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THE FIRST EIGENVALUE OF SOME (p,q)-LAPLACIAN AND GEOMETRIC ESTIMATES

Shahroud Azami

Abstract. We study the nonlinear eigenvalue problem for some of the (p,q)-Laplacian on compact manifolds with zero boundary condition. In particular, we obtain some geometric estimates for the first eigenvalue.

1. Introduction

Let M be a compact domain in a complete, simply connected, Riemannian manifold $\mathcal{M}$ of constant sectional curvature. Studying the eigenvalues of geometric operators is an important tool for Riemannian manifolds. There are many mathematicians who investigate properties of the spectrum of Laplacian and estimate the spectrum in terms of the other geometric quantities of M (see [4], [5], [6], [7], [9], [11], [12]). In [1], Abolarinwa obtained the classical estimates of Faber-Krahn inequality and Cheeger-type inequality for the first eigenvalue of p-Laplacian operator. In this paper we study the principal eigenvalue of some (p,q)-Laplacian on Riemannian manifold and reprove some of classical results introduced in above for the (p,q)-Laplacian of any $p>1,\ q>1$.

1.1. Eigenvalues of (p,q)-Laplacian

Let (M,g) be a compact domain in a complete, simply connected, Riemannian manifold $\mathcal{M}$ of constant sectional curvature and $f: M\longrightarrow \mathbb{R}$ be a smooth function on M or $f\in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of f for $1<p<\infty$ is defined as

$$\begin{aligned}(1)\qquad \triangle_p f &= div(|\nabla f|^{p-2}\nabla f)\\ &= |\nabla f|^{p-2}\Delta f+(p-2)|\nabla f|^{p-4}(Hessf)(\nabla f,\nabla f),\end{aligned}$$

where $f(x)\neq 0$ for $x\in M$, $f(x)=0$ on ∂M,

$$(Hessf)(X,Y)=\nabla(\nabla f)(X,Y)=Y.(X.f)-(\nabla_Y X).f,\quad X,Y\in\mathcal{X}(M)$$

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and in local coordinate, we have

$$(Hessf)(\partial_i,\partial_j)=\partial_i\partial_j f-\Gamma^k_{ij}\partial_k f.$$

When $p=2$, $\Delta=\Delta_2$ is the natural Laplace-Beltrami operator $\Delta f=div(grad\, f)$. In this paper, at the first, we consider the nonlinear system introduced in [8], that is

$$\begin{cases}\Delta_p u=-\lambda|u|^{\alpha}|v|^{\beta}v \ \text{ in } M\\ \Delta_q v=-\lambda|u|^{\alpha}|v|^{\beta}u \ \text{ in } M\\ (u,v)\in W^{1,p}(M)\times W^{1,q}(M),\end{cases}\tag{2}$$

where $p>1,\ q>1$ and α,β are real numbers satisfying

$$\alpha>0,\ \beta>0,\quad \frac{\alpha+1}{p}+\frac{\beta+1}{q}=1.\tag{3}$$

We say that λ is an eigenvalue of (2), whenever for some $u\in W_0^{1,p}(M)$ and $v\in W_0^{1,q}(M)$,

$$\int_M|\nabla u|^{p-2}\langle\nabla u,\nabla\phi\rangle d\mu=\lambda\int_M|u|^{\alpha}|v|^{\beta}v\phi d\mu,\tag{4}$$

$$\int_M|\nabla v|^{q-2}\langle\nabla v,\nabla\psi\rangle d\mu=\lambda\int_M|u|^{\alpha}|v|^{\beta}u\psi d\mu,\tag{5}$$

where $\phi\in W^{1,p}(M)$, $\psi\in W^{1,q}(M)$ and $W_0^{1,p}(M)$ is the closure of $C_0^\infty(M)$ in Sobolev space $W^{1,p}(M)$. The pair (u,v) is called eigenfunctions. A first positive eigenvalue $\lambda_{1,p,q}$ of (2) obtained as

$$\inf\{A(u,v):(u,v)\in W_0^{1,p}(M)\times W_0^{1,q}(M),\ B(u,v)=1\},$$

where

$$A(u,v)=\frac{\alpha+1}{p}\int_M|\nabla u|^p d\mu+\frac{\beta+1}{q}\int_M|\nabla v|^q d\mu,$$
$$B(u,v)=\int_M|u|^{\alpha}|v|^{\beta}uvd\mu.$$

