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A CLASSIFICATION RESULT AND CONTACT STRUCTURES IN ORIENTED CYCLIC 3-ORBIFOLDS

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Abstract. We prove every oriented compact cyclic 3-orbifold has a contact structure. There is another proof in the web by Daniel Herr in his uploaded thesis which depends on open book decompositions, ours is independent of that. We define overtwisted contact structures, tight contact structures and Lutz twist on oriented compact cyclic 3-orbifolds. We show that every contact structure in an oriented compact cyclic 3-orbifold contactified by our method is homotopic to an overtwisted structure with the overtwisted disc intersecting the singular locus of the orbifolds. In course of proving the above results we prove a classification result for compact oriented cyclic-3 orbifolds which has not been seen by us in literature before.

1. Introduction

Contact structures are generally known on manifolds. They are maximally non-integrable co-dimension one distributions. Contact structures can also be defined on orbifolds by going to the level of charts and keeping the distributions invariant under local group actions. Though a lot of work has been carried out on contact manifolds little is known about contact orbifolds. In this paper we provide contact structures to compact oriented cyclic 3-orbifolds.

Cyclic orbifolds are orbifolds where the local group acting on an orbifold chart are cyclic and action on each chart in some atlas is orientation preserving. The compact 3-dimensional version of these orbifolds are topologically manifolds and we call them oriented if the manifold is orientable with an orientation. We prove Martinet-like theorem on these orbifolds.

The main idea of the proof is that since the singular locus of the orbifold is a link we can get a rotationally invariant contact structure transverse to the link by Eliashberg's extension theorem or using Martinet theorem for manifolds on the smooth 3-manifold (the unique smooth structure (up to diffeomorphism) it

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gets by virtue of it topologically being a 3-manifold). Then we prove a classification result for these orbifolds. We prove any such cyclic orbifold will be a quotient of cyclic local group actions on the 3-manifold which are global cyclic group actions along the neighborhoods of the singular link components. By virtue of this classification the rotationally-symmetric manifold contact structure near the link induces a contact structure on the orbifold structure. The above classification has not been seen by us in literature before.

Since the underlying topological space is a manifold and since the contact structure on the orbifold structure induces a contact structure on the manifold structure (in this case) we can generalize overtwisted structures, tight structures and Lutz-twists to these orbifolds. This opens many questions solved in contact manifold theory to these orbifolds.

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2. Contact structure

We give a short exposition of contact structures and contact topology for a detailed exposition the reader may consult [3].

Definition 2.1. A contact structure ζ in a $2n+1$ dimensional manifold is a $2n$ dimensional maximally non-integrable plane distribution. In dimension 3 it means around each point p if we take two locally defined linearly independent vector fields X_p and Y_p lying in ζ, then their lie bracket does not lie in ζ. We call the manifold M with a contact structure ζ a contact manifold (M, ζ).

Remark 2.1. When the distribution is co-orientable we can define a global form α such that $\alpha \wedge (d\alpha)^n$ is a no-where zero top form of the $2n+1$ dimensional manifold and the kernel of α is ζ. In particular, $\alpha \wedge (d\alpha)^n$ defines a volume form and hence an orientation. In dimension 3 a contact manifold is always orientable (See Remark 2.1.12 in [3]).

Definition 2.2. A knot in a 3-manifold M is an embedding of a circle in the manifold.

Definition 2.3. A knot K in a contact 3-manifold (M, ζ) is called Legendrian if it is tangent to the contact structure. It is called transverse if it is transverse to the contact structure.

Definition 2.4. A framing of a knot K in a oriented 3-manifold is a framing of its normal bundle.

Definition 2.5. A link in a 3-manifold M is a submanifold whose connected components are knots.

Definition 2.6. A contact framing of a Legendrian knot K where the first of the frame vectors, is tangent to the contact structure.

Definition 2.7. A surface framing of a Legendrian knot K with respect to a surface Σ whose boundary is K is a framing where first of the frame vectors is tangent to the surface.

