Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 237–245
https://doi.org/10.4134/CKMS.c170041
pISSN: 1225-1763 / eISSN: 2234-3024

# VARIOUS CENTROIDS AND SOME CHARACTERIZATIONS OF CATENARY CURVES

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Abstract. For every interval $[a, b]$, we denote by $(\bar{x}_A, \bar{y}_A)$ and $(\bar{x}_L, \bar{y}_L)$ the geometric centroid of the area under a catenary curve $y = k\cosh((x-c)/k)$ defined on this interval and the centroid of the curve itself, respectively. Then, it is well-known that $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$.

In this paper, we fix an end point, say 0, and we show that one of $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$ for every interval with an end point 0 characterizes the family of catenaries among nonconstant $C^2$ functions.

## 1. Introduction

For the catenary given by $f(x) = k\cosh((x-c)/k)$ with a positive constant $k$, it is well-known that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. For a positive $C^1$ function $f(x)$ defined on an interval $I$ and an interval $[a, b] \subset I$, we consider the area $A(a, b)$ over the interval $[a, b]$ and the arc length $L(a, b)$ of the graph of $f(x)$. Then, the function $f(x) = k\cosh((x-c)/k), k > 0$ satisfies for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$. This property characterizes the family of catenaries among nonconstant $C^2$ functions ([13]). Thus, we have the following.

**Proposition 1.1.** *For a nonconstant positive $C^2$ function $f(x)$ defined on an interval $I$, the followings are equivalent.*

(1) *There exists a positive constant $k$ such that for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$.*
(2) *The function $f(x)$ satisfies $f(x) = k\sqrt{1 + f'(x)^2}$, where $k$ is a positive constant.*

Received February 11, 2017; Accepted May 11, 2017.
2010 *Mathematics Subject Classification.* 52A10, 53A04.
*Key words and phrases.* centroid, perimeter centroid, area, arc length, catenary.
The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2015020387).

(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

For a positive $C^1$ function $f(x)$ defined on an interval $I$ and an interval $[a, b] \subset I$, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a, b), \bar{y}_A(a, b))$ and $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a, b), \bar{y}_L(a, b))$ the geometric centroid of the area under the graph of $f(x)$ defined on this interval and the centroid of the graph itself, respectively. Then, for a catenary curve we have the following ([13]).

**Proposition 1.2.** *A catenary curve* $f(x) = k\cosh((x - c)/k)$ *satisfies the following properties.*

(1) *For every interval* $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
(2) *For every interval* $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.

Conversely, in a recent paper [10], it was shown that one of $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$ for all interval $[a, b]$ characterizes the family of catenaries among nonconstant $C^2$ functions as follows.

**Proposition 1.3.** *For a nonconstant positive* $C^2$ *function* $f(x)$ *defined on an interval* $I$*, the followings are equivalent.*

(1) *For every interval* $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
(2) *For every interval* $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.
(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In this paper, we consider intervals with a fixed end point, say 0. For a nonzero real number $x$, we denote by $I_x$ the interval defined by

$$I_x = \begin{cases} [0, x], & \text{if } x > 0, \\ [x, 0], & \text{if } x < 0. \end{cases} \tag{1.1}$$

We also denote by $A(x)$, $L(x)$, $(\bar{x}_A(x), \bar{y}_A(x))$ and $(\bar{x}_L(x), \bar{y}_L(x))$ the area under the graph of $f(x)$, the arc length of the graph of $f(x)$, the geometric centroid of the area under the graph of $f(x)$ and the centroid of the graph itself over the interval $I_x$, respectively. Then, it is trivial to show the following.

**Proposition 1.4.** *For a nonconstant positive* $C^2$ *function* $f(x)$ *defined on an interval* $I$ *containing* 0*, the followings are equivalent.*

(1) *There exists a positive constant* $k$ *such that for every nonzero real number* $x$, $A(x) = kL(x)$.
(2) *The function* $f(x)$ *satisfies* $f(x) = k\sqrt{1 + f'(x)^2}$*, where* $k$ *is a positive constant.*
(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

As a result, in Section 3 we establish the following characterization theorem for catenary curves.