In what follows, we use some identities arising from the co-area formula ([3], [10]). Suppose that $M(t)=\{x\in M: f(x)>t\}$ and $N(t)=\{x\in M: f(x)=t\}$, where t is a regular value of $|f|$. We denote by $dN(t)$ the Riemannaian measure of the set $N(t)$. For any $0<f\in C_0^\infty(M)$ and $1\le p\le\infty$, we have

$$\int_M|\nabla f|^p d\mu=\int_0^\infty(\int_{N(t)}|\nabla f|^{p-1}dN(t))dt$$

and

$$\mathrm{Vol}(M(t))=\int_{M(t)}d\mu=\int_0^\infty(\int_{N(t)}\frac{1}{|\nabla f|}dN(t))dt,$$
$$\mathrm{Vol}(\partial M(t))=\int_{N(t)}|\nabla f|d\mu=\int_0^\infty dN(t)dt.$$

Theorem 1.1. *Let M be a domain in a complete, simply connected Riemannian manifold $\mathcal{M}$ of constant sectional curvature. Let $B_\eta(x)$ be a geodesic ball of radius η, centred $x \in M$ in $\mathcal{M}$, such that* $\mathrm{Vol}(M) = \mathrm{Vol}(B_\eta(x))$. *Then the following estimate holds for* (2),

$$\lambda_{1,p,q}(M) \geq \lambda_{1,p,q}(B_\eta(x)).$$

The equality holds if and only if $M = B_\eta(x)$.

Proof. Let $v > 0$, $u > 0$ be the pair of eigenfunctions associated with $\lambda_{1,p,q}(M)$. Using a symmetrization procedure, we define a radially decreasing function $u^* : B_\eta \to \mathbb{R}^+$. Let $M_t = \{x \in M : u(x) > t\}$, $N_t = \{x \in M : u(x) = t\}$ and $N_t^* = \{x \in B_\eta : u^*(x) = t\}$ such that $\mathrm{Vol}(M_t) = \mathrm{Vol}(B_\eta)$ for each t. Let dN_t and dN_t^* are the $(n-1)$-dimensional volume elements of sets N_t and N_t^* respectively. By Hölder inequality we have

$$\int_{N_t} |\nabla u|^{p-1} dN_t \geq \frac{\left(\int_{N_t} dN_t\right)^p}{\left(\int_{N_t} |\nabla u|^{-1} dN_t\right)^{p-1}}. \tag{6}$$

Since $\mathrm{Vol}(M_t) = \mathrm{Vol}(B_\eta)$ and by

$$\frac{d}{dt}\mathrm{Vol}(M_t) = -\int_{N_t} \frac{1}{|\nabla u|} dN_t, \quad \frac{d}{dt}\mathrm{Vol}(B_\eta) = -\int_{N_t^*} \frac{1}{|\nabla u^*|} dN_t^*,$$

we get

$$\left(\int_{N_t} \frac{1}{|\nabla u|} dN_t\right)^k = \left(\int_{N_t^*} \frac{1}{|\nabla u^*|} dN_t^*\right)^k$$

for any real number k. Now, by the co-area formula for gradient

$$\begin{aligned}
\int_M |\nabla u|^p d\mu &= \int_0^\infty \left(\int_{N_t} |\nabla u|^{p-1} dN_t\right) dt \geq \int_0^\infty \frac{\left(\int_{N_t} dN_t\right)^p}{\left(\int_{N_t} |\nabla u|^{-1} dN_t\right)^{p-1}} dt \\
&\geq \int_0^\infty \frac{\left(\int_{N_t^*} dN_t{}^*\right)^p}{\left(\int_{N_t^*} |\nabla u^*|^{-1} dN_t^*\right)^{p-1}} dt = \int_0^\infty \left(\int_{N_t^*} |\nabla u^*|^{p-1} dN_t^*\right) dt \\
&= \int_{B_\eta} |\nabla u^*|^p d\mu.
\end{aligned}$$

Similarly, $\int_M |\nabla v|^q d\mu \geq \int_{B_\eta} |\nabla v^*|^q d\mu$, where v^* defined as u^*. Therefore

$$\begin{aligned}
\lambda_{1,p,q}(M) &= \frac{\alpha+1}{p}\int_M |\nabla u|^p d\mu + \frac{\beta+1}{q}\int_M |\nabla v|^q d\mu \\
&\geq \frac{\alpha+1}{p}\int_{B_\eta} |\nabla u^*|^p d\mu + \frac{\beta+1}{q}\int_{B_\eta} |\nabla v^*|^q d\mu \\
&\geq \lambda_{1,p,q}(B_\eta).
\end{aligned}$$