Definition 2.8. A characteristic foliation of a surface Σ in a contact 3-manifold (M, ζ) is a singular foliation formed by the intersection of the tangent space of Σ with ζ.

Definition 2.9. An overtwisted disc D is a disc in a contact 3-manifold (M, ζ) with Legendrian boundary K such that the contact framing of K does not twist with respect to surface framing and the characteristic foliation has exactly one interior singular point.

Definition 2.10. A contact structure which has a overtwisted disc is called a overtwisted contact structure. It is called tight if it is not overtwisted.

Example 2.2. We give an example of an overtwisted contact structure ζ_{ot} in $\mathbb{R}^3$. It can be found also in [3] Section 4.5. We define the structure by the following equation in standard cylindrical coordinates

$$\cos(r)dz + r\sin(r)d\phi = 0. \tag{2.1}$$

Take the disc D= $\{r \,|\, r \leq \pi, z = 0\}$. The characteristic foliation is singular at 0 and along the boundary. If we perturb this disc near the boundary, the characteristic foliation loses its singularity at the boundary and the boundary becomes a leaf of the foliation making it an overtwisted disc of the structure ζ_{ot} and thus ζ_{ot} is an overtwisted structure. We call the perturbed or unperturbed disc D, Δ the standard overtwisted disc and ζ_{ot} standard overtwisted structure.

2.1. Lutz twist

We consider a oriented 3-manifold with a contact structure ζ and a transverse oriented knot then it is known in a tubular neighborhood of the knot ζ is the kernel of the form $dz + r^2 d\theta$ where z is along the knot direction and r and θ are coordinates along meridian direction of the tubular neighborhood which coincides with the normal bundle of the knot in this case. We say $\zeta^{'}$ is obtained from ζ with a Lutz twist if on $S^1 \times D^2$ the new contact structure ζ is defined by

$$\zeta' = \ker(h_1(r)dz + h_2(r)d\phi) \tag{2.2}$$

and ζ' coincides with ζ outside the solid torus. Conditions are

(1) $h_1(r) = -1, h_2(r) = -r^2$near $r = 0$,
(2) $h_1(r) = 1$ and $h_2(r) = r^2$ near $r = 1$,
(3) $(h_1(r), h_2(r))$ is never parallel to $({h_1}', {h_2}')$ for $r > 0$.

2.2. Full Lutz twist

Changing condition 1 and adding some more conditions we obtained by a full Lutz twist.

$$h_1(r) = 1, \tag{2.3}$$

$$h_2(r) = r^2 \tag{2.4}$$

for $r \in [0, \epsilon] \cup [1-\epsilon, 1]$. By Lemma 4.5.3 in [3], we have the following lemma.

Lemma 2.3. *A full Lutz twist does not change the homotopy class of ζ as a 2-plane field.*

It can be seen $h_2(r)$ is zero in two places (for details Fig. 4.11 [3]). Taking r_0 the smaller radius where $h_2(r) = 0$ consider the following embedding ϕ : ${D^2}_{r_0} \rightarrow S^1 \times D^2$. $(r, \phi) \rightarrow (z(r), r, \phi)$ where $z(r)$ is a smooth function with $z(r_0) = 0$ and $z(r) > 0$ for $0 \leq r < r_0$ and $z^{'}(r) = 0$ for only $r = 0$. It can be seen $\phi(\delta(D^2_{r_0}))$ is Legendrian and its contact framing and surface framing do not twist with respect to each other and with some more conditions and work we can show $\phi(D^2_{r_0})$ is an overtwisted disk (for more details Section 4.5 [3]).

Proposition 2.4. *We thus conclude that any contact structure is homotopic to an overtwisted contact structure obtained by a full Lutz twist.*

3. Orbifolds

We give a brief introduction to orbifolds following the treatment in [1].

Definition 3.1. Let X be a topological space, and fix $n \geq 0$.

