Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 207–236
https://doi.org/10.4134/CKMS.c170121
pISSN: 1225-1763 / eISSN: 2234-3024

# RECURSION FORMULAS FOR $q$-HYPERGEOMETRIC AND $q$-APPELL SERIES

Vivek Sahai and Ashish Verma

Abstract. We obtain recursion formulas for $q$-hypergeometric and $q$-Appell series. We also find recursion formulas for the general double $q$-hypergeometric series. It is shown that these recursion relations can be expressed in terms of $q$-derivatives of the respective $q$-hypergeometric series.

## 1. Introduction

Recently, recursion formulas for multivariable hypergeometric functions have received considerable attention. The first paper in this direction was by Opps, Saad and Srivastava [3] giving the recursion formulas for the Appell function $F_2$, followed by the work of Wang [12], who studied the recursion relations of all Appell functions. The authors have carried out a systematic study of recursion formulas of multivariable hypergeometric functions including fourteen three variable Lauricella functions, three Srivastava's triple hypergeometric functions and four $k$-variable Lauricella functions [5] and Exton's triple hypergeometric functions [6]. The recursion formulas for the general triple hypergeometric function [10] were obtained in [7]. These results unified and generalized the results in [5] for the three variable hypergeometric function. In [8], recursion formulas for general Kampé de Fériet series and Srivastava and Daoust multivariable hypergeometric function [9] were presented.

In the present paper, we begin a study of recursion formulas satisfied by basic (or $q$-) hypergeometric series. In particular, we shall obtain the recursion formulas for generalized basic hypergeometric series ${}_r\phi_s$ and $q$-Appell series. We also find recursion formulas for the general double $q$-hypergeometric series [1]. The recursion formulas for three and higher variable $q$-series will be taken up in a subsequent paper.

Received March 25, 2017; Accepted August 4, 2017.
2010 *Mathematics Subject Classification.* Primary 33D15, 33D70.
*Key words and phrases.* recursion formulas, generalized basic hypergeometric series, $q$-Appell series, general double $q$-hypergeometric series.

The generalized basic (or $q$-) hypergeometric series is defined by [1]

$$
\begin{aligned}
&{}_r\phi_s(a_1,\dots,a_r;b_1,\dots,b_s;q,x)\\
(1)\qquad &= \sum_{m=0}^{\infty} \frac{(a_1;q)_m\cdots(a_r;q)_m}{(b_1;q)_m\cdots(b_s;q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\frac{x^m}{(q;q)_m},
\end{aligned}
$$

with $\binom{m}{2}=\frac{m(m-1)}{2}$, where $q\neq 0$ when $r>s+1$.

The series ${}_r\phi_s$ terminates if one of the numerator parameters is of the form $q^{-n}$, $n=0,1,2,\dots$ and $q\neq 0$. If $0<|q|<1$, the series ${}_r\phi_s$ converges absolutely for all $x$ if $r\le s$, and for $|x|<1$ if $r=s+1$. The series ${}_r\phi_s$ also converges absolutely if $|q|>1$ and $|x|<\frac{|b_1\cdots b_s|}{|a_1\cdots a_r|}$, unless it terminates.

Jackson introduced the $q$-analogues of the four Appell double hypergeometric series defined by [2]

$$(2)\qquad \Phi^{(1)}(a;b,b';c;q;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a;q)_{m_1+m_2}(b;q)_{m_1}(b';q)_{m_2}}{(q;q)_{m_1}(q;q)_{m_2}(c;q)_{m_1+m_2}}x^{m_1}y^{m_2};$$

$$(3)\qquad \Phi^{(2)}(a;b,b';c,c';q;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a;q)_{m_1+m_2}(b;q)_{m_1}(b';q)_{m_2}}{(q;q)_{m_1}(q;q)_{m_2}(c;q)_{m_1}(c';q)_{m_2}}x^{m_1}y^{m_2};$$

$$(4)\qquad \Phi^{(3)}(a,a';b,b';c;q;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a;q)_{m_1}(a';q)_{m_2}(b;q)_{m_1}(b';q)_{m_2}}{(q;q)_{m_1}(q;q)_{m_2}(c;q)_{m_1+m_2}}x^{m_1}y^{m_2};$$

$$(5)\qquad \Phi^{(4)}(a,b;c,c';q;x,y)=\sum_{m_1,m_2=0}^{\infty}\frac{(a;q)_{m_1+m_2}(b;q)_{m_1+m_2}}{(q;q)_{m_1}(q;q)_{m_2}(c;q)_{m_1}(c';q)_{m_2}}x^{m_1}y^{m_2}.$$

The series in (2)–(5) are absolutely convergent when $|q|,|x|,|y|<1$, by the comparison test with the series $\sum_{m_1,m_2=0}^{\infty}x^{m_1}y^{m_2}$, and their sums are called the $q$-Appell functions.

