Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 337–344
https://doi.org/10.4134/CKMS.c170091
pISSN: 1225-1763 / eISSN: 2234-3024

BOWEN'S DECOMPOSITION THEOREM FOR TOPOLOGICALLY ANOSOV HOMEOMORPHISMS ON NONCOMPACT AND NON-METRIZABLE SPACES

Ruchi Das, Tarun Das, and Sejal Shah

Abstract. We extend Bowen's decomposition theorem to topologically Anosov homeomorphisms on first countable, locally compact, paracompact, Hausdorff spaces which are not necessarily metrizable and not necessarily compact

1. Introduction

We first recall the following remarkable result in dynamical systems.

Theorem 1.1 (Spectral Decomposition, [7]). *Let $f : M \to M$ be an Axiom A diffeomorphism of a compact C^∞ Riemannian manifold M. One can write the non-wandering set $\Omega(f) = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$, where the Ω_i are pairwise disjoint closed sets with*

A. *$f(\Omega_i) = \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive*;
B. *$\Omega_i = X_{1,i} \cup X_{2,i} \cup \cdots \cup X_{n_i,i}$ with $X_{j,i}$'s pairwise disjoint closed sets, $f(X_{j,i}) = X_{j+1,i}$, $(f(X_{n+1,i}) = X_{1,i})$ and $f^{n_i}|_{X_{j,i}}$ topologically mixing.*

In the above Theorem, part A is due to Smale [16] and part B is due to Bowen [6]. A simple corollary of this spectral decomposition theorem is that an Anosov diffeomorphism of a connected manifold is topologically mixing - a connected manifold cannot be decomposed into disjoint closed sets. Using Bowen's decomposition theorem, Sigmund [15], was able to prove that the set of Bernoulli measures is dense among the invariant measures.

Smale's spectral decomposition theorem has been extended to homeomorphisms on compact metric spaces [3]. It states that the set of non-wandering points of an expansive homeomorphism f with the shadowing property on a compact metric space can be decomposed into a finite pairwise disjoint closed and f-invariant sets such that the restriction of f to these sets is topologically transitive. Such sets are called the basic sets. Further each basic set admits

Received March 9, 2017; Accepted May 30, 2017.
2010 *Mathematics Subject Classification.* Primary 37B20, 54H20.
Key words and phrases. topological shadowing, topological expansivity, chain recurrent set, topological mixing.

a decomposition into disjoint closed subsets such that the restriction of some iterate of f to these subsets is topologically mixing [3].

Yang [18], further extended Smale's spectral decomposition theorem to noncompact metrizable spaces with the additional requirement that the chain recurrent set be compact. In [5], a generalization of Smale's spectral decomposition theorem is obtained for piecewise-monotonic maps wherein the decomposition is made in terms of basic and solenoidal sets. In [1], authors use their main result to prove that any homoclinic class of a generic diffeomorphism has a spectral decomposition in the sense of Bowen.

It is well known that some dynamical properties of homeomorphisms on compact metrizable spaces may not be dynamical properties on noncompact spaces. For example a map may have expansivity or shadowing with respect to one metric but not with respect to another metric that induces the same topology on a noncompact metrizable space. One can refer Examples 6 and 7 of [10]. The second author and others have extended Smale's spectral decomposition theorem to those spaces which are not necessarily metrizable and not necessarily compact. They have obtained the following result.

Theorem 1.2 ([10]). *Let X be a first countable, locally compact, paracompact, Hausdorff space and $f : X \to X$ an expansive homeomorphism with the shadowing property. Then the non-wandering set $\Omega(f)$ can be written as a union of disjoint closed invariant sets (called as basic sets for f) on which f is topologically transitive. If X is compact, then this decomposition is finite.*

The goal of this paper is to extend Bowen's Theorem to first countable, locally compact, paracompact, Hausdorff spaces (not necessarily metrizable and not necessarily compact). This paper is organized as follows. In Section 2, we give preliminaries required for the development of the paper. In Section 3, giving necessary details of stable and unstable sets, we prove our main theorem.