1. An n-dimensional orbifold chart on X is given by a connected open subset $\tilde{U} \subset \mathbb{R}^n$, a finite group G of smooth diffeomorphisms of $\tilde{U}$, and a map $\phi : \tilde{U} \rightarrow$ X so that ϕ is G-invariant and induces a homeomorphism of $\tilde{U}/G$ onto an open subset $U \subset X$.
2. An embedding $\lambda(\tilde{U}, G, \phi) \rightarrow (\tilde{V}, H, \psi)$ between two such charts is a smooth embedding $\lambda : \tilde{U} \rightarrow \tilde{V}$ with $\psi\lambda = \phi$.
3. An orbifold atlas on X is a family of such charts $\mathcal{U} = (\tilde{U}, G, \phi)$, which cover X and satisfy the compatibility condition: given any two charts $(\tilde{U}, G, \phi)$ for $U = \phi(\tilde{U}) \subset X$ and $(\tilde{V}, H, \psi)$ for $V = \psi(\tilde{V}) \subset X$ and a point $x \in U \cap V$ there exists an open neighborhood $W \subset U \cap V$ of x and a chart $(\tilde{W}, K, \mu)$ for W such that there are embeddings $\lambda_1 : (\tilde{W}, K, \mu) \rightarrow (\tilde{U}, G, \phi)$ and $\lambda_2 : (\tilde{W}, K, \mu) \rightarrow (\tilde{V}, H, \phi)$. Every embedding of charts induces injective homomorphisms $\lambda : K \rightarrow G$.
4. An atlas $\mathcal{U}$ is said to refine another atlas Υ if for every chart in $\mathcal{U}$ there exists an embedding into some chart of Υ. Two orbifold atlases are said to be equivalent if they have a common refinement.

Definition 3.2. An effective orbifold X of dimension n is a paracompact Hausdorff space **X** equipped with an equivalence class $[\mathcal{U}]$ of n-dimensional orbifold atlases such that each local group acts effectively in local charts.

Throughout this paper we always assume that our orbifolds are effective.

Definition 3.3. If the finite group actions on all the charts are free, then $\mathbf{X}$ is locally Euclidean, hence a manifold. Points in an effective orbifold where there are non-trivial stabilizers form the *singular locus*.

Definition 3.4. Let $\mathbf{X} = (X, \mathcal{U})$ and $\mathbf{Y} = (Y, \Upsilon)$ be orbifolds. A map $f : X \to Y$ is said to be smooth if for any point $x \in X$ there are charts $(\tilde{U}, G, \phi)$ around x and $(\tilde{V}, G, \psi)$, around $f(x)$ with the property that f maps $U = \phi(\tilde{U})$ into $V = \psi(\tilde{U})$ and can be lifted to a smooth map $\tilde{f} : \tilde{U} \to \tilde{V}$ satisfying $\psi \tilde{f} = f \phi$.

Definition 3.5. The isotropy subgroup of a point $\tilde{x}$ in the chart $(\tilde{U}, G, \phi)$ is $G_{\tilde{x}} = \{g \in G \mid g\tilde{x} = \tilde{x}\}$. If $\tilde{y}$ is a point in the G-orbit of $\tilde{x}$, then $G_{\tilde{y}}$ is conjugate to $G_{\tilde{x}}$ in G. If G is Abelian, $G_{\tilde{x}} = G_{\tilde{y}}$ and we denote $G_{\tilde{x}}$ by G_x where $x = \phi(\tilde{x})$. G_x is called the isotropy group of x.

3.1. Tangent bundle of an orbifold

We describe a tangent bundle for an orbifold $\mathbf{X} = (X, \mathcal{U})$. Given a chart $(\tilde{U}, G, \phi)$ we consider a tangent bundle $\mathcal{T}\tilde{U}$. Since G acts smoothly on $\tilde{U}$, hence it acts smoothly on $\mathcal{T}\tilde{U}$. Indeed if $(\tilde{u}, v)$ a typical element of $\mathcal{T}\tilde{U}$, then $g(\tilde{u}, v) = (g\tilde{u}, dg_{\tilde{u}}(v))$ (here d stands for the differential). Moreover the projection map $\mathcal{T}\tilde{U} \to \tilde{U}$ is equivariant from which we get a projection $p : \mathcal{T}\tilde{U}/G \to U$ by using the map ϕ. If $x = \phi(\tilde{x})$,

