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# A NOTE ON LIGHTLIKE HYPERSURFACES OF A GRW SPACE-TIME

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Abstract. This note provides a study of lightlike hypersurfaces of a generalized Robertson-Walker(GRW) space-time with a certain screen distribution, which are integrable and have good properties. Focus is to investigate geometric features from the relation of the second fundamental forms between lightlike hypersurfaces and leaves of the integrable screen distribution. Also, we shall apply those results on lightlike hypersurfaces of a GRW space-time to lightlike hypersurfaces of a Robertson-Walker(RW) space-time.

## 1. Introduction

A generalized Robertson-Walker(GRW) space-time is a Lorentzian warped product of an open interval $I$ and a riemannian manifold $(F, g_F)$ with Lorentzian metric $\bar{g} = -dt^2 + f^2(t)g_F$, which is denoted by $\bar{M} := (I \times_f F, \bar{g})$, where $f$ is a differentiable positive function defined on $I$. When $F$ has constant curvature $c$, $\bar{M}$ is called a Robertson-Walker(RW) space-time, which will be denoted by $\bar{M}(c, f)$. In this article we will give a study of lightlike hypersurfaces with a specific screen distribution of a GRW space-time. There are difficulties to study lightlike geometries since the induced metrics are degenerate. This leads to the fact that the tangent bundle $TM$ of a lightlike hypersurface $M$ and its normal bundle $TM^\perp$ intersect each other, which generates a distribution called radical distribution along $M$. Moreover the screen distribution is not unique and the lightlike geometry depends on its choice. Fortunately, the second fundamental form is independent of the choice of screen distributions and so the results on totally geodesic and totally umbilical lightlike hypersurfaces. It is, therefore, worthy to look for a screen distribution with good properties. Recently some authors dealt with the construction of relevant screen distributions and acquired useful and desirable results ([5, 6]).

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In this point of view we construct an integrable screen distribution, which leads the first coordinate vector field $\partial_t$ on a GRW space-time to lie in a hyperbolic plane spanned by lightlike vector fields $\xi$ and $N$ (see (4.2)). As a result we can obtain a relation between the second fundamental forms of lightlike hypersurfaces of a GRW space-time and leaves of the screen distribution. The relation plays an important role throughout this paper.

## 2. A brief review on GRW space-times

Consider an $(n+2)$-dimensional GRW space-time

$$\bar{M} := (I \times_f F, \bar{g}), \bar{g} = -dt^2 + f^2(t) g_F,$$

where $f$ is a smooth positive function on $I$, and $(F, g_F)$ is an $(n+1)$-dimensional riemannian manifold with riemannian metric $g_F$.

Let $\pi$ and $\sigma$ be the natural projections of $I \times F$ onto $I$ and $F$, respectively. Let $\mathfrak{L}(I)$ and $\mathfrak{L}(F)$ be the set of horizontal and vertical lifts of vector fields on $I$ and $F$ to $I \times_f F$, respectively. Let $\partial_t \in \mathfrak{L}(I)$ denote the horizontal lift vector field to $I \times_f F$ of the standard vector field $\frac{d}{dt}$ on $I$.

For each vector $X$ tangent to $\bar{M}$ we put

$$X = \phi_X \partial_t + \hat{X}, \tag{2.1}$$

where $\phi_X = -\bar{g}(X, \partial_t)$ and $\hat{X}$ is the vertical component of $X$.

The following lemma is well-known ([10]).

**Lemma 2.1.** *Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$. For vectors fields $X, Y$ on $\mathfrak{L}(F)$ we have*
(i) $\bar{\nabla}_{\partial_t} \partial_t = 0$,
(ii) $\bar{\nabla}_{\partial_t} X = \bar{\nabla}_X \partial_t = (\ln f)' X$,
(iii) $\bar{g}(\bar{\nabla}_X Y, \partial_t) = -\bar{g}(X, Y)(\ln f)'$.

Using this lemma and (2.1), we obtain for any vector field $X$ on $\bar{M}$

$$\bar{\nabla}_X \zeta = f' X, \tag{2.2}$$

where $\zeta = f \partial_t$. In particular $\mathcal{L}_\zeta g = 2f' g$, where $\mathcal{L}_\zeta$ is the Lie derivative along $\zeta$. Hence $\zeta$ is a conformal Killing vector field.