2. Preliminaries

We shall consider throughout topological spaces which are first countable, locally compact, paracompact, Hausdorff and maps which are homeomorphisms. Let X be a non-empty set and let $\triangle_X = \{(x,x) \,|\, x \in X\}$, called as the *diagonal* of $X \times X$. For a subset M of $X \times X$ define $M^T = \{(y,x) \,|\, (x,y) \in M\}$. Set M is called *symmetric* if $M^T = M$. The set $M[x] = \{y \in X | (x,y) \in M\}$ is called the *cross section* of M at $x \in X$. A set $M \subset X \times X$ is *proper* if for any compact subset S, the set $M[S] = \cup_{x\in S} M[x]$ is compact. Let U be a neighborhood of $\triangle_X$. Then $U \cap U^T$ is a symmetric neighborhood of $\triangle_X$. Thus one can work with symmetric neighborhoods without loss of generality. Let $U^n = \{(x,y) \,|$ there exist $z_0 = x, z_1, \ldots, z_n = y \in X$ such that $(z_{i-1}, z_i) \in U$ for $i = 1, 2, \ldots, n\}$. Intuitively, if U is a topological equivalent of an $\frac{\epsilon}{n}$-ball at a point, then U^n would be a topological equivalent of an ϵ-ball at a point. For a dynamical system (X, f), we denote $f \times f$ by F. The theory of uniform spaces was introduced by A. Weil in [17].

Definition ([11])**.** Let X be a non-empty set. A *uniform structure* on X is a non-empty set $\mathcal{U}$ of subsets of $X \times X$ satisfying the following conditions:

i. if $U \in \mathcal{U}$, then $\triangle_X \subset U$,
ii. if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}$,
iii. if $U \in \mathcal{U}$ and $V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$,
iv. if $U \in \mathcal{U}$, then $U^T \in \mathcal{U}$,
v. if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The elements of $\mathcal{U}$ are called the *entourages* of the uniform structure and the pair $(X, \mathcal{U})$ is called a *uniform space*. Every uniform space comes equipped with uniform topology based on $\{D[x] \,|\, (D, x) \in \mathcal{U} \times X\}$. Note that every locally compact Hausdorff space is Tychonoff, so has a uniform structure. Paracompactness implies that the neighborhoods of the diagonal form a uniformity. Also, local compactness and paracompactness imply that the proper neighborhoods form a base for the uniformity [11]. Also, for any point $x \in X$ and any neighborhood G of x in X we can find a neighborhood U of $\triangle_X$ such that $U[x] \subset G$. If f is a homeomorphism on a paracompact, locally compact space X, then X can be decomposed into clopen f-invariant subsets each of which is σ-compact as well as locally compact [2].

Recently metric dynamical properties like expansivity, shadowing, specification have been extended to the non-metric setting [10], [13]. Authors in [12] have defined and studied positively expansive and expansive measures on uniform spaces. The notions of Devaney chaos and entropy are also studied for uniform spaces ([4], [8], [14]).

Let (X, d) be a metric space and $f : X \to X$ be a homeomorphism. The map f is said to be *metric expansive* if there exists a constant $c > 0$ such that $x \neq y$ in X implies $d(f^n(x), f^n(y)) > c$ for some integer n [3]. The constant c is called an *expansive constant* for f. For $\delta > 0$, a δ*-chain* is a sequence $\{x_0, x_1, \ldots, x_n\} (n \geq 1)$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, 1, \ldots, n-1$. A δ*-pseudo-orbit* is a bi-infinite δ-chain. A sequence $\{x_i\}$ is said to be ϵ*-traced* by a point x in X if $d(f^i(x), x_i) < \epsilon$, holds for each i. The map f is said to have the *metric shadowing property* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that each δ-pseudo-orbit for f is ϵ-traced by some point of X [3].

We work here with spaces which are not necessarily compact and not necessarily metrizable. A homeomorphism $f : X \to X$ is *topologically expansive* if there exists a closed neighborhood D of $\triangle_X$ such that for any pair of distinct points $x, y \in X$, there exists an $n \in \mathbb{Z}$ such that $F^n(x, y) \notin D$; D is called an *expansive neighborhood* for f [10]. It is observed that metric expansivity implies topological expansivity but not vice versa. Also, for compact metric spaces both metric and topological expansivity are equivalent. If $f : X \to X$ is an expansive homeomorphism of a first countable, locally compact, paracompact Hausdorff space, then there is a proper expansive neighborhood [10].

