Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 293–303
https://doi.org/10.4134/CKMS.c170116
pISSN: 1225-1763 / eISSN: 2234-3024

ON SLANT CURVES IN S-MANIFOLDS

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Abstract. In this paper, we consider biharmonic slant curves in S-space forms. We obtain a main theorem, which gives us four different cases to find curvature conditions for these curves. We also give examples of slant curves in $\mathbb{R}^{2n+s}(-3s)$.

1. Introduction

J. Eells and L. Maire suggested k-harmonic maps in 1983 [6]. Following their idea, G. Y. Jiang obtained bitension field equation in 1986 [11]. On the other hand, in [4], Chen defined a biharmonic submanifold of Euclidean space as $\Delta H = 0$, where H is the mean curvature vector field and Δ is the Laplace operator. If the ambient space is Euclidean, then Jiang's and Chen's results coincide.

J. T. Cho, J. Inoguchi and J. E. Lee defined slant curves in Sasakian manifolds as a generalization of Legendre curves in 2006 [5]. In a 3-dimensional Sasakian manifold, they proved that a non-geodesic curve is slant if and only if the ratio of $(\tau \pm 1)$ and k is constant, where k and τ are the geodesic curvature and torsion of the curve, respectively. In their study, they also gave examples of a slant helix and a non-helix slant curve.

D. Fetcu studied biharmonic Legendre curves in Sasakian space forms in 2008 [8]. He proved the non-existence of such a curve in 7-dimensional 3-Sasakian manifold. In the same paper, he also obtained parametric equations for some biharmonic Legendre curves in 7-dimensional sphere. Furthermore, D. Fetcu and C. Oniciuc considered biharmonic submanifolds of Sasakian space forms in 2009 [9]. Their method of studying Legendre curves leads the idea of four cases in our present paper.

Motivated by these studies, we focus our interest on biharmonic slant curves in S-space forms. We obtain curvature characterizations of these kinds of curves. The paper is organized as follows: In Section 2, we give brief introduction about biharmonic maps and S-space forms. In Section 3, we define slant

Received March 23, 2017; Accepted November 3, 2017.
2010 *Mathematics Subject Classification.* 53C25, 53C40, 53A04.
Key words and phrases. S-manifold, slant curve, biharmonic curve, Frenet curve.

curves of S-manifolds and give two non-trivial examples. Finally, in Section 4, we find curvature characterizations of slant curves in S-space forms.

2. Biharmonic maps and S-space forms

Let $\phi : (M, g) \to (N, h)$ be a smooth map between two Riemannian manifolds (M, g) and (N, h). The *energy functional of* ϕ is given by

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \, \upsilon_g.$$

ϕ is called *harmonic* if it is a critical point of its energy functional [7]. Moreover, ϕ is said to be a *biharmonic map* if it is a critical point of its bienergy functional

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 \, \upsilon_g.$$

Here, $\tau(\phi)$ is *the first tension field* of ϕ given by $\tau(\phi) = trace\nabla d\phi$. The biharmonic map equation [11]

$$\tau_2(\phi) = -J^{\phi}(\tau(\phi)) = -\Delta\tau(\phi) - traceR^N(d\phi, \tau(\phi))d\phi = 0,$$

is derived using the *Euler-Lagrange equation of the bienergy functional* $E_2(\phi)$, where J^{ϕ} denotes the Jacobi operator of ϕ. Harmonic maps are directly biharmonic. Thus, we call non-harmonic biharmonic maps *proper biharmonic*.

Let (M, g) be a $(2m+s)$-dimensional Riemann manifold. It is called *framed metric manifold* [16] with a *framed metric structure* $(\varphi, \xi_\alpha, \eta^\alpha, g), \alpha \in \{1, \ldots, s\}$, if it satisfies the following equations:

$$\varphi^2 = -I + \sum_{\alpha=1}^{s} \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \quad \varphi(\xi_\alpha) = 0, \quad \eta^\alpha \circ \varphi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \eta^\alpha(X)\eta^\alpha(Y), \tag{2.2}$$

$$d\eta^\alpha(X, Y) = g(X, \varphi Y) = -d\eta^\alpha(Y, X), \quad \eta^\alpha(X) = g(X, \xi). \tag{2.3}$$