When $F$ has constant curvature $c$, we denoted the RW space-time by $\bar{M}(c, f)$ and $S_{t_0} := \pi^{-1}(t_0)$ is called be a *spacelike slice* which is a riemannian hypersurface of constant curvature in $\bar{M}(c, f)$ with metric $f^2(t_0) g_c$. The standard choices for $F$ are $S^n$, $E^n$ and $H^n$, with curvature 1, 0, −1, respectively. The curvature tensor $\bar{R}$ of $\bar{M}(c, f)$ is given as follows ([7]):

**Lemma 2.2.** *For any vector fields $X, Y, Z$ on $\bar{M}(c, f)$*

$$\begin{aligned} \bar{R}(X,Y)Z &= \alpha\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \\ &\quad + \beta\{\phi_X \phi_Z Y - \phi_Y \phi_Z X + (\phi_X \bar{g}(Y,Z) - \phi_Y \bar{g}(X,Z))\partial_t\}, \end{aligned} \tag{2.3}$$

*where* $\alpha = \frac{f'^2 + c}{f^2}$, $\beta = \frac{f f'' - (f'^2 + c)}{f^2}$.

*Remark* 2.3. (i) $\bar{M}(c,f)$ is of constant curvature if and only if $\beta = 0$,
(ii) $\bar{M}(c,f)$ is flat if and only if $f(t) = at + b(c = -a^2)$,
(iii) $\bar{M}(c,f)$ has constant curvature $k^2 > 0$ if and only if $f(t) = ae^{kt} + be^{-kt}, c = 4k^2ab$,
(iv) $\bar{M}(c,f)$ has constant curvature $-k^2 < 0$ if and only if $f(t) = a\sin(kt) + b\cos(kt)$, $c = -4k^2(a^2+b^2)$,

where (i) follows from Lemma 2.2 and (ii) $\sim$ (iv) follow from solutions of the differential equation $\beta = 0$.

## 3. Basics on lightlike hypersurfaces

In this section, we review some results from the general theory of lightlike hypersurfaces of a semi-riemannian manifold (cf. [4, 7]).

Let $(M, g)$ be an $(n+1)$-dimensional lightlike hypersurface of an $(n+2)$-dimensional semi-riemannian manifold $(\bar{M}, \bar{g})$ with constant index $q(1 \le q \le n+1)$. Then the so called *radical distribution* $Rad(TM) = TM \cap TM^{\perp}$ is of rank one (therefore $Rad(TM) = TM^{\perp}$), and the induced metric $g$ on $M$ is degenerate and has constant rank $n$, where $TM^{\perp}$ denotes the normal bundle over $M$. Also, a complementary vector bundle of $Rad(TM)$ in $TM$ is a non-degenerate distribution of rank $n$(called a *screen distribution*) over $M$, denoted by $S(TM)$. Thus we have the orthogonal direct sum

$$TM = S(TM) \perp Rad(TM). \tag{3.1}$$

Let $tr(TM)$ be a complementary (but not orthogonal) vector bundle (called a *transversal vector bundle*) to $TM$ in $T\bar{M} \mid M$. It is known that for any non-zero section $\xi \in \Gamma(TM^{\perp})$ on a coordinate neighborhood $\mathcal{U} \subset M$ there exists a unique null section $N$ of the transversal vector bundle $tr(TM)$ on $\mathcal{U}$ such that

$$\bar{g}(N,\xi) = 1, \bar{g}(N,N) = \bar{g}(N,X) = 0, \quad \forall X \in \Gamma(S(TM) \mid_{\mathcal{U}}), \tag{3.2}$$

where $\Gamma(\bullet)$ denotes the module of smooth sections of the vector bundle $\bullet$. Thus we have the decomposition.

$$T\bar{M} = S(TM) \perp (TM^{\perp} \oplus tr(TM)) = TM \oplus tr(TM). \tag{3.3}$$

Throughout the paper the manifolds we consider are supposed to be para-compact, smooth and connected. Now let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and $P$ be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$. According to the decomposition (3.3) and (3.1), we write the local Gauss and Weingarten formulas as follows. For any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) = \nabla_X Y + B(X,Y)N, B(X,Y) := \bar{g}(h(X,Y),\xi), \tag{3.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^t_X N = -A_N X + \tau(X)N, \ \tau(X) := \bar{g}(\nabla^t_X N, \xi), \tag{3.5}$$