**Theorem 1.5.** *For a nonconstant positive $C^2$ function $f(x)$ defined on an interval $I$ containing 0, the followings are equivalent.*

(1) *For every nonzero real number $x \in I$, $\bar{x}_L(x) = \bar{x}_A(x)$.*
(2) *For every nonzero real number $x \in I$, $\bar{y}_L(x) = 2\bar{y}_A(x)$.*
(3) *For some $k > 0$ and $c \in \mathbb{R}$,*

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In order to prove the above mentioned main theorem, first of all in Section 2 we prove some lemmas. In particular, Lemma 2.1 establishes a sufficient and necessary condition for two positive $C^2$ functions to be proportional.

To find the centroid of polygons, see [3]. For the perimeter centroid of a polygon, we refer [2]. In [11], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [12]. The relationships between various centroids of a quadrangle were given in [5, 9].

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([14]). Some characterizations of parabolas using these properties were given in [4, 6, 7].

Two higher dimensional generalizations of Proposition 1.1 were established in [1]. In [8], it was shown that among nonconstant $C^2$ functions $f : \mathbb{R}^2 \to \mathbb{R}$ with isolated singularities, $S = kV$, $k \in \mathbb{R}$ characterizes the family of catenary rotation surfaces $f(x, y) = k \cosh(r/k)$, $r = |(x, y)|$, where $V$ and $S$ denote the volume and the surface area of the graph of $z = f(x, y)$ over a rectangular domain.

## 2. Some lemmas

In this section, first of all we prove the following lemma which is useful in the proof of Theorems stated in Section 1.

**Lemma 2.1.** *We denote by $f(x)$ and $g(x)$ two positive $C^2$ functions defined on an interval $I$ containing $0 \in \mathbb{R}$. Suppose that $f(x)$ and $g(x)$ satisfy the following property:*

$$\frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt} = \frac{\int_0^x tg(t)dt}{\int_0^x g(t)dt}, \quad x \in I,\ x \neq 0. \tag{2.1}$$

*Then the ratio of $f(x)$ and $g(x)$ is constant, that is, for some constant $k \in \mathbb{R}$ we have $f(x) = kg(x)$.*

*Proof.* Suppose that $f(x)$ and $g(x)$ satisfy (2.1). Then for all $x \in I$ we get

$$\int_0^x g(t)dt \int_0^x tf(t)dt = \int_0^x f(t)dt \int_0^x tg(t)dt. \tag{2.2}$$

By differentiating (2.2) with respect to the variable $x$, we obtain

$$g(x)\int_0^x tf(t)dt + xf(x)\int_0^x g(t)dt = f(x)\int_0^x tg(t)dt + xg(x)\int_0^x f(t)dt. \tag{2.3}$$

We denote by $k(x)$ the ratio of $f(x)$ and $g(x)$, that is, we let

$$k(x) = \frac{f(x)}{g(x)}. \tag{2.4}$$

Then the function $k(x)$ is a $C^2$ function on the interval $I$. It follows from (2.3) that

$$\int_0^x tf(t)dt + xk(x)\int_0^x g(t)dt = k(x)\int_0^x tg(t)dt + x\int_0^x f(t)dt. \tag{2.5}$$

Differentiating (2.5) with respect to $x$ gives

$$(xk(x))'\int_0^x g(t)dt = k'(x)\int_0^x tg(t)dt + \int_0^x f(t)dt. \tag{2.6}$$

By differentiating (2.6) with respect to $x$ once more, we obtain

$$(xk(x))''\int_0^x g(t)dt = k''(x)\int_0^x tg(t)dt. \tag{2.7}$$

First, suppose that the open set $I_1$ defined by $I_1 = \{x \in I \mid k''(x) \neq 0, x \neq 0\}$ is nonempty. Then, on the open set $I_1$, from (2.7) we get

$$\int_0^x tg(t)dt = h(x)\int_0^x g(t)dt, \tag{2.8}$$

where

$$h(x) = \frac{(xk(x))''}{k''(x)}. \tag{2.9}$$

It follows from (2.8) that the function $h(x)$ is a $C^2$ function on the open set $I_1$. Differentiating (2.8) with respect to $x$ shows that