Here, φ is a $(1,1)$ tensor field of rank $2m$; $\xi_1, \ldots, \xi_s$ are vector fields; $\eta^1, \ldots, \eta^s$ are 1-forms and g is a Riemannian metric on M; $X, Y \in TM$ and $\alpha, \beta \in \{1, \ldots, s\}$. $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is also said to be *framed* φ*-manifold* [13] or *almost* r*-contact metric manifold* [15]. $(\varphi, \xi_\alpha, \eta^\alpha, g)$ is called S*-structure*, when the Nijenhuis tensor of φ is equal to $-2d\eta^\alpha \otimes \xi_\alpha$ for all $\alpha \in \{1, \ldots, s\}$ [2].

In case of $s = 1$, a framed metric structure becomes an almost contact metric structure and an S-structure becomes a Sasakian structure. For an S-structure, the following equations are valid [2]:

$$(\nabla_X \varphi)Y = \sum_{\alpha=1}^{s} \left\{ g(\varphi X, \varphi Y)\xi_\alpha + \eta^\alpha(Y)\varphi^2 X \right\}, \tag{2.4}$$

$$\nabla \xi_\alpha = -\varphi, \ \alpha \in \{1, \ldots, s\}. \tag{2.5}$$

In Sasakian case ($s = 1$), (2.5) can be directly obtained from (2.4).

A plane section in T_pM is called a *φ-section* if there exists a vector $X \in T_pM$ orthogonal to $\xi_1, \ldots, \xi_s$ such that $\{X, \varphi X\}$ span the section. The sectional curvature of a φ-section is called a *φ-sectional curvature*. An S-manifold of constant φ-sectional curvature has the curvature tensor R given by

$$\begin{aligned} R(X,Y)Z = & \sum_{\alpha,\beta} \{\eta^\alpha(X)\eta^\beta(Z)\varphi^2 Y - \eta^\alpha(Y)\eta^\beta(Z)\varphi^2 X \\ & -g(\varphi X, \varphi Z)\eta^\alpha(Y)\xi_\beta + g(\varphi Y, \varphi Z)\eta^\alpha(X)\xi_\beta\} \\ & + \frac{c+3s}{4}\{-g(\varphi Y, \varphi Z)\varphi^2 X + g(\varphi X, \varphi Z)\varphi^2 Y\} \\ & + \frac{c-s}{4}\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X + 2g(X, \varphi Y)\varphi Z\} \end{aligned} \tag{2.6}$$

for $X, Y, Z \in TM$ [3]. An S-manifold of constant φ-sectional curvature c is called an *S-space form* and it is denoted by $M(c)$. If $s = 1$, an S-space form turns into a Sasakian space form [1].

3. Slant curves of S-manifolds

Let $\gamma : I \to M$ be a unit-speed curve in an n-dimensional Riemannian manifold (M, g). γ is called a *Frenet curve of osculating order* r; if there exist orthonormal vector fields $v_1, v_2, \ldots, v_r$ along γ satisfying Frenet equations given by

$$\begin{aligned} v_1 &= \gamma', \\ \nabla_{v_1} v_1 &= k_1 v_2, \\ \nabla_{v_1} v_2 &= -k_1 v_1 + k_2 v_3, \\ &\cdots \\ \nabla_{v_1} v_r &= -k_{r-1} v_{r-1}, \end{aligned} \tag{3.1}$$

where $k_1, \ldots, k_{r-1}$ are positive functions and $1 \le r \le n$.

i) A *geodesic* is a Frenet curve of osculating order 1.

ii) A *circle* is a Frenet curve of osculating order 2 if k_1 is a non-zero positive constant.

iii) A *helix of order* r is a Frenet curve of osculating order $r \ge 3$ if $k_1, \ldots, k_{r-1}$ are non-zero positive constants. A helix of order 3 is shortly called a *helix*.

A submanifold of an S-manifold is called an *integral submanifold* if $\eta^\alpha(X) = 0$, $\alpha \in \{1, \ldots, s\}$, where X denotes tangent vectors of the submanifold [12]. A 1-dimensional integral submanifold of an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is called a *Legendre curve of* M. More precisely, a curve $\gamma : I \to M = (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ is a Legendre curve if $v_1 \perp \xi_\alpha$ for all $\alpha = 1, \ldots, s$, where v_1 is the tangent vector field of γ [14].