Let D and E be neighborhoods of $\triangle_X$. A D*-chain* is a sequence $\{x_0, x_1, \ldots, x_n\}$ $(n \geq 1)$ such that $(f(x_{i-1}), x_i) \in D$ for $i = 1, 2, \ldots, n$. A D*-pseudo-orbit* is a bi-infinite D-chain. A D-pseudo-orbit $\{x_i\}$ is said to be *E-traced* by a point $y \in X$ if $(f^i(y), x_i) \in E$ for all $i \in \mathbb{Z}$. A homeomorphism $f : X \to X$ is said to have the *topological shadowing property* if for every neighborhood E of $\triangle_X$, there exists a neighborhood D of $\triangle_X$ such that every D-pseudo-orbit is E-traced by some point $y \in X$ [10]. It is observed that for general topological spaces metric shadowing and topological shadowing are independent concepts. However, for compact metrizable spaces topological shadowing and metric shadowing are equivalent [10]. If f is topologically expansive and has topological shadowing property, then f is called a *topologically Anosov map.*

3. Bowen's decomposition theorem for topologically Anosov homeomorphisms

In this section we prove Bowen's decomposition theorem for topologically Anosov homeomorphisms on first countable, locally compact, paracompact, Hausdorff spaces.

Let $x \in X$ and B be a neighborhood of $\triangle_X$. The *local stable set of x relative to B* is $W^s_B(x) = \{y \in X : F^i(x, y) \in B$ for all $i \geq 0\}$ and the *local unstable set of x relative to B* is $W^u_B(x) = \{y \in X : F^i(x, y) \in B$ for all $i \leq 0\}$ [10]. The *stable set* of x is $W^s(x) = \{y \in X|$ for all neighborhood B of $\triangle_X$, there exists $n \in \mathbb{N}$ such that $F^i(x, y) \in B$ for all $i \geq n\}$ and the *unstable set* of x is $W^u(x) = \{y \in X|$ for all neighborhood B of $\triangle_X$, there exists $n \in \mathbb{N}$ such that $F^i(x, y) \in B$ for all $i \leq -n\}$ [10]. One can observe that $f(W^\sigma(p)) = W^\sigma(f(p))$ for $\sigma = u, s$. Also,

$$W^s(x) \subset \cup_{n\geq 0} f^{-n}(W^s_D(f^n(x))), \quad W^u(x) \subset \cup_{n\geq 0} f^{n}(W^u_D(f^{-n}(x))),$$

where D is a neighborhood of $\triangle_X$. We now recall Lemma 19 and Proposition 20 proved in [10].

Lemma 19 ([10])**.** *Let f be a topologically expansive self homeomorphism on a first countable, locally compact, paracompact Hausdorff space. If B is a proper expansive neighborhood for f and x is a periodic point, then $W^\sigma_B(x) \subset W^\sigma(x)$, where $\sigma = s, u$.*

Proposition 20 ([10])**.** *Let f be a topologically expansive self homeomorphism with topological shadowing property on a first countable, locally compact, paracompact Hausdorff space. Then we can find neighborhoods B and D of $\triangle_X$ and a continuous map $t : D \to X$ such that*

1. *$W^s_B(x) \cap W^u_B(y)$ contains at most one point for any $x, y \in X$,*
2. *$W^s_B(x) \cap W^u_B(y) = \{t(x, y)\}$ if $(x, y) \in D$,*
3. *$W^s_B(x) \cap D[x] = \{y \,|\, y = t(x, y)$, where $(x, y) \in D\}$,*
4. *$W^u_B(x) \cap D[x] = \{y \,|\, y = t(y, x)$, where $(x, y) \in D\}$.*

Conley's Fundamental Theorem of dynamical systems initiated the study of chain recurrence classes [9]. Chain recurrence is one type of recurrence with "errors" allowed along the orbit. However some of these classes gained interest with the advent of Smale's spectral decomposition theorem. Let $x, y \in X$. If there is an A-chain from x to y and a B-chain from y to x for all neighborhoods A, B of $\triangle_X$, then we write $x \sim y$. The set $CR(f) = \{x \in X : x \sim x\}$ is called the *chain recurrent set* for f. The relation $\sim$ induces an equivalence relation on $CR(f)$; the equivalence classes are called *chain components* of f [10].