A $q$-analogue of the general double hypergeometric series is defined as follows [1]

$$
\begin{aligned}
(6)\qquad &\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}a_A : b_B; c_C\\ d_D : e_E; f_F\end{matrix}; q; x, y\right]\\
&= \sum_{m_1,m_2=0}^{\infty}\frac{(a_A;q)_{m_1+m_2}(b_B;q)_{m_1}(c_C;q)_{m_2}}{(d_D;q)_{m_1+m_2}(q,e_E;q)_{m_1}(q,f_F;q)_{m_2}}\\
&\quad\times\left[(-1)^{m_1+m_2}q^{\binom{m_1+m_2}{2}}\right]^{D-A}\left[(-1)^{m_1}q^{\binom{m_1}{2}}\right]^{1+E-B}\left[(-1)^{m_2}q^{\binom{m_2}{2}}\right]^{1+F-C}x^{m_1}y^{m_2},
\end{aligned}
$$

where $a_A$ abbreviates the array of $A$ parameters $a_1$, $a_2$,…, $a_A$, etc. and $q\neq 0$ when $\min(D-A,1+E-B,1+F-C)<0$. The series (6) converges absolutely for $|x|,|y|<1$ when $\min(D-A,1+E-B,1+F-C)\ge 0$ and $|q|<1$. The series in (6) is called the $q$-Kampé de Fériet series when $B=C$ and $E=F$.

## 2. Recursion formulas for ${}_r\phi_s$

In this section, we obtain recursion formulas for $q$-hypergeometric series ${}_r\phi_s$ given by (1). Following abbreviated notations are used. We, for example, write ${}_r\phi_s$ for the generalized $q$-hypergeometric series ${}_r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x)$ and ${}_r\phi_s(a_lq^{n_1},xq^{n_2})$ stands for the series ${}_r\phi_s(a_1,\ldots,a_{l-1},a_lq^{n_1},a_{l+1},\ldots,a_r;$ $b_1,\ldots,b_s;q,xq^{n_2})$, etc. Throughout the paper, the symbol $n$ denotes a non-negative integer.

**Theorem 2.1.** *The following recursion formulas hold true for the numerator parameter $a_l$, $l=1,\ldots,r$, of the basic hypergeometric series ${}_r\phi_s$:*

$$\begin{aligned}{}_r\phi_s(a_lq^n) = {}_r\phi_s + (-1)^{1+s-r}a_lx\frac{\prod_{i=1,i\neq l}^r(a_i;q)_1}{\prod_{i=1}^s(b_i;q)_1}\sum_{n_1=1}^{n}q^{n_1-1}\\ \times {}_r\phi_s(a_1q,\ldots,a_{l-1}q,a_lq^{n_1},a_{l+1}q,\ldots,a_rq,b_1q,\ldots,b_sq,xq^{1+s-r}),\end{aligned} \tag{7}$$

$$\begin{aligned}{}_r\phi_s(a_lq^{-n}) = {}_r\phi_s - (-1)^{1+s-r}a_lx\frac{\prod_{i=1,i\neq l}^r(a_i;q)_1}{\prod_{i=1}^s(b_i;q)_1}\sum_{n_1=1}^{n}q^{-n_1}\\ \times {}_r\phi_s(a_1q,\ldots,a_{l-1}q,a_lq^{-n_1+1},a_{l+1}q,\ldots,a_rq,b_1q,\ldots,\\ b_sq,xq^{1+s-r}).\end{aligned} \tag{8}$$

*Proof.* From definition of the $q$-hypergeometric series ${}_r\phi_s$ and the relation

$$(a_lq;q)_m = \left[1+\frac{a_l(1-q^m)}{1-a_l}\right](a_l;q)_m, \tag{9}$$

we get the following contiguous relation:

$$\begin{aligned}{}_r\phi_s(a_lq) = {}_r\phi_s + (-1)^{1+s-r}a_lx\frac{\prod_{i=1,i\neq l}^r(a_i;q)_1}{\prod_{i=1}^s(b_i;q)_1}\\ \times {}_r\phi_s(a_1q,\ldots,a_rq,b_1q,\ldots,b_sq,xq^{1+s-r}).\end{aligned} \tag{10}$$