Now, we can give the following new definition:

Definition 3.1. Let γ be a unit-speed curve in an S-manifold

$$M = (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g).$$

We call γ a *slant curve*, if there exists a constant angle θ such that $\eta^\alpha(v_1) = \cos\theta$ for all $\alpha = 1, \dots, s$. Here θ is called the *contact angle of* γ.

From the definition, it is obvious that every Legendre curve is slant with contact angle $\frac{\pi}{2}$.

We can state the following essential proposition for slant curves:

Proposition 3.1. *Let $M = (M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold. If θ is the contact angle of a non-geodesic unit-speed slant curve in M, then*

$$\frac{-1}{\sqrt{s}} < \cos\theta < \frac{1}{\sqrt{s}}.$$

Proof. Let γ be a non-geodesic unit-speed slant curve with contact angle θ in M. Using equation (2.2), we find

$$\begin{aligned} g(\varphi v_1, \varphi v_1) &= g(v_1, v_1) - \sum_{\alpha=1}^{s} \eta^\alpha(v_1)\eta^\alpha(v_1) \\ &= 1 - s\cos^2\theta. \end{aligned}$$

Since g is non-degenerate, we have $1 - s\cos^2\theta \geq 0$. The equality case leads to a contradiction since γ is non-geodesic. Thus, we have

$$\cos^2\theta < \frac{1}{s}.$$

□

Now, we will obtain non-trivial examples of slant curves in $\mathbb{R}^{2n+s}(-3s)$. Let us consider $M = \mathbb{R}^{2n+s}$ with coordinate functions $\{x_1, \dots, x_n, y_1, \dots, y_n, z_1, , \dots z_s\}$ and define

$$\xi_\alpha = 2\frac{\partial}{\partial z_\alpha}, \ \alpha = 1, \dots, s,$$

$$\eta^\alpha = \frac{1}{2}\left(dz_\alpha - \sum_{i=1}^{n} y_i dx_i\right), \ \alpha = 1, \dots, s,$$

$$\varphi X = \sum_{i=1}^{n} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{n} X_i \frac{\partial}{\partial y_i} + \left(\sum_{i=1}^{n} Y_i y_i\right)\left(\sum_{\alpha=1}^{s} \frac{\partial}{\partial z_\alpha}\right),$$

$$g = \sum_{\alpha=1}^{s} \eta^\alpha \otimes \eta^\alpha + \frac{1}{4}\sum_{i=1}^{n} \left(dx_i \otimes dx_i + dy_i \otimes dy_i\right),$$

where

$$X = \sum_{i=1}^{n} \left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}\right) + \sum_{\alpha=1}^{s} \left(Z_\alpha \frac{\partial}{\partial z_\alpha}\right) \in \chi(M).$$

It is known that $\left(\mathbb{R}^{2n+s},\varphi,\xi_\alpha,\eta^\alpha,g\right)$ is an S-space form with constant φ-sectional curvature $c=-3s$ and it is denoted by $\mathbb{R}^{2n+s}(-3s)$ [10]. The vector fields

$$X_i=2\frac{\partial}{\partial y_i},\ X_{n+i}=\varphi X_i=2(\frac{\partial}{\partial x_i}+y_i\sum_{\alpha=1}^{s}\frac{\partial}{\partial z_\alpha}),\ \xi_\alpha=2\frac{\partial}{\partial z_\alpha}$$

form a g-orthonormal basis and the Levi-Civita connection is calculated as

$$\nabla_{X_i}X_j=\nabla_{X_{n+i}}X_{n+j}=0,\nabla_{X_i}X_{n+j}=\delta_{ij}\sum_{\alpha=1}^{s}\xi_\alpha,\nabla_{X_{n+i}}X_j=-\delta_{ij}\sum_{\alpha=1}^{s}\xi_\alpha,$$

$$\nabla_{X_i}\xi_\alpha=\nabla_{\xi_\alpha}X_i=-X_{n+i},\nabla_{X_{n+i}}\xi_\alpha=\nabla_{\xi_\alpha}X_{n+i}=X_i$$

