution of PDE (4.2) using the method of images stated in Remark 3.1.

4.2. Derivation of the double discount barrier options formula with a stochastic interest rate: The method of images and Mellin transform approaches

To solve PDE (4.2), from Remark 3.1, we consider the following related PDE in an unrestricted domain

$$\frac{\partial\tilde{Q}}{\partial t}+\frac{1}{2}\hat{\sigma}^2(t)v^2\frac{\partial^2\tilde{Q}}{\partial v^2}=0,\quad \tilde{Q}(T,v)=(v-K)^{+}\mathbf{1}_{\{L<v<H\}}\tag{4.3}$$

on domain $\{(t,v)\,:\,0\leq t\leq T,\,0\leq v<\infty\}$.

Using the Mellin transform approach, we have

$$\begin{aligned}
\tilde{Q}(t,v)=&\int_L^H\frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)^{+}du\\
=&\int_{L\wedge K}^H\frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du\\
=&\int_{L\wedge K}^\infty\frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du\\
&-\int_{H}^\infty\frac{1}{2\sqrt{\pi\alpha}}e^{-\frac{\alpha}{4}}\left(\frac{v}{u}\right)^{\frac{1}{2}}\frac{1}{u}e^{-\frac{1}{4\alpha}(\ln\frac{v}{u})^2}(v-K)du\\
=&\,v\,\Phi\left(d_1(t,\frac{v}{L\wedge K})\right)-(L\wedge K)\Phi\left(d_2(t,\frac{v}{L\wedge K})\right)\\
&-\left(v\,\Phi\left(d_1(t,\frac{v}{H})\right)-K\Phi\left(d_2(t,\frac{v}{H})\right)\right),
\end{aligned}\tag{4.4}$$

where $a \wedge b := \begin{cases} b, & \text{if } a \geq b \\ a, & \text{if } a < b. \end{cases}$

Contrary to the image operator of the discount single barrier option with the stochastic interest rate described in Section 3, the image operator of the discount double barrier option with the stochastic interest rate is complex and is expressed by any sequential sum of successive image operators. From Bucheon and Konstandatos [4], we use the sum of the continuative image operators to derive the semi-analytic formula of PDE (4.1) with respect to $\tilde{Q}(t, v)$.

Lemma 4.1. *Let $\mathcal{K}_L^H$ denote the doubly infinite chain of image operators*

$$\begin{aligned} \mathcal{K}_L^H = I - \mathcal{I}_L + \mathcal{I}_{HL} - \mathcal{I}_{LHL} + \mathcal{I}_{HLHL} + \cdots \\ - \mathcal{I}_H + \mathcal{I}_{LH} - \mathcal{I}_{HLH} + \mathcal{I}_{LHLH} + \cdots . \end{aligned} \tag{4.5}$$

Then, the solution of PDE (4.2) *yields $Q(t, v) = \mathcal{K}_L^H[\tilde{Q}(t, v)]$, where $\tilde{Q}(t, v)$ is given by* (4.4).

Proof. First, because $\mathcal{K}_L^H$ is the infinite chain of the image operators, by Remark 3.1, we have $\mathcal{L}\,\mathcal{K}_L^H[\tilde{Q}(t, v)] = 0$. Additionally, if we define

$$\mathcal{K}_{HL} = I - \mathcal{I}_H + \mathcal{I}_{HL} - \mathcal{I}_{LHL} + \cdots ,$$

then, by the symbol of consecutive image operators mentioned in Bucheon and Konstandatos [4], $\mathcal{I}_{ba} = \mathcal{I}_b \mathcal{I}_a$, the factorization of $\mathcal{K}_L^H$ satisfies

$$\mathcal{K}_L^H = (I - \mathcal{I}_L)\mathcal{K}_{HL},$$

and we obtain $\mathcal{K}_L^H[\tilde{Q}(t, L)] = \mathcal{K}_L^H[\tilde{Q}(t, H)] = 0$.

Finally, let $\mathcal{K}_L^H[\tilde{Q}(T, v)] = \tilde{Q}(T, v) + \text{image sequence of } \tilde{Q}(T, v)$ in (4.5). From (4.5), $\tilde{Q}(T, v) = h(v)$ for $L < v < H$ and from the property (4) of the image operator in Remark 3.1, the image sequence of $\tilde{Q}(T, v)$ vanishes at an outside interval (L, H). Hence, $\mathcal{K}_L^H[\tilde{Q}(T, v)] = h(v)$.

Therefore, $\mathcal{K}_L^H[\tilde{Q}(t, v)]$ is the solution of the PDE (4.3) and, then, $U(t, v) = \mathcal{K}_L^H[\tilde{Q}(t, v)]$. □

Remark 4.1. For positive constants α, β, and γ, we consider the corresponding image operator $\mathcal{I}_\alpha$, $\mathcal{I}_\beta$, and $\mathcal{I}_\gamma$ respectively. Then, by Lemma 3.5 in Buchen and Konstandatos [4], we obtain

$$\mathcal{I}_{\gamma\beta\alpha} = \mathcal{I}_{\frac{\gamma\alpha}{\beta}}, \tag{4.6}$$

where $\mathcal{I}_{\gamma\beta\alpha} = \mathcal{I}_\gamma \mathcal{I}_\beta \mathcal{I}_\alpha$.