$$p^{-1}(x) = \{G(\tilde{x}, v) \subset \mathcal{T}\tilde{U}/G\}. \tag{3.1}$$

It is evident that this fiber is homeomorphic to $\mathcal{T}_{\tilde{x}}\tilde{U}/G_{\tilde{x}}$ where as $G_{\tilde{x}}$ is the isotropy subgroup of the G action at $\tilde{x}$. This means we have constructed locally a bundle like object where the fiber is not necessarily a vector space, but rather a quotient of the form $\mathbb{R}^n/G_0$ where $G_0 \subset Gl_n(\mathbb{R})$. To construct the tangent bundle on an orbifold $\tilde{X} = (X, \mathcal{U})$, we simply need to glue together the bundles defined over the charts. Our resulting space will be an orbifold, with an atlas $\mathcal{TU}$ comprising local charts $(\mathcal{T}\tilde{U}, G, \phi)$ over $\mathcal{T}U = \mathcal{T}\tilde{U}/G$ for each $(\tilde{U}, G, \phi) \in \mathcal{U}$: We observe that the gluing maps $\lambda_{12} = \lambda_2 {\lambda_1}^{-1}$ are smooth, so we can use the transition functions $d\lambda_{12} : \mathcal{T}\lambda_1(\tilde{W}) \to \mathcal{T}\lambda_2(\tilde{W})$ to glue $\mathcal{T}\tilde{U}/G \to U$ to $\mathcal{T}\tilde{U}/H \to H$. In other words, we define the space $\mathcal{T}X$ as an identification space $\bigsqcup_{\tilde{U} \in v}(\mathcal{T}\tilde{U}/G)/\sim$ where we give it the minimal topology that will make the natural maps $\mathcal{T}\tilde{U}/G \to \mathcal{T}X$ homeomorphisms onto open subsets of $\mathcal{T}X$. We summarize this in the next remark.

Remark 3.1. The tangent bundle of an n-dimensional orbifold $\mathbf{X}$ denoted by $\mathcal{T}\mathbf{X} = (\mathcal{T}X, \mathcal{TU})$ has the structure of a $2n$-dimensional orbifold. Moreover, the natural projection $p : \mathcal{T}X \to X$ defines a smooth map of orbifolds, with fibers $p^{-1}(x) = \mathcal{T}_{\tilde{x}}\tilde{U}/G_{\tilde{x}}$.

3.2. Example: Teardrop orbifold

A teardrop orbifold is a topological sphere with an orbifold singularity at north pole. Consider the standard embedding of the sphere in $\mathbb{R}^3$ with $(0,0,1)$ as the north pole and $(0,0,-1)$ as the south pole. We give two orbifold charts. Let $U = \{(x,y,z) \in S^2 \mid z < \frac{1}{3}\}$ and $V = \{(x,y,z) \in S^2 \mid z > -\frac{1}{3}\}$ are two open sets around the south pole and north pole respectively. We define a chart $(\tilde{U}, G, \phi)$ where $\tilde{U} = U$, G is the trivial group and ϕ identity map. Around north pole we take the open set V and define an orbifold chart $(\tilde{V}, H, \psi)$ where $\tilde{V} = V$, $H = \mathbb{Z}_3$ and ψ is a triple cover branched at the north pole. Now if $z \in U \cap V$ we take neighborhood so small that the maps ϕ and ψ cover these neighborhood evenly. Let us call this neighborhood W_z and with the chart (W_z, I, id) and the maps λ_1 and λ_2 are mere inclusions. Thus we have an orbifold structure which is different from the manifold structure of the sphere and we call it a teardrop orbifold. From the above description it is fairly easy to construct its tangent bundle.