The *non-wandering set* of f is $\Omega(f) = \{x \in X \mid$ for any open neighborhood G of x in X, $f^n(G) \cap G \neq \emptyset$ for some $n > 0\}$ [7]. Let f be a self homeomorphism of a first countable, locally compact, paracompact, Hausdorff space then both $CR(f)$ and the chain component R of f are non-empty, closed and f-invariant subsets of X. It is known that if f has the topological shadowing property, then $\Omega(f) = CR(f)$. For a topologically Anosov map f, the set $Per(f)$ of periodic points of f is dense in $CR(f)$ and chain components are both open and closed in $CR(f)$.

Theorem 3.1. *Let X be a first countable, locally compact, paracompact, Hausdorff space and $f : X \to X$ be an expansive homeomorphism with the shadowing property. Then there exist a subset S of a basic set R and $k > 0$ such that $f^k(S) = S$, $S \cap f^j(S) = \phi$ $(0 < j < k)$, $f^k|_S$ is topologically mixing and $R = \cup_{j=0}^{k-1} f^j(S)$.*

Proof. Let A be a proper expansive neighborhood for f. Let B be a symmetric neighborhood of $\triangle_X$ such that $B^3 \subset A$ and let $E = B \cap F^{-1}(B)$ which is also a symmetric neighborhood of $\triangle_X$. Since f has the shadowing property, there exists a neighborhood D of $\triangle_X$ such that every D-pseudo orbit is E-traced by some point of X.

By Theorem 1.2, $\Omega(f)$ can be written as a disjoint union of basic sets. Let R be a basic set and let $S = \overline{W^u(p) \cap R}$, where $p \in R \cap Per(f)$ is a periodic point with period m. Then S is a closed subset of R. We now prove that S is open in R.

Note that $D[S]$ is an open set containing S. Let $q \in D[S]$ be a periodic point of period n. Then $(p, q), (q, p) \in D$. By Lemma 19 and Proposition 20 in [10], $W^u(p) \cap W^s(q) \neq \phi$. Let $y \in W^u(p) \cap W^s(q)$. Since p and q are periodic points, the tracing point y is also periodic [10]. Therefore $y \in W^u(p) \cap W^s(q) \cap R$. For $k > 0$, $f^{kmn}(y) \in f^{kmn}(W^u(p)) = W^u(f^{kmn}(p)) = W^u(p)$. Thus we have for every $k > 0$,

$$f^{kmn}(y) \in W^u(p) \cap R. \tag{1}$$

Let G be an open set in X containing q. Then there exists a neighborhood U of $\triangle_X$ such that $U[q] \subset G$. Now $y \in W^s(q)$ implies that for the above neighborhood U of $\triangle_X$, there exists $l \in \mathbb{N}$ such that $F^i(q, y) \in U$ for all $i \geq l$. This further implies that $F^{jmn}(q, y) \in U$ for large j, i.e., for large j,

$$(q, f^{jmn}(y)) \in U. \tag{2}$$

By (1) and (2), $q \in U[W^u(p) \cap R]$. Then there exists $z \in W^u(p) \cap R$ such that $(z, q) \in U$. Without loss of generality, we can assume that U is symmetric.

Therefore $(q,z) \in U$ and hence $z \in U[q]$. Since $U[q] \subset G$, we have $z \in G$. Thus every neighborhood G of q has a non-empty intersection with $W^u(p) \cap R$ proving that $q \in \overline{W^u(p) \cap R} = S$. This prove that $D[S] \subset S$ implying $S = D[S]$. Hence S is an open set.