Remark 4.2. For integer $n > 0$, let $\mathcal{K}_{LH}^n := \mathcal{I}_{LH}\mathcal{I}_{LH}\cdots\mathcal{I}_{LH}$ (n-LH pairs). Using Remark 4.1 and induction, we have

$$\mathcal{K}_{LH}^n = \mathcal{I}_H \mathcal{I}_{\frac{H^{n+1}}{L^n}} \text{ and } \mathcal{K}_{LH}^{-n} = \mathcal{I}_H \mathcal{I}_{\frac{H^{-n+1}}{L^{-n}}}. \tag{4.7}$$

The proof of (4.7) is described by Lemma 3.7 in Buchen and Konstandatos [4]. Then, Remark 4.1 and Remark 4.2 lead to the following lemma.

Lemma 4.2. (a) $\mathcal{K}_{LH}^{n} = \mathcal{K}_{HL}^{-n}$.

(b) $$\mathcal{K}_L^H = (I - \mathcal{I}_L) \sum_{n=-\infty}^{\infty} \mathcal{K}_{LH}^n = (I - \mathcal{I}_H) \sum_{n=-\infty}^{\infty} \mathcal{K}_{LH}^n$$
$$= (I - \mathcal{I}_L) \sum_{n=-\infty}^{\infty} \mathcal{K}_{HL}^n = (I - \mathcal{I}_H) \sum_{n=-\infty}^{\infty} \mathcal{K}_{HL}^n.$$

Proof. The proof of this lemma follows from Corollary 3.8, Lemma 3.9, and Corollary 3.10 in Buchen and Konstandatos [4]. □

Hence, by Lemma 4.1, we obtain

$$Q(t,v) = \mathcal{K}_L^H[\tilde{Q}(t,v)] = (I - \mathcal{I}_L) \sum_{n=-\infty}^{\infty} \mathcal{K}_{LH}^n[\tilde{Q}(t,v)]. \tag{4.8}$$

Theorem 4.1. *The price of the discount double barrier call option with a Hull-White interest rate, defined by in* (4.1)*, is expressed by*

$$P(t,x,r) = \sum_{n=-\infty}^{\infty} \left[\frac{B\,H^n}{L^n} \tilde{Q}(t, \frac{H^{2n}}{L^{2n}} \frac{x}{B}) - x \frac{H^n}{L^{n+1}} \tilde{Q}(t, \frac{H^{2n}}{L^{2n-2}} \frac{x}{B}) \right], \tag{4.9}$$

where $\tilde{Q}(t,\cdot)$ *is defined by* (4.4)*, and* $B = B(t,r;T)$ *is the bond price from time* t *to the expiry time* T *given by* (2.3) *to* (2.4)*.*

Proof. By (4.8),

$$Q(t,v) = \sum_{n=-\infty}^{\infty} \mathcal{K}_{LH}^n[\tilde{Q}(t,v)] - \sum_{n=-\infty}^{\infty} \mathcal{I}_L \mathcal{K}_{LH}^{-n}[\tilde{Q}(t,v)]. \tag{4.10}$$

Then, from $\mathcal{K}_{LH}^n = \mathcal{I}_H \mathcal{I}_{\frac{H^{n+1}}{L^n}}$ in Remark 4.2 and the image function of $\tilde{Q}(t,v)$ expressed by $\mathcal{I}_H \tilde{Q}(t,v) = \frac{v}{H} \tilde{Q}(t, \frac{H^2}{v})$ in Theorem 3.1,

$$\mathcal{K}_{LH}^n[\tilde{Q}(t,v)] = \mathcal{I}_H \left[\frac{L^n}{H^{n+1}} v \tilde{Q}(t, \frac{H^{2n+2}}{L^{2n}} \frac{1}{v}) \right] = \frac{L^n}{H^n} \tilde{Q}(t, \frac{H^{2n}}{L^{2n}} v), \tag{4.11}$$

and, similarly, by $\mathcal{K}_{LH}^{-n} = \mathcal{I}_H \mathcal{I}_{\frac{H^{-n+1}}{L^{-n}}}$ in Remark 4.2,

$$\mathcal{I}_L \mathcal{K}_{LH}^{-n}[\tilde{Q}(t,v)] = \mathcal{I}_L \left[\frac{L^{-n}}{H^{-n}} \tilde{Q}(t, \frac{H^{-2n}}{L^{-2n}} v) \right] = \frac{H^n}{L^{n+1}} v \tilde{Q}(t, \frac{L^{2n}}{H^{2n}} \frac{L^2}{v}). \tag{4.12}$$

Therefore,

$$Q(t,v) = \sum_{n=-\infty}^{\infty} \frac{L^n}{H^n} \tilde{Q}(t, \frac{H^{2n}}{L^{2n}} v) - \sum_{n=-\infty}^{\infty} v \frac{H^n}{L^{n+1}} \tilde{Q}(t, \frac{L^{2n+2}}{H^{2n}} \frac{1}{v}), \tag{4.13}$$

where $\tilde{Q}(t,\cdot)$ is given by (4.4).

Finally, from $Q(t,v) = \frac{P(t,x,r)}{B(t,r;T)}$, $v = \frac{x}{B(t,r;T)}$, and (4.13), we obtain the desired results in Theorem 4.1. □

5. Concluding remarks

This paper investigated whether an exact form solution for the discount European barrier option with a Hull-White stochastic interest rate can be expressed by cumulative normal distribution functions, and whether the semi-analytic solution for the discount double barrier option under a Hull-White stochastic interest rate can be described by the infinite sum of cumulative normal distribution functions. The solutions were derived using the method of images and Mellin transform approach. The Mellin transform method resolves the complexity of the calculation compared to the probabilistic techniques, Fourier transforms, and the method of change of variables in other types of options as well as the barrier options. Additionally, research using Mellin transform methods on option pricing is ongoing.

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