Definition 3.6. A metric in an orbifold $(X, \mathcal{U})$ with tangent bundle $(\mathcal{T}X, \mathcal{T}\mathcal{U})$ is a section of $\mathcal{T}^*X \otimes \mathcal{T}^*X$ which is symmetric and positive definite. Every effective orbifold with an orbifold structure as defined above can be given a metric by the partition of unity.

Remark 3.2. By taking exponential map in the chart we can have charts where the local groups are subgroup of the orthogonal group.

4. Cyclic orbifold

Definition 4.1. A cyclic orbifold is an effective orbifold where the local groups are cyclic. A locally orientable cyclic orbifold is a cyclic orbifold with an atlas where the local group actions are orientation preserving. From the above remark the local groups are finite cyclic subgroups of $SO(n)$.

Remark 4.1. All locally orientable cyclic 3-orbifolds are topologically manifolds. To see this at a singular point (points where there is a non trivial stabilizer) the action fixes an axis and rotates transverse 2-planes in charts. Since a 2-plane quotients to a topological manifold under such action the topological space is a manifold. Thus for the locally orientable cyclic 3-orbifold $\mathbf{X} = (X, \mathcal{U})$ the topological space X is a 3-manifold.

Definition 4.2. An oriented compact locally orientable cyclic 3-orbifold $\mathbf{X}$ is a locally orientable cyclic orbifold such that the underlying topological space X is oriented and compact.

Lemma 4.2. *The singular locus S of a locally orientable compact cyclic* 3-*orbifold* $\mathbf{X}$ *with the topological space X is a link.*

Proof. By local orientability condition around a singular point the action is of a cyclic subgroup of $SO(3)$. The fixed point set of the local action is an axis which is an one manifold. Since the singular locus is a closed set and the topological space X is compact the connected components are knots. □

Remark 4.3. The order of the local group along a connected component of the singular link does not change as we can see from above it is locally constant.

Remark 4.4. From now on by cyclic 3-orbifolds we mean locally orientable cyclic 3-orbifolds.

4.1. Example

Tear drop orbifold described in 3.2 is a cyclic orbifold. In three dimensions take a solid torus lying in a 3-manifold and replace it by a Z_n quotient where the group acts as rotation in meridian disc directions. The resulting space has a cyclic orbifold structure.

5. Contact orbifold

Definition 5.1. A contact orbifold is an orbifold where there is a local group invariant contact structure in each orbifold chart and which is invariant under embedding of charts.

6. Compact oriented cyclic orbifolds have contact structure

We prove in this section that:

Theorem 6.1. *Any compact oriented cyclic 3-orbifold has a contact structure.*

To prove this we are required to prove a classification theorem for oriented cyclic 3-orbifold.

Theorem 6.2. *Every oriented cyclic 3-orbifold can be reduced to a quotient of local actions which are global cyclic group actions along the neighborhoods of the singular link components and their actions are rotations along discs transverse to the link with respect to a suitable metric and coordinates compatible with respect to orbifold coordinates.*

Proof. Take a compact oriented cyclic orbifold $\mathbf{X}$ with a topological space X and singular link S. Take a point x on the singular link. Take an orbifold chart $\mathcal{U} = (\tilde{U_x}, G, \phi)$ where G is a finite cyclic subgroup of $SO(3)$ and $\phi((\tilde{U_x})$ is an open set in X around the point x. The G action fixes $\phi^{-1}(S)$ and preserves a 2-plane in tangent space of $\phi^{-1}(x)$ (which is a vector space and covers the orbifold tangent space of x which is not a vector space) transverse to the vector along the lift of the singular link. If x lies in the overlap of image of two charts $\mathcal{U} = (\tilde{U_x}, G, \phi)$ and $\mathcal{V} = (\tilde{V_x}, H, \psi)$ the tangent space of $\phi^{-1}(x)$ and $\psi^{-1}(x)$ are connected by $d(\lambda_2 {\lambda_1}^{-1})$ where $\lambda_1 : \tilde{W_x} \to \tilde{U_x}$ and $\lambda_2 : \tilde{W_x} \to \tilde{V_x}$ are embeddings of another chart $\tilde{W_x}$ around x to the above two charts and d means the differential (for definition of embedding of charts please refer to 3.1. It should also be noted that these embeddings induce isomorphisms of local groups since the order of each group involved is same as the order of singularity along the link component). It is obvious from the definition of embeddings that the transition function $d(\lambda_2{\lambda_1}^{-1})$ maps the direction along the singular link S in one chart to the direction of the singular link S in the other chart and the G invariant plane to the H invariant plane. If we take any other embedding λ_3 and λ_4 the two differentials $d(\lambda_2{\lambda_1}^{-1})$ and $d(\lambda_4{\lambda_3}^{-1})$ will differ by a composition of $2k\pi/n$ rotation (they are smooth hence differ by a group element and n is the order of this connected component of the link). So changing λ_4 suitably (i.e., multiplying by a $2k\pi/n$ rotation or an element of the local group) we get embeddings which give rise to transition functions of these link-transverse planes. Thus we can glue trivialization of 2-planes into a bundle B around the connected component of the singular link.