We now show that $f(S) = \overline{W^u(f(p)) \cap R}$. Clearly, $\overline{f(W^u(p) \cap R)} \subset f(S)$ and $f(S) = f(\overline{W^u(p) \cap R}) \subset \overline{f(W^u(p) \cap R)}$. We therefore have $f(S) = \overline{f(W^u(p) \cap R)}$. Now, $f(S) = \overline{f(W^u(p)) \cap R} = \overline{W^u(f(p)) \cap R}$. Since p is a periodic point of period m, we have $f^m(S) = S$. We can find $0 < k \leq m$ such that $f^k(S) = S$. Note that $R = \cup_{i=0}^{k-1} f^i(S)$. For if $R - \cup_{i=0}^{k-1} f^i(S) \neq \phi$, then for $x \in R - \cup_{i=0}^{k-1} f^i(S)$, there exists an open set V containing x such that $V \subset R - \cup_{i=0}^{k-1} f^i(S)$ as $R - \cup_{i=0}^{k-1} f^i(S)$ is an open set. For non-empty open sets S and V, by transitivity of $f_{|R}$ there exists an integer $a \in \mathbb{Z}$ such that $f^a(S) \cap V \neq \phi$ implying $f^i(S) \cap V \neq \phi$ for some $0 \leq i \leq k-1$ as $f^k(S) = S$. This contradicts the choice of V. Hence $R = \cup_{i=0}^{k-1} f^i(S)$.

Let $q \in S$ be a periodic point of period n and let $H = \overline{W^u(q) \cap R}$. We now prove that $H = S$. Let D be the neighborhood of the diagonal as defined earlier. Then $D[S] = S$ and $W^u_D(q) = \{z \in X \mid F^i(q,z) \in D \text{ for all } i \leq 0\}$ implies that $W^u_D(q) \subset S$. Since q has period n, $W^u(q) \subset \cup_{j \geq 0} f^{mnj}(W^u_D(f^{-mnj}(q))) = \cup_{j \geq 0} f^{mnj}(W^u_D(q))$. Now $W^u_D(q) \subset S$ implies $f^{mnj}(W^u_D(q)) \subset f^{mnj}(S) = S$ for all $j \geq 0$. This further implies that $\cup_{j \geq 0} f^{mnj}(W^u_D(q)) \subset S$ and hence $W^u(q) \subset S$. Thus $H = \overline{W^u(q) \cap R} \subset S$.

Suppose $p \notin H$. Now $q \in S$ implies $q \in \overline{W^u(p) \cap R}$. Therefore for every neighborhood W of $\triangle_X$, $W[q] \cap (W^u(p) \cap R) \neq \phi$. Let $\alpha \in W[q] \cap (W^u(p) \cap R)$. Then $\alpha \in W[q]$ implies $(q, \alpha) \in W$. Without loss of generality, we can assume that $(\alpha, q) \in W$ and hence $q \in W[\alpha]$. Thus $W[\alpha] \cap H \neq \phi$ for every neighborhood W of $\triangle_X$. Since H is closed therefore $\alpha \in H$. On the other hand, $\alpha \in W^u(p)$ implies that for every neighborhood C of $\triangle_X$, there exists $l \in \mathbb{N}$ such that $F^i(p, \alpha) \in C$ for all $i \leq -l$. This further implies that $F^{-mnlj}(p, \alpha) \in C$ for every $j \geq 0$, i.e., $(p, f^{-mnlj}(\alpha)) \in C$ for every $j \geq 0$. Since $p \notin H$ there exists $j \geq 0$ such that $f^{-mnlj}(\alpha) \notin H$ implying $\alpha \notin f^{mnlj}(H) = H$, which is a contradiction. Hence $S = H$.

Note that $f^i(S) \cap f^j(S) = \phi$, $0 \leq i < j < k$. For if $f^i(S) \cap f^j(S) \neq \phi$ for some i, j such that $0 \leq i, j < k$, then choose $x \in f^i(S) \cap f^j(S) \cap Per(f) \cap R$. Such an x exists as $f^i(S) \cap f^j(S)$ is an open subset of R and $Per(f)$ is dense in R. By the preceding argument $f^i(S) = \overline{W^u(f^i(p)) \cap R} = \overline{W^u(x) \cap R}$ and $f^j(S) = \overline{W^u(f^j(p)) \cap R} = \overline{W^u(x) \cap R}$ implying $f^i(S) = f^j(S)$.