(see [10]). Let $\gamma:I\to\mathbb{R}^{2n+s}(-3s)$ be a slant curve with contact angle θ. Let us denote

$$\gamma(t)=\left(\gamma_1(t),\ldots,\gamma_n(t),\gamma_{n+1}(t),\ldots,\gamma_{2n}(t),\gamma_{2n+1}(t),\ldots,\gamma_{2n+s}(t)\right),$$

where t is the arc-length parameter. The tangent vector field of γ is

$$v_1=\gamma_1'\frac{\partial}{\partial x_1}+\cdots+\gamma_n'\frac{\partial}{\partial x_n}+\gamma_{n+1}'\frac{\partial}{\partial y_1}+\cdots+\gamma_{2n}'\frac{\partial}{\partial y_n}+\gamma_{2n+1}'\frac{\partial}{\partial z_1}+\cdots+\gamma_{2n+s}'\frac{\partial}{\partial z_\alpha}.$$

In terms of the g-orthonormal basis, v_1 can be written as

$$\begin{aligned}v_1=\frac{1}{2}&\left[\gamma_{n+1}'X_1+\cdots+\gamma_{2n}'X_n+\gamma_1'X_{n+1}+\cdots+\gamma_n'X_{2n}\right.\\&+\left(\gamma_{2n+1}'-\gamma_1'\gamma_{n+1}-\cdots-\gamma_n'\gamma_{2n}\right)\xi_1+\cdots\\&\left.+\left(\gamma_{2n+s}'-\gamma_1'\gamma_{n+1}-\cdots-\gamma_n'\gamma_{2n}\right)\xi_s\right].\end{aligned}$$

Since γ is slant, we obtain

$$\eta^\alpha(v_1)=\frac{1}{2}\left(\gamma_{2n+\alpha}'-\gamma_1'\gamma_{n+1}-\cdots-\gamma_n'\gamma_{2n}\right)=\cos\theta$$

for all $\alpha=1,\ldots,s$. Thus, we have

$$\gamma_{2n+1}'=\cdots=\gamma_{2n+s}'=\gamma_1'\gamma_{n+1}+\cdots+\gamma_n'\gamma_{2n}+2\cos\theta.$$

Since γ is a unit-speed curve, we can write

$$\left(\gamma_1'\right)^2+\cdots+\left(\gamma_{2n}'\right)^2=4\left(1-s\cos^2\theta\right).$$

So, we have the following examples:

Example 3.1. Let $n=1$ and $s=2$. Then, $\gamma:I\to\mathbb{R}^4(-6)$, $\gamma(t)=\left(\sqrt{2}t,0,t,t\right)$ is a slant circle with $k_1=\sqrt{2}$ and its contact angle is $\frac{\pi}{3}$.

Example 3.2. The curve $\gamma:I\to\mathbb{R}^4(-6)$, $\gamma(t)=(\gamma_1(t),\gamma_2(t),\gamma_3(t),\gamma_4(t))$ is a slant curve with contact angle θ, where

$$\gamma_1(t)=c_1+2\sqrt{-\cos2\theta}\int_{t_0}^{t}\cos u(p)dp,$$

$$\gamma_2(t) = c_2 + 2\sqrt{-\cos 2\theta}\int_{t_0}^{t}\sin u(p)dp,$$

$$\begin{aligned}\gamma_3(t) &= \gamma_4(t) + c_3\\ &= c_4 + 2t\cos\theta\\ &\quad + 2\sqrt{-\cos 2\theta}\int_{t_0}^{t}\cos u(q)\left(c_2 + 2\sqrt{-\cos 2\theta}\int_{t_0}^{q}\sin u(p)dp\right)dq,\end{aligned}$$

$$\cos\theta \in \left(-1/\sqrt{2}, 1/\sqrt{2}\right),$$

$t_0 \in I$, c_1, c_2, c_3 and c_4 are arbitrary constants.