Remark 6.3. Since there exist charts around each point in the singular link component which is a replica of part of the tangent space of a point lying in the link component (since we are taking exponential maps to construct charts), embeddings are linear maps in some coordinates thus their differentials coincide with actual maps along the above link transverse planes. Thus the bundle transition functions are actually transition functions of the orbifold in the transverse plane direction near the link points.

Since around each point on the singular link of the total space of the above bundle B gets coordinates from a neighborhood chart we get coordinates of the total space around the point that are compatible with local charts of the orbifold. Now covering the connected component of the link by such neighborhoods in the total space of B by tube lemma like arguments we will get a tubular neighborhood of the connected component of the link such that the bundle coordinates are compatible with the coordinates of the orbifold $\mathbf{X}$. Now since the coordinates of the orbifold structure induce a smooth manifold structure on the complement of the link S by creating similar tubular neighborhoods around other components of the link we get a smooth manifold structure by gluing the tubular neighborhoods with the complement of the link by pasting points away from the link. (the gluing is possible because the chart branched cover and the covered space near the link component are homeomorphic around the points of the link, and diffeomorphic away from the link component (both homeomorphic to a D^2 (disc) bundle over S^1 because action of cyclic groups by rotation on meridian discs quotients to discs).)

Since the topological space resulting from gluing is homeomorphic to the X and since X is an oriented manifold the bundle B is an oriented bundle. Thus B can be trivialized. Take an element g of a local group and such that it acts as a $2\pi/n$ rotation in the respective chart and since

$$d(\lambda_2{\lambda_1}^{-1})dg = d(\eta(g))d(\lambda_2{\lambda_1}^{-1}) \tag{6.1}$$

(where η is a group isomorphism) and since the bundle B is orientable $d(\eta(g))$ acts as a $2\pi/n$ rotation on the link-transverse planes on the tangent bundle of the other chart. To see the above let v belong to a fiber in a trivialization of B corresponding to the first chart. Then $d(g)v$ and v form a $2\pi/n$ sector in the fiber. Then $d(\lambda_2{\lambda_1}^{-1})dgv$ and $d(\lambda_2{\lambda_1}^{-1})v$ will form a $2k\pi/n$ sector by the above equation 6.1. Since the transition map is one-one and $\eta(g)$ is group homomorphism we have $k = 1$. So $d(\eta(g))$ will be a $2\pi/n$ or a $-2\pi/n$ rotation, but since the bundle is orientable $\eta(g)$ is a $2\pi/n$ rotation in the other chart. So the action of the local groups can be made global on the tubular neighborhood around the connected component of the link. Giving a metric by gluing the standard metric in the charts in the tubular neighborhood, which is invariant under the action of this group we get ∂r and $\partial\theta$ coordinates where action of the group elements are just translations by constants in the theta direction. □

Now we give the proof of Theorem 6.1.

Proof of Theorem 6.1. We define a contact structure around the tubular neighborhood (derived in Theorem 6.2) given by $dz + r^2 d(\theta)$ where z direction is the link direction.