$$xg(x) - h(x)g(x) = h'(x)\int_0^x g(t)dt. \tag{2.10}$$

Second, suppose that the open set $I_2$ defined by $I_2 = \{x \in I_1 \mid h'(x) \neq 0, x \neq 0\}$ is nonempty. Then, on the open set $I_2$, from (2.10) we get

$$(j(x)g(x))' = g(x), \tag{2.11}$$

where

$$j(x) = \frac{x - h(x)}{h'(x)}. \tag{2.12}$$

It also follows from (2.10) that the function $j(x)$ is a $C^2$ function on the open set $I_2$.

On the other hand, together with (2.8), (2.1) implies

$$\int_0^x tf(t)dt = h(x)\int_0^x f(t)dt, \quad x \in I_1, \tag{2.13}$$

where $h(x)$ is given in (2.9). Differentiating (2.13) with respect to $x$ gives

$$xf(x) - h(x)f(x) = h'(x)\int_0^x f(t)dt. \tag{2.14}$$

Hence as in the discussions above, on the open set $I_2$, from (2.14) we obtain

$$(j(x)f(x))' = f(x), \tag{2.15}$$

where $j(x)$ is defined in (2.12).

It follows from (2.11) and (2.15) that

$$j(x)g'(x) = g(x) - g(x)j'(x) \tag{2.16}$$

and

$$j(x)f'(x) = f(x) - f(x)j'(x), \tag{2.17}$$

respectively. For the function $k(x) = f(x)/g(x)$, (2.16) and (2.17) show that on the open set $I_2$,

$$j(x)k'(x) = 0. \tag{2.18}$$

Now, we claim that the derivative $k'(x)$ of the function $k(x)$ vanishes on $I_2$. Otherwise, on a nonempty open set $I_3$ contained in $I_2$, the function $j(x)$ vanishes. Then, it follows from (2.12) that on $I_3$, $h(x) = x$. Hence (2.10) shows that on $I_3$, $\int_0^x g(t)dt = 0$, which is a contradiction. Thus, together with (2.18), this contradiction yields that on the open set $I_2$, $k'(x)$ must vanish, which completes the proof of the above claim.

But, the vanishing of the derivative $k'(x)$ of $k(x)$ on $I_2$ contradicts to the hypothesis on $I_1$. Therefore, this contradiction implies that $I_2$ is empty, that is, on the open set $I_1$, $h'(x)$ must vanish. Hence, it follows from (2.10) that $h(x) = x$ on $I_1$, which leads to a contradiction. This contradiction shows that $I_1$ is empty, that is, on the whole domain $I$, $k''(x)$ must vanish.

Since $k''(x) = 0$ on $I$, (2.7) implies that $(xk(x))'' = 2k'(x) + xk''(x)$ also vanishes on $I$. Combining them, we see that $k'(x) = 0$ on $I$. Thus the function $k(x) = f(x)/g(x)$ is a constant $k$. This completes the proof of Lemma 2.1. □

Now, we prove the following lemma which is crucial in the proof of Theorem 1.5.

**Lemma 2.2.** *We consider two positive $C^2$ functions $f(x)$ and $g(x)$ defined on an interval $I$ containing $0 \in \mathbb{R}$ and denote by $k(x)$ the ratio $k(x) = f(x)/g(x)$ of $f(x)$ and $g(x)$. Suppose that $f(x)$ and $g(x)$ satisfy the following*:

$$\frac{\int_0^x f(t)g(t)dt}{\int_0^x f(t)dt} = \frac{\int_0^x g(t)^2dt}{\int_0^x g(t)dt}, \quad x \in I,\ x \neq 0. \tag{2.19}$$

*Then, on the open set $I_1 = \{x \in I \mid k'(x) \neq 0\}$, we have $g'(x) = 0$.*

*Proof.* Suppose that $f(x)$ and $g(x)$ satisfy (2.19). Then for all $x \in I$ we get

$$\int_0^x g(t)dt \int_0^x f(t)g(t)dt = \int_0^x f(t)dt \int_0^x g(t)^2 dt. \tag{2.20}$$