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY) = \nabla^*_X PY + C(X,PY)\xi, \tag{3.6}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{3.7}$$

where $h$ and $h^*$ are the second fundamental forms of $M$ and $S(TM)$, $B$ and $C$ are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, $\nabla^*$ is a metric connection on $\Gamma(S(TM))$, $A^*_\xi$ the local shape operator on $\Gamma(S(TM))$ and $\tau$ is a 1-form on $TM$. Note that the local second fundamental form $B$ is independent of the choice of screen distribution $S(TM)$. The two local second fundamental forms of $M$ and $S(TM)$ are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{3.8}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{3.9}$$

Note that in general, $A_N$ is not symmetric with respect to $g$ and the local second fundamental form $B$ satisfies

$$B(X,\xi) = 0, \quad \forall X \in \Gamma(TM). \tag{3.10}$$

A lightlike hypersurface $M$ is *totally umbilical* in $\bar{M}$ if there is a smooth function $\lambda$ such that

$$B(X,Y) = \lambda g(X,Y),\ \forall X, Y \in \Gamma(TM).$$

A screen distribution $S(TM)$ is *totally umbilical* in $M$ if there exists a smooth function $\mu$ such that

$$C(X,PY) = \mu g(X,PY),\ \forall X, Y \in \Gamma(TM).$$

In case $\mu = 0$ (resp. $\lambda = 0$) we say that $S(TM)$ (resp. $M$) is *totally geodesic* in $M$ (resp. $\bar{M}$) ([3, 4, 7, 9]). The results on totally geodesic and totally umbilical lightlike hypersurfaces do not depend on the screen distribution of $M$.

From (3.8) and (3.10), we see that $A^*_\xi \xi = 0$. Hence in terms of the shape operators, we may state as follows (cf. [3]);

*Remark* 3.1. (i) $M$ is totally umbilical if and only if there exists a function $\lambda$ such that $A^*_\xi X = \lambda X$ for every $X \in \Gamma(S(TM))$,

(ii) $S(TM)$ is totally umbilical in $M$ if and only if there is a function $\mu$ such that $A_N X = \mu X$ for every $X \in \Gamma(S(TM))$ and $A_N \xi = 0$.

Furthermore, the induced linear connection $\nabla$ is not a metric connection. Indeed we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y) \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$, where $\eta$ is a differential 1-form locally defined on $M$ by

$$\eta(X) = \bar{g}(X,N), \quad \forall X \in \Gamma(TM).$$

Denote by $\bar{R}$ and $R$ the curvature tensor of $\bar{\nabla}$ and $\nabla$, respectively. Then the Gauss and Weingarten formulas in (3.4) and (3.5) imply

$$\begin{aligned} \bar{R}(X,Y)Z &\\ = R(X,Y)Z &+ B(X,Z)A_N Y - B(Y,Z))A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y)\}N \end{aligned} \tag{3.12}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$, where we set

$$(\nabla_X B)(Y,Z) = XB(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

Also we obtain from (3.6) and (3.7)

(3.13)

$$\begin{aligned} &R(X,Y)PZ \\ &= R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ &\quad + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X)\}\xi \end{aligned}$$

where we set

(3.14) $$(\nabla_X C)(Y,PZ) = XC(Y,PZ) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ),$$

and $R^*$ denotes the curvature tensor with respect to the metric connection $\nabla^*$ on $\Gamma(S(TM))$.

From (3.12) we get

(3.15) $$\bar{g}(\bar{R}(X,Y)Z, N) = g(R(X,Y)Z, N).$$

By using (3.7) and calculating the right hand side of (3.15) we obtain

(3.16) $$\bar{g}(\bar{R}(X,Y)\xi, N) = C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y).$$

## 4. Lightlike hypersurfaces with a specific screen distribution

**A**. Let $M$ be an $(n+1)$-dimensional lightlike hypersurface of an $(n+2)$-dimensional GRW space-time $\bar{M} = (I \times_f F, \bar{g})$. Note that ${\partial_t}^\perp$ is spacelike. We consider the screen distribution of rank $n$ over $M$ given by ${\partial_t}^\perp \cap TM$, which we denote it by $S(\partial_t)$. Then we can take the unique null vector field $\xi \in \Gamma(TM)$ such that

(4.1) $$\bar{g}(\partial_t, \xi) = 1$$

(for details see [6]). Moreover the lightlike transversal vector field to $S(\partial_t)$ is given by ([4])

(4.2) $$N = \partial_t + \frac{1}{2}\xi.$$

Then a pair $(\xi, N)$ satisfies (3.2).