4. Biharmonic slant curves in *S*-space forms

Now, let us take an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$ and a curve $\gamma : I \to M$ which is a slant Frenet curve of osculating order r. Differentiating

$$\eta^\alpha(v_1) = \cos\theta \tag{4.1}$$

and using (3.1), we find

$$\eta^\alpha(v_2) = 0,\ \alpha \in \{1, \ldots, s\}. \tag{4.2}$$

Then, (2.1) and (4.2) give us

$$\varphi^2 v_2 = -v_2. \tag{4.3}$$

Using equations (2.1), (2.2), (2.3), (2.6), (3.1), (4.2) and (4.3), we can easily calculate

$$\begin{aligned}\nabla_{v_1}\nabla_{v_1}v_1 &= -k_1^2 v_1 + k_1' v_2 + k_1 k_2 v_3,\\ \nabla_{v_1}\nabla_{v_1}\nabla_{v_1}v_1 &= -3k_1k_1'v_1 + \left(k_1'' - k_1^3 - k_1k_2^2\right)v_2\\ &\quad + \left(2k_1'k_2 + k_1k_2'\right)v_3 + k_1k_2k_3v_4,\\ R(v_1, \nabla_{v_1}v_1)v_1 &= -k_1\left[s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta)\right]v_2\\ &\quad - 3k_1\frac{(c-s)}{4}g(\varphi v_1, v_2)\varphi v_1.\end{aligned}$$

So, we get

$$\begin{aligned}\tau_2(\gamma) &= \nabla_{v_1}\nabla_{v_1}\nabla_{v_1}v_1 - R(v_1, \nabla_{v_1}v_1)v_1\\ &= -3k_1k_1'v_1\\ &\quad + \left(k_1'' - k_1^3 - k_1k_2^2 + k_1\left[s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta)\right]\right)v_2\\ &\quad + (2k_1'k_2 + k_1k_2')v_3 + k_1k_2k_3v_4\\ &\quad + 3k_1\frac{(c-s)}{4}g(\varphi v_1, v_2)\varphi v_1.\end{aligned} \tag{4.4}$$

Let $k = \min\{r, 4\}$. From (4.4), the curve γ is proper biharmonic if and only if $k_1 > 0$ and

(1) $c = s$ or $\varphi v_1 \perp v_2$ or $\varphi v_1 \in span\{v_2, \ldots, v_k\}$; and

(2) $g(\tau_2(\gamma), v_i) = 0$ for any $i = 1, \ldots, k$.

So we can state the following theorem:

Theorem 4.1. *Let γ be a slant curve of osculating order r in an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$ and $k = \min\{r, 4\}$. Then γ is proper biharmonic if and only if*

(1) $c = s$ or $\varphi v_1 \perp v_2$ or $\varphi v_1 \in span\{v_2, \ldots, v_k\}$; and

(2) the first k of the following equations are satisfied (replacing $k_k = 0$):

$$k_1 = constant > 0,$$

$$k_1^2 + k_2^2 = s^2 \cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) + \frac{3(c-s)}{4}\left[g(\varphi v_1, v_2)\right]^2,$$

$$k_2' + \frac{3(c-s)}{4} g(\varphi v_1, v_2) g(\varphi v_1, v_3) = 0,$$

$$k_2 k_3 + \frac{3(c-s)}{4} g(\varphi v_1, v_2) g(\varphi v_1, v_4) = 0.$$

We have four cases to investigate results of Theorem 4.1.

Case I. $c = s$.

Let $c = s$. Then, γ is proper biharmonic if and only if following equations hold:

$$k_1 = \text{constant} > 0,$$
$$k_1^2 + k_2^2 = s,$$
$$k_2 = \text{constant},$$
$$k_2 k_3 = 0.$$

Using these last four equations, we can state the following theorem:

Theorem 4.2. *Let γ be a slant curve in an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \ldots, s\}$, $c = s$. Then γ is proper biharmonic if and only if either γ is a circle with $k_1 = \sqrt{s}$, or a helix with $k_1^2 + k_2^2 = s$. Moreover, if γ is Legendre, then $2m + s > 3$.*

Remark 4.1. If $2m+s = 3$, then $m = s = 1$. So M is a 3-dimensional Sasakian space form. Since a Legendre curve in a Sasakian 3-manifold has torsion 1 (see [1]), we can write $k_1 > 0$ and $k_2 = 1$, which contradicts $k_1^2 + k_2^2 = s = 1$. Hence γ cannot be proper biharmonic.

Case II. $c \neq s$, $\varphi v_1 \perp v_2$.