By differentiating (2.20) with respect to the variable $x$, we obtain

$$g(x)\int_0^x f(t)g(t)dt + f(x)g(x)\int_0^x g(t)dt = f(x)\int_0^x g(t)^2dt + g(x)^2\int_0^x f(t)dt. \tag{2.21}$$

We denoted by $k(x)$ the ratio of $f(x)$ and $g(x)$, that is, we put

$$k(x) = \frac{f(x)}{g(x)}. \tag{2.22}$$

Then the function $k(x)$ is a $C^2$ function on the interval $I$. It follows from (2.21) that

$$\int_0^x f(t)g(t)dt + f(x)\int_0^x g(t)dt = k(x)\int_0^x g(t)^2dt + g(x)\int_0^x f(t)dt. \tag{2.23}$$

Differentiating (2.23) with respect to $x$ gives

$$f'(x)\int_0^x g(t)dt = k'(x)\int_0^x g(t)^2dt + g'(x)\int_0^x f(t)dt. \tag{2.24}$$

First, suppose that the open set $I_1$ defined by $I_1 = \{x \in I \,|\, k'(x) \neq 0\}$ is nonempty. Then, on the open set $I_1$, from (2.24) we get

$$\phi(x)\int_0^x g(t)dt = \int_0^x g(t)^2dt + \psi(x)\int_0^x f(t)dt, \tag{2.25}$$

where we put

$$\phi(x) = \frac{f'(x)}{k'(x)}, \quad \psi(x) = \frac{g'(x)}{k'(x)}. \tag{2.26}$$

By differentiating (2.25) with respect to $x$, we obtain

$$\phi'(x)\int_0^x g(t)dt = \psi'(x)\int_0^x f(t)dt. \tag{2.27}$$

Next, suppose that the open set $I_2$ defined by $I_2 = \{x \in I_1 \,|\, \psi'(x) \neq 0, x \neq 0\}$ is nonempty. Then, on the open set $I_2$, from (2.27) we get

$$\int_0^x g(t)dt = h(x)\int_0^x f(t)dt, \tag{2.28}$$

where

$$h(x) = \frac{\psi'(x)}{\phi'(x)}. \tag{2.29}$$

It follows from (2.28) that the function $h(x)$ is a $C^2$ function on the open set $I_2$. Differentiating (2.28) with respect to $x$ shows that

$$g(x) - h(x)f(x) = h'(x)\int_0^x f(t)dt. \tag{2.30}$$

Finally, suppose that the open set $I_3$ defined by $I_3 = \{x \in I_2 \mid h'(x) \neq 0, x \neq 0\}$ is nonempty. Then, on the open set $I_3$, from (2.30) we get

$$(j(x)g(x))' = f(x), \tag{2.31}$$

where

$$j(x) = \frac{1 - k(x)h(x)}{h'(x)}. \tag{2.32}$$

It follows from (2.30) that the function $j(x)$ is a $C^2$ function on the open set $I_3$.

On the other hand, together with (2.28), (2.19) implies

$$\int_0^x g(t)^2 dt = h(x) \int_0^x f(t)g(t)dt, \quad x \in I_2, \tag{2.33}$$

where $h(x)$ is given in (2.29). Differentiating (2.33) with respect to $x$ gives

$$g(x)^2 - h(x)f(x)g(x) = h'(x) \int_0^x f(t)g(t)dt. \tag{2.34}$$

Hence as in the discussions above, on the open set $I_3$, from (2.34) we obtain

$$\big(j(x)g(x)^2\big)' = f(x)g(x), \tag{2.35}$$

where $j(x)$ is defined in (2.32) and we use (2.22).

On the open set $I_3$, it follows from (2.31) and (2.35) that

$$j(x)g'(x) = 0, \tag{2.36}$$

and hence

$$f(x) = (j(x)g(x))' = j'(x)g(x). \tag{2.37}$$

Together with $f(x) = k(x)g(x)$, this shows that

$$j'(x) = k(x), \quad x \in I_3. \tag{2.38}$$

Since $k(x) > 0$, (2.36) and (2.38) imply that $g'(x)$ vanishes on $I_3$. This shows that on $I_3$, $\psi(x)$ (and hence $h(x)$) vanishes, which contradicts to the hypothesis on $I_2$. This contradiction yields that $I_3$ is empty, that is, on the open set $I_2$, $h(x)$ is a constant $h_0$.