Multiplying (4.2) by $f$ and differentiating in the direction $X \in \Gamma(TM)$, we have

(4.3) $$(Xf)N + f(-A_N X + \tau(X)N) = f'X + \frac{1}{2}\{(Xf) + f(-A^*_\xi X - \tau(X))\}\xi,$$

where we have used (2.2).

**Proposition 4.1.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then we have for any $X \in \Gamma(TM)$*

(i) $\tau(X) = (\ln f)'\eta(X)$,
(ii) $A_N X = -(\ln f)'PX + \frac{1}{2}A^*_\xi X$,
(iii) $\phi_X = -\eta(X)$.

*Proof.* Comparing the transversal part and radical part in (4.3) yields

$$Xf + f\tau(X) = 0,\ 2f'\eta(X) + Xf - f\tau(X) = 0,$$

respectively. From these equations we have

$$\tau(X) = (\ln f)'\eta(X),\ X \in \Gamma(TM). \tag{4.4}$$

Since $A_N X \in \Gamma(S(\partial_t))$ for any $X \in \Gamma(TM)$, (ii) follows from taking the screen distribution part in (4.3).

Substituting (4.2) into $\phi_X = -\bar{g}(X, \partial_t)$ gives

$$\phi_X = -\eta(X), \quad \forall X \in \Gamma(TM). \tag{4.5}$$

□

**Corollary 4.2.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then*
(i) $\phi_\xi = -1$ *and* $\phi_N = \frac{1}{2}$,
(ii) $A_N\xi = 0$,
(iii) $\phi_X = 0$, $X \in \Gamma(S(\partial_t))$, *i.e., the screen distribution $S(\partial_t)$ is tangent to spacelike slices of $\bar{M}$.*

*Proof.* From (4.5) we get $\phi_\xi = -\eta(\xi) = -1$. On the other hand, from (4.2) we have $\phi_N = 1 + \frac{1}{2}\phi_\xi$. Hence $\phi_N = \frac{1}{2}$. (ii) and (iii) follow from (ii) and (iii) in Proposition 4.1, respectively. □

**Theorem 4.3.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then*
(i) *The screen distribution $S(\partial_t)$ is integrable,*
(ii) *$S(\partial_t)$ is totally umbilical in $M$ if and only if $M$ is totally umbilical in $\bar{M}$.*

*Proof.* It follows from (ii) in Proposition 4.1 that the shape operator of $M$ is symmetric with respect to $g$, i.e., $g(A_NX, Y) = g(A_NY, X)$ for any $X, Y \in \Gamma(S(\partial_t))$. This is equivalent to the integrability of $S(\partial_t)$ (see [4]). The statement (ii) is also obtained from Proposition 4.1(ii) and Corollary 4.2(ii). □

**Theorem 4.4.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then*
(i) *$M$ is totally umbilical in $\bar{M}$ if and only if any leaf $M'$ of $S(\partial_t)$ is totally umbilical in a spacelike slice $S_t$ of $\bar{M}$,*
(ii) $A_E X = A^*_\xi X + A_N X,\ \ X \in \Gamma(S(\partial_t))$.

*Proof.* Let $E$ be a unit normal to $M'$ in $S_t$. Then from the decomposition (3.3) $E$ lies in span$\{\xi, N\}$. Noting that $\bar{g}(E, \partial_t) = 0$ and $\bar{g}(E, E) = 1$, we obtain

$$E = \frac{1}{2}\xi + N. \tag{4.6}$$

For any $X, Y \in \Gamma(S(\partial_t))$, define $h_1(X, Y) = \bar{g}(\bar{\nabla}_X Y, E)E$ and $h_2(X, Y) = -\bar{g}(\bar{\nabla}_X Y, \partial_t)\partial_t$ by the second fundamental forms of $M'$ in $S_t$ and $S_t$ in $\bar{M}$, respectively. Then for any $X, Y \in \Gamma(S(\partial_t))$

$$h^*(X, Y) + h(X, Y) = h_1(X, Y) + h_2(X, Y). \tag{4.7}$$

Using (3.8) and (3.9), comparing both sides with respect to the decomposition (3.3) we get

$$A_E X = 2A_N X + (\ln f)' X, \quad A_E X = A^*_\xi - (\ln f)' X, \quad X \in \Gamma(S(\partial_t)), \tag{4.8}$$

where we have used that $h_2(X, Y) = (\ln f)' \bar{g}(X, Y)\partial_t$, which is obtained from (2) in Lemma 2.1. (i) follows from the second equation. (ii) follows from adding two equations. □