Let us assume that $g(\varphi v_1, v_2) = 0$. Theorem 4.1 gives us

$$k_1 = \text{constant} > 0, \tag{4.5}$$

$$k_1^2 + k_2^2 = s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta),$$

$$k_2 = \text{constant},$$
$$k_2 k_3 = 0.$$

We have the following proposition in this case:

Proposition 4.1. *Let γ be a slant curve of osculating order 3 in an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$ and $\varphi v_1 \perp v_2$. Then $\{v_1, v_2, v_3, \varphi v_1, \nabla_{v_1} \varphi v_1, \xi_1, \dots, \xi_s\}$ is linearly independent at any point of γ. Hence $m \geq 3$.*

Therefore, we can give the following theorem:

Theorem 4.3. *Let γ be a slant curve in an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, $c \neq s$ and $\varphi v_1 \perp v_2$. Then γ is proper biharmonic if and only if either*

(1) $m \geq 2$ and γ is a circle with $k_1 = \frac{1}{2}\sqrt{c + 3s - (c-s)s\cos^2\theta}$, where $c > -3s + (c-s)s\cos^2\theta$ and $\{v_1 = v_1, v_2, \varphi v_1, \nabla_{v_1}\varphi v_1, \xi_1, \dots, \xi_s\}$ is linearly independent; or

(2) $m \geq 3$ and γ is a helix with $k_1^2 + k_2^2 = \frac{c+3s-(c-s)s\cos^2\theta}{4}$, where $c > -3s + (c-s)s\cos^2\theta$ and $\{v_1, v_2, v_3, \varphi v_1, \nabla_{v_1}\varphi v_1, \xi_1, \dots, \xi_s\}$ is linearly independent.

Case III. $c \neq s$, $\varphi v_1 \parallel v_2$.

In this case, $\varphi v_1 = \pm\sqrt{1 - s\cos^2\theta}\, v_2$, $g(\varphi v_1, v_2) = \pm(1 - s\cos^2\theta)$, $g(\varphi v_1, v_3) = 0$ and $g(\varphi v_1, v_4) = 0$. Using Theorem 4.1, γ is biharmonic if and only if

$$k_1 = \text{constant} > 0,$$
$$k_1^2 + k_2^2 = c - s\cos^2\theta(c-s),$$
$$k_2 = \text{constant},$$
$$k_2 k_3 = 0.$$

Let us choose $\varphi v_1 = \sqrt{1 - s\cos^2\theta}\, v_2$. Equation (2.1) gives us

$$\sqrt{1 - s\cos^2\theta}\,\varphi v_2 = \varphi^2 v_1 = -v_1 + \sum_{\alpha=1}^{s} \eta^\alpha(v_1)\xi_\alpha = -v_1 + \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha. \tag{4.6}$$

Using (2.1), (2.2), (2.3) and (2.4), we find

$$\nabla_{v_1}\varphi v_1 = -s\cos\theta\, v_1 + \sum_{\alpha=1}^{s}\xi_\alpha + k_1 \varphi v_2. \tag{4.7}$$

From (4.6) and (4.7), we have

$$\begin{aligned}\nabla_{v_1}\varphi v_1 &= -s\cos\theta\, v_1 + \sum_{\alpha=1}^{s}\xi_\alpha + k_1\left[\frac{-1}{\sqrt{1 - s\cos^2\theta}} v_1 + \frac{\cos\theta}{\sqrt{1 - s\cos^2\theta}}\sum_{\alpha=1}^{s}\xi_\alpha\right] \\ &= \sqrt{1 - s\cos^2\theta}\,(-k_1 v_1 + k_2 v_3).\end{aligned} \tag{4.8}$$

Using (3.1) and (4.8), we can write

$$\left(1 + \frac{k_1\cos\theta}{\sqrt{1 - s\cos^2\theta}}\right)\left(-s\cos\theta\, v_1 + \sum_{\alpha=1}^{s}\xi_\alpha\right) = k_2\sqrt{1 - s\cos^2\theta}\, v_3, \tag{4.9}$$

which gives us the following theorem:

Theorem 4.4. *Let γ be a slant curve in an S-space form $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,\dots,s\}$, $c\neq s$ and $\varphi v_1\parallel v_2$. Then γ is proper biharmonic if and only if it is one of the following*:

i) *a Legendre helix with the Frenet frame field*

$$\left\{v_1,\varphi v_1,\frac{1}{\sqrt{s}}\sum_{\alpha=1}^{s}\xi_\alpha\right\}$$

and $k_1=\sqrt{c-s}$ and $k_2=\sqrt{s}$, where $c>s$;

ii) *a non-Legendre slant circle with the Frenet frame field*

$$\left\{v_1,\frac{\varphi v_1}{\sqrt{1-s\cos^2\theta}}\right\}$$

and

$$k_1=\frac{-\sqrt{1-s\cos^2\theta}}{\cos\theta}=\sqrt{c-s\cos^2\theta(c-s)};$$

iii) *a non-Legendre slant helix with the Frenet frame field*

$$\left\{v_1,\frac{\varphi v_1}{\sqrt{1-s\cos^2\theta}},\frac{1}{\sqrt{s}\sqrt{1-s\cos^2\theta}}\left(\sum_{\alpha=1}^{s}\xi_\alpha-s\cos\theta v_1\right)\right\}$$

and

$$k_1^2+k_2^2=c-s\cos^2\theta(c-s).$$

Case IV. $c\neq s$, $\varphi v_1\nparallel v_2$ and $g(\varphi v_1,v_2)\neq 0$.

Finally, let $(M^{2m+s},\varphi,\xi_\alpha,\eta^\alpha,g)$ be an S-space form, $\alpha\in\{1,\dots,s\}$ and $\gamma: I\to M$ a slant curve of osculating order r, where $4\leq r\leq 2m+s$ and $m\geq 2$. In this case, γ is biharmonic if and only if $\varphi v_1\in span\,\{v_2,v_3,v_4\}$. We denote the angle function between φv_1 and v_2 by $\mu(t)$; which means $g(\varphi v_1,v_2)=\sqrt{1-s\cos^2\theta}\cos\mu(t)$. If we differentiate $g(\varphi v_1,v_2)$ along γ and use (2.1), (2.3), (3.1), (4.7), we obtain

$$\begin{aligned}-\sqrt{1-s\cos^2\theta}\mu'(t)\sin\mu(t)&=\nabla_{v_1}g(\varphi v_1,v_2)=g(\nabla_{v_1}\varphi v_1,v_2)+g(\varphi v_1,\nabla_{v_1}v_2)\\&=g(-s\cos\theta v_1+\sum_{\alpha=1}^{s}\xi_\alpha+k_1\varphi v_2,v_2)\\&\quad+g(\varphi v_1,-k_1v_1+k_2v_3)\\&=k_2g(\varphi v_1,v_3).\end{aligned}\tag{4.10}$$

We can write $\varphi v_1=g(\varphi v_1,v_2)v_2+g(\varphi v_1,v_3)v_3+g(\varphi v_1,v_4)v_4$. So, the equations in Theorem 4.1 become

$$\begin{aligned}k_1&=\text{constant}>0,\\k_1^2+k_2^2&=s^2\cos^2\theta+\frac{c+3s}{4}(1-s\cos^2\theta)+\frac{3(c-s)}{4}\left[g(\varphi v_1,v_2)\right]^2,\end{aligned}$$

$$k_2' + \frac{3(c-s)}{4} g(\varphi v_1, v_2) g(\varphi v_1, v_3) = 0,$$
$$k_2 k_3 + \frac{3(c-s)}{4} g(\varphi v_1, v_2) g(\varphi v_1, v_4) = 0.$$

Multiplying the third equation of the above system with $2k_2$ and using (4.10), we have

$$2k_2 k_2' + \sqrt{1 - s\cos^2\theta}\,\frac{3(c-s)}{4}(-2\mu' \cos\mu \sin\mu) = 0,$$

which gives us

$$k_2^2 = -\sqrt{1 - s\cos^2\theta}\,\frac{3(c-s)}{4} \cos^2\mu + \omega_0, \tag{4.11}$$

where ω_0 is a constant. If we write (4.11) in the second equation, we have

$$k_1^2 = s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) + \frac{3(c-s)}{4}\Big(1 - s\cos^2\theta + \sqrt{1 - s\cos^2\theta}\Big)\cos^2\mu + \omega_0.$$