If $M = B_\eta(x)$ is the geodesic ball of radius η in $\mathcal{M}$, then equality is hold and conversely if the equality is hold, then $\int_M |\nabla u|^p d\mu = \int_{B_\eta} |\nabla u^*|^p d\mu$ which implies that $M = B_\eta(x)$. $\square$

We denote the Cheeger's isoperimetric constant by

$$\mathcal{C}(M) = \inf_{M_c} \frac{\mathrm{Vol}(\partial M_c)}{\mathrm{Vol}(M_c)}, \quad M_c \subset\subset M,$$

where the infimum is taken all over manifold M_c with compact closure in M and ∂M_c is assumed to be smooth.

Theorem 1.2. *Let M be a compact manifold with smooth boundary in a complete Riemannain manifold. Then for the first eigenvalue of* (2) *we have*

$$\lambda_{1,p,q}(M) \geq \frac{\alpha+1}{p}(\frac{\mathcal{C}(M)}{p})^p \int_M |u|^p d\mu + \frac{\beta+1}{q}(\frac{\mathcal{C}(M)}{q})^q \int_M |v|^q d\mu,$$

where (u, v) be the pair of positive eigenfunctions corresponding to $\lambda_{1,p,q}(M)$.

Proof. Let (u, v) be the pair of eigenfunctions corresponding to $\lambda_{1,p,q}(M)$ where $u > 0,\ v > 0$. By applying co-area formula for $\varphi \in C_0^\infty(M)$ we have

$$\begin{aligned}
\int_M |\nabla\varphi| d\mu &= \int_{-\infty}^{\infty} (\int_{N_t} dN_t) dt \\
&= \int_{-\infty}^{\infty} \mathrm{Vol}(\partial M_t) dt \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{Vol}(\partial M_t)}{\mathrm{Vol}(M_t)} \mathrm{Vol}(M_t) dt \\
&\geq \inf_t \left(\frac{\mathrm{Vol}(\partial M_t)}{\mathrm{Vol}(M_t)}\right) \int_{-\infty}^{\infty} \mathrm{Vol}(M_t) dt \\
&= \inf_t \left(\frac{\mathrm{Vol}(\partial M_t)}{\mathrm{Vol}(M_t)}\right) \int_M \varphi d\mu = \mathcal{C}(M) \int_M \varphi d\mu.
\end{aligned}$$

Suppose $\varphi = u^p$, by using the Hölder inequality

$$\begin{aligned}
\mathcal{C}(M) \int_M |u|^p d\mu \leq \int_M |\nabla u^p| d\mu &= p \int_M |u^{p-1} \nabla u| d\mu \\
&\leq p \left(\int_M |u|^p d\mu\right)^{\frac{p-1}{p}} \left(\int_M |\nabla u|^p d\mu\right)^{\frac{1}{p}}
\end{aligned}$$

by this we get

$$\int_M |\nabla u|^p d\mu \geq (\frac{\mathcal{C}(M)}{p})^p \int_M |u|^p d\mu.$$

Hence

$$\lambda_{1,p,q}(M) \geq \frac{\alpha+1}{p}(\frac{\mathcal{C}(M)}{p})^p \int_M |u|^p d\mu + \frac{\beta+1}{q}(\frac{\mathcal{C}(M)}{q})^q \int_M |v|^q d\mu. \quad \square$$

Another class of (p,q)-Laplacian is defined as follows

(7) $\Delta_p u+\Delta_q u=div\left((|\nabla u|^{p-2}+|\nabla u|^{q-2})\nabla u\right),\ u\in W=W_0^{1,p}(M)\cap W_0^{1,q}(M),$

where $1<q<p<\infty$ and introduced in [2]. We say that $\lambda\in\mathbb{R}$ is an eigenvalue of (7) if there exists $u\in W,\ u\neq 0$ such that

$$-\Delta_p u-\Delta_q u=\lambda|u|^{p-2}u \tag{8}$$

or equivalently

$$\int_M |\nabla u|^{p-2}\nabla u.\nabla v d\mu+\int_M |\nabla u|^{q-2}\nabla u.\nabla v d\mu=\lambda\int_M |u|^{p-2}uv d\mu \tag{9}$$

for any $v\in W^{1,p}(M)\cap W^{1,q}(M)$. The first positive eigenvalue $\lambda_{1,p,q}(M)$ of (7) is obtained as

$$\lambda_{1,p,q}(M)=\inf\left\{\int_M |\nabla u|^p d\mu+\int_M |\nabla u|^q d\mu : \int_M |u|^p d\mu=1\right\}. \tag{10}$$