**Theorem 4.5.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then $M$ is totally umbilical in $\bar{M}$ if and only if any leaf $M'$ of $S(\partial_t)$ is totally umbilical in $\bar{M}$ in such a way that $M'$ is immersed in $\bar{M}$ as a spacelike submanifold of codimension* 2.

*Proof.* Note that $M'$ lies in a spacelike slice and any spacelike slice is totally umbilcal in $\bar{M}$, i.e.,

$$A_{\partial_t} X = -(\ln f)' X, \ \forall X \in \Gamma(S(\partial_t)). \tag{4.9}$$

Let $\mathbf{n}$ be a unit normal to $M'$ in $\bar{M}$. Then $\mathbf{n} \in \mathrm{span}\{E, \partial_t\}$. Putting $\mathbf{n} = aE + b\partial_t$ for some functions on $M'$, we have

$$A_{\mathbf{n}} X = aA_E X + bA_{\partial t} X, \ \forall X \in \Gamma(S(\partial_t)).$$

Substituting (4.8) and (4.9) into this equation, we obtain

$$A_{\mathbf{n}} X = aA^*_\xi X - (a + b)(\ln f)' X,$$

which completes the proof. □

**Proposition 4.6.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. If $\eta$ is parallel, i.e., $\bar{\nabla}_X \eta = 0$ for any $X \in \Gamma(TM)$, then the warping function $f$ is constant.*

*Proof.* Differentiating $\eta(Y) = \bar{g}(Y, N)$ in the direction $X$ and our assumption yield

$$-\bar{g}(Y, A_N X) + \tau(X)\bar{g}(Y, N) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{4.10}$$

Putting $X = \xi = Y$ in this equation, we have $\tau(\xi) = 0$. But $\tau(\xi) = (\ln f)'$. Hence $f$ is constant. □

**Corollary 4.7.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. If $\eta$ is parallel, then $S(\partial_t)$ and $M$ are both totally geodesic in $M$ and $\bar{M}$, respectively.*

*Proof.* Since the warping function $f$ is constant, it follows from (i) in Proposition 4.1 that $\tau(X) = 0$ for any $X \in \Gamma(TM)$. Hence we obtain from (4.10) that $A_N X = 0$, which and (ii) in Proposition 4.1 imply that $A^*_\xi X = 0$ for any $X \in \Gamma(TM)$. Therefore we complete the proof. □

*Remark* 4.8. If the warping function $f$ is constant, then Proposition 4.1(ii) shows that $A_N X = \frac{1}{2} A^*_\xi X$, $X \in \Gamma(TM)$, i.e., $(M, S(\partial_t))$ is screen globally conformal with constant conformal weight $\frac{1}{2}$. In this case, $M$ is globally a product $\mathcal{C} \times M'$, where $\mathcal{C}$ is a null curve and $M'$ is a leaf of $S(\partial_t)$ ([1]).

**B**. If we take a vector field $\zeta = f\partial_t$ instead of $\partial_t$, then we can also take the unique null vector field $\bar{\xi} \in \Gamma(TM)$ such that

$$\bar{g}(\zeta, \bar{\xi}) = 1. \tag{4.11}$$

Moreover the lightlike transversal vector field denoted by $\bar{N}$ corresponding to $\bar{\xi}$ is given by

$$\bar{N} = \zeta + \frac{1}{2}f^2\bar{\xi}. \tag{4.12}$$

Differentiating (4.11) in the direction $X \in \Gamma(TM)$, we have

$$-A_{\bar{N}}X + \bar{\tau}(X)\bar{N} = f'X + f(Xf)\bar{\xi} - \frac{1}{2}f^2(A^*_{\bar{\xi}}X + \bar{\tau}(X)\xi), \tag{4.13}$$

where we have used (2.2) and $\bar{\tau}$ is a 1-form with respect to $\bar{\xi}$.