Hence μ is a constant. From (4.10) and (4.11), we have $g(\varphi v_1, v_3) = 0$ and $k_2 =$constant> 0. Since $\|\varphi v_1\| = \sqrt{1 - s\cos^2\theta}$ and $\varphi v_1 = \sqrt{1 - s\cos^2\theta}\cos\mu v_2$ $+ g(\varphi v_1, v_4)v_4$, we obtain $g(\varphi v_1, v_4) = \sqrt{1 - s\cos^2\theta}\sin\mu$. Because of the fact that $\varphi v_1 \nparallel v_2$ and $g(\varphi v_1, v_2) \neq 0$, it is obvious that $\mu \in (0, 2\pi) \backslash \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$. As a result, we can give the following theorem:

Theorem 4.5. *Let $\gamma : I \to M$ be a slant curve of osculating order r in an S-space form $(M^{2m+s}, \varphi, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, \dots, s\}$, where $r \geq 4$, $m \geq 2$, $c \neq s$, $\varphi v_1 \nparallel v_2$ and $g(\varphi v_1, v_2) \neq 0$. Then γ is proper biharmonic if and only if*

$$k_\lambda = \textit{constant} > 0, \ \ \lambda \in \{1, 2, 3\},$$
$$k_1^2 + k_2^2 = s^2\cos^2\theta + \frac{c+3s}{4}(1 - s\cos^2\theta) + \frac{3(c-s)}{4}(1 - s\cos^2\theta)\cos^2\mu,$$
$$k_2 k_3 = \frac{3(s-c)}{8}(1 - s\cos^2\theta)\sin 2\mu,$$

where $\varphi v_1 = \sqrt{1 - s\cos^2\theta}\cos\mu v_2 + \sqrt{1 - s\cos^2\theta}\sin\mu v_4$, $\mu \in (0, 2\pi) \backslash \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ is a constant.

References

[1] C. Baikoussis and D. E. Blair, *On Legendre curves in contact* 3*-manifolds*, Geom. Dedicata **49** (1994), no. 2, 135–142.

[2] D. E. Blair, *Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$*, J. Differential Geometry **4** (1970), 155–167.

[3] J. L. Cabrerizo, L. M. Fernadez, and M. Fernandez, *The curvature of submanifolds of an S-space form*, Acta Math. Hungar. **62** (1993), no. 3-4, 373–383.

[4] B.-Y. Chen, *A report on submanifolds of finite type*, Soochow J. Math. **22** (1996), no. 2, 117–337.

[5] J. T. Cho, J. Inoguchi, and J.-E. Lee, *On slant curves in Sasakian 3-manifolds*, Bull. Austral. Math. Soc. **74** (2006), no. 3, 359–367.
[6] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, **50**, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1983.
[7] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
[8] D. Fetcu, *Biharmonic Legendre curves in Sasakian space forms*, J. Korean Math. Soc. **45** (2008), no. 2, 393–404.
[9] D. Fetcu and C. Oniciuc, *Explicit formulas for biharmonic submanifolds in Sasakian space forms*, Pacific J. Math. **240** (2009), no. 1, 85–107.
[10] I. Hasegawa, Y. Okuyama, and T. Abe, *On pth Sasakian manifolds*, J. Hokkaido Univ. Ed. Sect. II A **37** (1986), no. 1, 1–16.
[11] G. Y. Jiang, 2-*harmonic maps and their first and second variational formulas*, Chinese Ann. Math. Ser. A **7** (1986), no. 4, 389–402.
[12] J.-S. Kim, M. K. Dwivedi, and M. M. Tripathi, *Ricci curvature of integral submanifolds of an S-space form*, Bull. Korean Math. Soc. **44** (2007), no. 3, 395–406.
[13] H. Nakagawa, *On framed f-manifolds*, Kōdai Math. Sem. Rep. **18** (1966), 293–306.
[14] C. Özgür and S. Güvenç, *On biharmonic Legendre curves in S-space forms*, Turkish J. Math. **38** (2014), no. 3, 454–461.
[15] J. Vanžura, *Almost r-contact structures*, Ann. Scuola Norm. Sup. Pisa (3) **26** (1972), 97–115.
[16] K. Yano and M. Kon, *Structures on Manifolds*, Series in Pure Mathematics, **3**, World Scientific Publishing Co., Singapore, 1984.

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