Similar to the proof of Theorem 1.1 we have:

Corollary 1.3. *Let M be a domain in a complete, simply connected Riemannian manifold $\mathcal{M}$ of constant sectional curvature and $B_\eta(x)$ be a geodesic ball of radius η in $\mathcal{M}$ such that* $\mathrm{Vol}(M)=\mathrm{Vol}(B_\eta(x))$. *Then the following estimate holds for* (10),

$$\lambda_{1,p,q}(M)\geq\lambda_{1,p,q}(B_\eta(x)).$$

The equality holds if and only if $M=B_\eta(x)$.

Theorem 1.4. *Let M be a compact manifold with smooth boundary in a complete Riemannain manifold. Then for* (10) *we have*

$$\lambda_{1,p,q}(M)\geq\left(\frac{\mathcal{C}(M)}{p}\right)^p+\left(\frac{\mathcal{C}(M)}{q}\right)^q, \tag{11}$$

where (u,v) be the pair of positive eigenfunctions corresponding to $\lambda_{1,p,q}(M)$.

Proof. Let $u>0$ be the eigenfunctions corresponding to $\lambda_{1,p,q}(M)$. From the proof of Theorem 1.2 we have

$$\int_M |\nabla u|^p d\mu\geq\left(\frac{\mathcal{C}(M)}{p}\right)^p\int_M |u|^p d\mu=\left(\frac{\mathcal{C}(M)}{p}\right)^p,$$

and

$$\int_M |\nabla u|^q d\mu\geq\left(\frac{\mathcal{C}(M)}{q}\right)^q\int_M |u|^q d\mu\geq\left(\frac{\mathcal{C}(M)}{q}\right)^q.$$

These complete the proof. □

Theorem 1.5. *Let $\mathcal{M}$ be an n dimensional complete simply connected Riemannian manifold of negative constant sectional curvature $-k$. Let M be a domain in $\mathcal{M}$. Then the follow inequality holds for* (10)

$$\lambda_{1,p,q}(M)\geq\left(\frac{(n-1)\sqrt{-k}}{p}\right)^p+\left(\frac{(n-1)\sqrt{-k}}{q}\right)^q.$$

Proof. Let $B_\eta(x)$ be a geodesic ball of radius η in $\mathcal{M}$ where $x \in M$ and $\mathrm{Vol}(M) = \mathrm{Vol}(B_\eta(x))$. For $y \notin B_\eta(x)$ we define $\rho(y) = d(x, y)$. $\rho(y)$ is differentiable with $||\nabla\rho|| = 1$, because of $\mathcal{M}$ is simply connected with negative sectional curvature. By the Laplacian comparison theorem $\Delta\rho \geq (n-1)\sqrt{-k}$. If ν is the outer unit normal along the boundary of $B_\eta(x)$, then $\langle\nabla\rho, \nu\rangle \leq 1$. Therefore

$$\begin{aligned}\mathrm{Vol}(\partial B_\eta(x)) &= \int_{\partial B_\eta} dA \\ &\geq \int_{\partial B_\eta} \langle\nabla\rho, \nu\rangle dA = \int_{B_\eta} \Delta\rho d\mu \\ &\geq (n-1)\sqrt{-k}\int_{B_\eta} dA = (n-1)\sqrt{-k}\mathrm{Vol}(B_\eta),\end{aligned}$$

which implies that

$$(n-1)\sqrt{-k} \leq \mathcal{C}(B_\eta(x)) = \frac{\mathrm{Vol}(\partial B_\eta(x))}{\mathrm{Vol}(B_\eta)}$$

thus (11) results that

$$\lambda_{1,p,q}(B_\eta(x)) \geq (\frac{(n-1)\sqrt{-k}}{p})^p + (\frac{(n-1)\sqrt{-k}}{q})^q.$$

Now, by Corollary 1.3, we get

$$\lambda_{1,p,q}(M) \geq (\frac{(n-1)\sqrt{-k}}{p})^p + (\frac{(n-1)\sqrt{-k}}{q})^q.$$

$\square$

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Shahroud Azami
Department of Mathematics
Faculty of Sciences
Imam Khomeini International University
Qazvin, Iran
Email address: azami@sci.ikiu.ac.ir