**Proposition 4.9.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then we have for the pair $(\bar{\xi}, \bar{N})$ and for any $X \in \Gamma(TM)$*
(i) $\bar{\tau}(X) = 0$,
(ii) $A_{\bar{N}}X = -f'PX + \frac{1}{2}f^2A^*_{\bar{\xi}}X$,
(iii) $f\phi_X = -\bar{\eta}(X)$.

*Proof.* The proof is similar to that of Proposition 4.1. □

**Corollary 4.10.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. Then for the pair $(\bar{\xi}, \bar{N})$ the Ricci tensor of the induced connection is symmetric.*

*Proof.* The proof follows from (i) in Proposition 4.8. □

*Remark* 4.11. (i) The Ricci tensor does not depend on the choice of the lightlike section $\xi$ ([2]).

(ii) We note that $B$ and $\tau$ depend on the section $\xi$, in our case $\bar{\xi} = \frac{1}{f}\xi$, it follows that $\bar{N} = fN$ and $\bar{B} = \frac{1}{f}B$ and $\tau(X) = \bar{\tau}(X) + (\ln f)'\eta(X)$.

**Proposition 4.12.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}$. For the pair $(\bar{\xi}, \bar{N})$ if $\bar{\eta}$ is parallel, then $S(\partial_t)$ is totally geodesic in $M$ and $M$ is totally umbilical in $\bar{M}$.*

*Proof.* The proof is similar to that of Corollary 4.7. □

## 5. Applications to lightlike hypersurfaces of a RW spacetime

In this section we shall investigate the structure equations of lightlike hypersurfaces with a specific screen distribution $S(\partial_t)$ of a Robertson-Walker space-time $\bar{M}(c, f)$ and apply those results acquired in the previous section to lightlike hypersurfaces of $\bar{M}(c, f)$.

Substituting (2.2) into (3.12), we have respectively

$$\begin{aligned} R(X,Y)Z &= \alpha(g(Y,Z)X - g(X,Z)Y) + B(Y,Z)A_NX - B(X,Z)A_NY \\ &\quad + \beta\{\phi_X\phi_ZY - \phi_Y\phi_ZX - \frac{1}{2}(\phi_Xg(Y,Z) - \phi_Yg(X,Z))\xi\}, \end{aligned} \tag{5.1}$$

$$\begin{aligned}(5.2)\qquad &\beta\{g(Y,Z)\phi_X - g(X,Z)\phi_Y\}\\ &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y)\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$.

Also from (3.13) and (5.1) we obtain respectively

$$\begin{aligned}(5.3)\qquad R^*(X,Y)PZ = {}&\alpha(g(Y,Z)PX - g(X,Z)PY) + B(Y,Z)A_N X\\ &- B(X,Z)A_N Y + C(Y,PZ)A^*_\xi X - C(X,PZ)A^*_\xi Y,\end{aligned}$$

$$\begin{aligned}(5.4)\qquad &(\alpha + \frac{\beta}{2})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\\ &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X)\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$, where we have used Proposition 4.1 (iii).

**Proposition 5.1.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$. If the second fundamental form $h$ of $M$ is parallel, i.e., $\nabla_X h = 0$ for any tangent vector field $X$ to $M$, then $\beta = 0$, i.e., $\bar{M}(c, f)$ is of constant curvature.*

*Proof.* Since $(\nabla_X h)(Y,Z) = 0$ for any $X, Y, Z \in \Gamma(TM)$ is equivalent to

$$(\nabla_X B)(Y,Z) + B(Y,Z)\tau(X) = 0,$$

it follows from (5.2) that

$$\beta\{g(Y,Z)\phi_X - g(X,Z)\phi_Y\} = 0.$$

This equation with $Y = Z = PZ \neq 0$ and $X = \xi$ gives $\beta = 0$. Hence Remark 2.3(i) shows that $\bar{M}(c, f)$ is of constant curvature. □

**Proposition 5.2.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$. If the screen second fundamental form $h^*$ is parallel, i.e., $\nabla_X h^* = 0$ for any tangent vector field $X$ to $M$, then $2\alpha + \beta = 0$.*

*Proof.* Assume that the screen second fundamental form $h^*$ is parallel, i.e., $(\nabla_X h)(Y,PZ) = 0$ for any $X, Y, Z \in \Gamma(TM)$, which is equivalent to

$$(\nabla_X C)(Y,PZ) - C(Y,PZ)\tau(X) = 0.$$

It follows from (5.4) that

$$(\alpha + \frac{\beta}{2})(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)) = 0,$$

from which putting $Y = Z = PZ \neq 0$ and $X = \xi$ gives $2\alpha + \beta = 0$. □