Now do the same thing for all components of the link. Taking a disjoint ball pull back on it the standard overtwisted structure with the standard overtwisted disc lying in it. We can extend these 2-plane fields (the local contact structure around link components and the disjoint ball) to global 2-plane fields and we can get a contact structure homotopic to the global plane field which extends the contact structure around the link on the whole manifold by using Eliashberg's extension theorem (see [2, Theorem 3.1.1]) by taking A as the link, L as the empty set and K as the one point set in the statement of the theorem. This structure is a contact structure in the orbifold coordinates (see Theorem 6.2) around the link (since the tubular neighborhood coordinates are compatible with orbifold coordinates) and is invariant under the action of the local groups which are global along these neighborhoods of the link component (this follows from the previous theorem). Since away from the link the manifold and orbifold coordinates are compatible we have a contact structure on the whole orbifold **X**.

Alternative argument: From the above discussions it is clear the orbifold X is topologically a manifold and has an unique smooth structure. For this smooth structure on X we get the existence of a contact structure by Martinet's theorem (see Theorem 4.1.1 in [3]). After an isotopy we can assume that the set S forms a transverse link, to which we can apply neighborhood theorem (see Example 2.5.16 in [3]) giving a desired contact form $dz + r^2 dq$. As this is rotationally invariant under our group action (due to the isotopy), we can now push this to the orbifold setting using the uniformization charts of the above proved theorem. □

Remark 6.4. The need for a neighborhood around the link component with compatible orbifold co-ordinates which extends to smooth manifold coordinate system is necessary because otherwise the structure $dz + r^2 d(\theta)$ would have lifted to a singular form in orbifold coordinates $(dz + r^{2/n} d(\theta/n))$, if we just go by definition without the above adjustment. This happens because the orbifold co-ordinates are branched cover over the transverse discs of actual manifold co-ordinates (the branched cover along the transverse discs can be best described by the map $w \to w^n$, hence the singular form).

Corollary 6.5. *The orbifold contact structure induces a contact structure in the manifold.*

Proof. Since the orbifold co-ordinates around the link components can be extended to a smooth manifold coordinate system the orbifold contact structure is a manifold contact structure transverse and rotationally invariant near the link components. □

Remark 6.6. The oriented condition is necessary for the space to have a contact structure since from the above corollary it is clear that an orbifold contact structure on these orbifolds induces a contact structure on the manifold structure thus making it an oriented manifold.

Remark 6.7. There are cyclic orbifolds whose topological space is non-oriented. For example take a non-oriented manifold. Take a small Euclidean open neighborhood and take a smaller solid torus. Remove the interior and replace it with a solid torus quotient of a Z_n action by rotation on meridian discs. This is a cyclic orbifold with a non-oriented manifold as a topological space.

7. Overtwisted and tight structures

Definition 7.1. An overtwisted contact structure in an oriented compact contact cyclic 3-orbifold is a contact structure which lifts to an overtwisted structure in the corresponding manifold structure. It is a tight structure if it is not overtwisted.

Definition 7.2. A Lutz twist in an oriented compact contact cyclic 3-orbifold for knots disjoint from the singular link S and knots lying in the singular link is a Lutz twist on the contact structure induced in the manifold. Since the twist parameters are rotationally invariant, on the orbifold contact structure given by our method the result of a Lutz twist is again another contact structure in orbifold level. In the same way we can define full Lutz twist.

Theorem 7.1. *Any contact structure induced by our method is homotopic to an overtwisted contact structure with the overtwisted disc intersecting the singular link S.*

Proof. Inspecting Lemma 4.5.3 in [3] the homotopy between a contact structure and the structure resulting from a full Lutz twist is rotation invariant. So around a knot in the singular link our contact structure can be changed to an overtwisted contact structure by a homotopy in the orbifold level. Since an overtwisted disc intersects the knot around which the twist is taken some overtwisted disc will intersect the knot in the singular link. □

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