We now prove that $f^m_{|S}$ is topologically mixing. Let U and V be non-empty open subsets of S. Then for $q \in V \cap Per(f)$, we have $S = \overline{W^u(q) \cap R}$. Since $f^m(S) = S$, we have $f^{mj}(S) = S$ and hence $U \cap W^u(f^{mj}(q)) \neq \phi$ for every $j \in \mathbb{Z}$. Suppose q is a periodic point of period n. Then for $0 \leq j \leq n-1$, there exists $z_j \in U \cap W^u(f^{mj}(q))$ such that $f^{-mnt}(z_j)$ converges to $f^{mj}(q)$ as $t \to \infty$. Clearly, $f^{mj}(q) \in f^{mj}(V)$. Fix $0 \leq j \leq n-1$. Then there exits $N_j > 0$ such that for every $t > N_j$, $f^{-mnt}(z_j) \in f^{mj}(V)$. Let $N = \max\{N_j \mid 0 \leq j \leq n-1\}$. Then for every $t \geq N$, we have $t = sn + j$ for some $s > 0$ and $0 \leq j \leq n-1$. For $s \geq N$, we have $f^{-mt}(z_j) = f^{-msn-mj}(z_j) \in V$. Thus $f^{mt}(U) \cap V \neq \phi$, for $t \geq nN$ proving that $f^m_{|S}$ is topologically mixing. □

References

[1] F. Abdenur and S. Crovisier, *Transitivity and topological mixing for C^1 diffeomorphisms*, Essays in Mathematics and its Applications, 1–16, Springer, Heidelberg, 2012.

[2] E. Akin and J. Auslander, *Compactifications of dynamical systems*, arXiv:1004.0323 (2010).

[3] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems*, North-Holland Publishing Co., Amsterdam, 1994.

[4] M. Awartani and S. Elaydi, *An extension of chaotic dynamics to general topological spaces*, Panamer. Math. J. **10** (2000), no. 2, 61–71.

[5] A. M. Blokh, *The spectral decomposition for one-dimensional maps*, Dynamics reported, 1–59, Dynam. Report. Expositions Dynam. Systems (N.S.), **4**, Springer, Berlin, 1995.

[6] R. Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397.

[7] ______, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Math. **470**, Springer-Verlag, 1975.

[8] T. Ceccherini-Silberstein and M. Coornaert, *Sensitivity and Devaney's chaos in uniform spaces*, J. Dyn. Control Syst. **19** (2013), no. 3, 349–357.

[9] C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics **38**. American Mathematical Society, Providence, R.I., 1978.

[10] T. Das, K. Lee, D. Richeson, and J. Wiseman, *Spectral decomposition for topologically Anosov homeomorphisms on noncompact and non-metrizable spaces*, Topology Appl. **160** (2013), no. 1, 149–158.

[11] J. Kelley, *General Topology*, D. Van Nostrand Company Inc., Toronto, New York, London, 1955.

[12] C. Morales and V. Sirvent, *Expansivity for measures on uniform spaces*, Trans. Amer. Math. Soc. **368** (2016), no. 8, 5399–5414.

[13] S. Shah, R. Das, and T. Das, *Specification property for topological spaces*, J. Dyn. Control Syst. **22** (2016), no. 4, 615–622.

[14] ______, *A note on uniform entropy for maps having topological specification property*, Appl. Gen. Topol. **17** (2016), no. 2, 123–127.

[15] K. Sigmund, *On mixing measures for axiom A diffeomorphisms*, Proc. Amer. Math. Soc. **36** (1972), 497–504.

[16] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[17] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actual. Sci. Ind., Hermann et Cie., Paris 551, 1937.

[18] R.-S. Yang, *Topological Anosov maps of non-compact metric spaces*, Northeast. Math. J. **17** (2001), 120–126.

Ruchi Das
Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi
Delhi - 110007, India
Email address: rdasmsu@gmail.com

Tarun Das
Department of Mathematics
Faculty of Mathematical Sciences
University of Delhi
Delhi - 110007, India
Email address: tarukd@gmail.com

Sejal Shah
Department of Mathematics
Faculty of Science
The Maharaja Sayajirao University of Baroda
Vadodara - 390002, Gujarat, India
Email address: sks1010@gmail.com