It follows from (2.30) that on $I_2$, $k(x) = f(x)/g(x) = 1/h_0$ is constant, which contradicts to the hypothesis on $I_1$. This contradiction shows that $I_2$ is empty, that is, on the open set $I_1$, $\psi(x)$ is a constant $b$.

Since $\psi'(x) = 0$, it follows from (2.27) that on the open set $I_1$, $\phi(x)$ is a constant $a$. Hence, (2.26) yields that on the open set $I_1$ for some constants $c$ and $d$

$$f(x) = ak(x) + c, \quad g(x) = bk(x) + d. \tag{2.39}$$

Thus, it follows from the definition of $k(x)$ that $k(x)$ is a root of the quadratic polynomial $q(t) = bt^2 + (d-a)t - c = 0$. If the quadratic polynomial $q(t) = bt^2 + (d-a)t - c = 0$ is nontrivial, then the ratio $k(x)$ must be constant on the open set $I_1$. This contradiction shows that $q(t)$ is trivial. That is, we have $b = 0$, $d = a$ and $c = 0$, and hence from (2.39) we get on the open set $I_1$

$$f(x) = ak(x), \quad g(x) = a. \tag{2.40}$$

This completes the proof of Lemma 2.2. □

## 3. Proof of Theorem 1.5

In this section, with the help of Lemmas in Section 2, we prove Theorem 1.5 stated in Section 1.

First, suppose that a positive nonconstant $C^2$ function $f(x)$ defined on an interval $I$ containing $0 \in \mathbb{R}$ satisfies $\bar{x}_L(x) = \bar{x}_A(x)$ for every nonzero real number $x$. Then for all $x \in I$ with $x \neq 0$, we have

$$\frac{\int_0^x tw(t)dt}{\int_0^x w(t)dt} = \frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}, \tag{3.1}$$

where we put

$$w(x) = \sqrt{1 + f'(x)^2}. \tag{3.2}$$

It follows from Lemma 2.1 that for some constant $k \in \mathbb{R}$ we have $f(x) = kw(x)$. Therefore Proposition 1.1 implies that for some $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x - c}{k}\right).$$

This completes the proof of (1) ⇒ (3).

Next, suppose that a nonconstant positive $C^2$ function $f(x)$ defined on an interval $I$ containing $0 \in \mathbb{R}$ satisfies $\bar{y}_L(x) = 2\bar{y}_A(x)$ for every nonzero real number $x$. Then for all $x \in I$ with $x \neq 0$, we have

$$\frac{\int_0^x f(t)w(t)dt}{\int_0^x w(t)dt} = \frac{\int_0^x f(t)^2dt}{\int_0^x f(t)dt}, \tag{3.3}$$

where $w(x)$ is given in (3.2). For the ratio $k(x) = w(x)/f(x)$, we consider the open set $I_1 = \{x \in I \mid k'(x) \neq 0\}$. Then, it follows from Lemma 2.2 that on the open set $I_1$, we have $f'(x) = 0$.

Suppose that the open set $I_1$ is not empty. Then on a fixed connected component $I_0$ of $I_1$, $f(x)$ is a constant $a$. Therefore, we have $w(x) = 1$, and hence on $I_0$, $k(x) = 1/a$. This contradiction shows that the open set $I_1$ must be empty. That is, the ratio $k(x)$ is a constant. Hence, for some constant $k \in \mathbb{R}$ we have $f(x) = kw(x)$. Therefore Proposition 1.1 implies that for some $c \in \mathbb{R}$,

$$f(x) = k \cosh\left(\frac{x - c}{k}\right).$$

This completes the proof of (2) ⇒ (3).

Conversely, it follows from Proposition 1.2 that (3) ⇒ (1) and (2). This completes the proof of Theorem 1.5.

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