**Proposition 5.3.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$. Then each 1-form $\tau$ induced by $S(\partial_t)$ is closed, i.e., $d\tau = 0$.*

*Proof.* From (5.1) we have

$$\bar{g}(R(X,Y)\xi, N) = 0$$

with the aid of (iii) in Proposition 4.1. Also Proposition 4.1(ii) gives

$$C(X, A^*_\xi Y) = C(Y, A^*_\xi X).$$

Hence (3.16) with these results shows that $d\tau = 0$. □

**Proposition 5.4.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$. Then*

$$(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z), \quad \forall X, Y \in \Gamma(S(\partial_t)), \forall Z \in \Gamma(TM),$$
$$(\nabla_X C)(Y,Z) = (\nabla_Y C)(X,Z), \quad \forall X, Y, Z \in \Gamma(S(\partial_t)).$$

*Proof.* These are clear from (5.2) and (5.4), respectively. □

**Proposition 5.5.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$. Then $M$ is totally umbilical, then each leaf $M'$ ($\dim M' > 2$) of the distribution $S(\partial_t)$ as a submanifold of each slice $(S_t, f^2(t)g_c)$ is of constant curvature.*

*Proof.* The assumption and Theorem 4.4(i) implies that each leaf $M'$ of $S(\partial_t)$ is totally umbilical in a spacelike slice $S_t$. But $S_t$ is of constant curvature for each $t \in I$. Hence Schur's theorem shows $M'$ is of constant curvature. In fact, putting $B(X,Y) = \lambda g(X,Y)$ for some smooth function $\lambda$ and $\forall\, X, Y \in \Gamma(TM)$. Then from (5.3) and (ii) in Proposition 4.1 we obtain

$$R^*(X,Y)Z = C(t,\lambda)\{g(Y,Z)X - g(X,Z)Y\}, \quad \forall X, Y, Z \in \Gamma(S(\partial_t)),$$

where $C(t,\lambda) = \lambda^2 - 2(\ln f)'\lambda + \alpha$. $C(t,\lambda)$ is constant for a fixed $t$. □

**Theorem 5.6.** *Let $(M, g, S(\partial_t))$ be a lightlike hypersurface of $\bar{M}(c, f)$ with $\dim M > 2$. If $\bar{M}(c, f)$ is a space of constant curvature, i.e., $\beta = 0$ and $M$ is totally umbilical, then $M$ is Einstein.*

*Proof.* For any $X, Y \in \Gamma(TM)$ the Ricci tensor on $M$ is given by

$$Ric(X,Y) = \sum_{i=1}^{n} g(R(E_i, X)Y, E_i) + \bar{g}(\bar{R}(\xi, X)Y, N),$$

where $\{\xi, E_1, \ldots, E_n\}$ is an orthonormal frame field in $M$ adapted to $S(\partial_t)$, that is, $\{E_1, \ldots, E_n\}$ $(n > 1)$ is the orthonormal frame field of $S(\partial_t)$. Substituting (2.3) and (5.10) into the equation, we obtain

$$Ric(X,Y) = (n+2)\alpha g(X,Y) + B(X,Y) tr A_N - g(A^*_\xi Y, A_N X).$$

Putting $B(X,Y) = \lambda g(X,Y)$ in this equation, we have from Proposition 4.1(ii)

$$Ric(X,Y) = \Lambda g(X,Y),$$

where $\Lambda = (n+2)\alpha + (n-1)\lambda\mu$, $\mu = \frac{\lambda}{2} - (\ln f)'$. Thus we complete the proof. □

*Remark* 5.7. In [8], the authors define a new lightlike hypersurface of a Robertson-Walker space-time $\bar{M}(c, f)$ as follows:

A lightlike hypersurface $(M, g, S(TM))$ of $\bar{M}(c, f)$ is said to be *horizontally lightlike hypersurface* if $\partial_t$ lies in span$\{\xi, N\}$.

This note shows that it is possible for a lightlike hypersurface $M$ to be a horizontally lightlike hypersurface $\bar{M}(c, f)$ by choosing an appropriate screen distribution.

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