Replacing $a_l$ by $a_lq$ in (10) and using (10), we have

$$\begin{aligned}{}_r\phi_s(a_lq^2) = {}_r\phi_s + (-1)^{1+s-r}a_lx\frac{\prod_{i=1,i\neq l}^r(a_i;q)_1}{\prod_{i=1}^s(b_i;q)_1}\sum_{n_1=1}^{2}q^{n_1-1}\\ \times {}_r\phi_s(a_1q,\ldots,a_{l-1}q,a_lq^{n_1},a_{l+1}q,\ldots,a_rq,b_1q,\ldots,b_sq,xq^{1+s-r}).\end{aligned} \tag{11}$$

Iterating this process $n$ times, we get (7). For the proof of (8) replace the parameter $a_l$ by $a_lq^{-1}$ in (10). This gives

$$\begin{aligned}{}_r\phi_s(a_lq^{-1}) = {}_r\phi_s - (-1)^{1+s-r}a_lx\frac{\prod_{i=1,i\neq l}^r(a_i;q)_1}{\prod_{i=1}^s(b_i;q)_1}q^{-1}\\ \times {}_r\phi_s(a_1q,\ldots,a_{l-1}q,a_{l+1}q,\ldots,a_rq,b_1q,\ldots,b_sq,xq^{1+s-r}).\end{aligned} \tag{12}$$

Iteratively, we get (8). □

Note that the contiguous relation given by Eqs. (10) and (12) were obtained by Swarttouw [11].

**Corollary 2.2.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-hypergeometric series ${}_2\phi_1(a,b;c;q,x)$:*

$$(13)\qquad {}_2\phi_1(aq^n,b;c;q,x) = {}_2\phi_1(a,b;c;q,x) + ax\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,{}_2\phi_1(aq^{n_1},bq;cq;q,x),$$

$$(14)\qquad {}_2\phi_1(aq^{-n},b;c;q,x) = {}_2\phi_1(a,b;c;q,x) - ax\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,{}_2\phi_1(aq^{-n_1+1},bq;cq;q,x).$$

Using contiguous relations (10) and (12), we get another form of the recursion formulas for ${}_r\phi_s$.

**Theorem 2.3.** *The following recursion formulas hold true for the numerator parameter $a_l$, $l=1,\dots,r$, of the basic hypergeometric series ${}_r\phi_s$:*

$$(15)\qquad {}_r\phi_s(a_l\,q^n) = \sum_{n_1=0}^{n}\begin{bmatrix} n\\ n_1\end{bmatrix}_q \frac{\prod_{i=1,i\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}} x^{n_1}a_l^{n_1}q^{n_1(n_1-1)}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+s-r} \times {}_r\phi_s(a_1q^{n_1},\dots,a_rq^{n_1},b_1q^{n_1},\dots,b_sq^{n_1},xq^{n_1(1+s-r)}),$$

$$(16)\qquad {}_r\phi_s(a_l\,q^{-n}) = \sum_{n_1=0}^{n}\begin{bmatrix} n\\ n_1\end{bmatrix}_{q^{-1}} \frac{\prod_{i=1,\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}} (-x)^{n_1}a_l^{n_1}q^{-\frac{n_1(n_1+1)}{2}}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+s-r} \times {}_r\phi_s(a_1q^{n_1},\dots,a_{l-1}q^{n_1},a_{l+1}q^{n_1},\dots,a_rq^{n_1},b_1q^{n_1},\dots,b_sq^{n_1},xq^{n_1(1+s-r)}),$$

*where $\left[\begin{smallmatrix} n\\ n_1\end{smallmatrix}\right]_q$ is the $q$-binomial coefficient defined by* [1]

$$\begin{bmatrix} n\\ n_1\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n-n_1}}$$

*for $n_1=0,1,\dots,n$.*

*Proof.* We prove (15) by applying mathematical induction on $n$. For $n=1$, the result (15) is true obviously. Assuming (15) is true for $n=m$, that is,

$$(17)\qquad {}_r\phi_s(a_l\,q^m) = \sum_{n_1=0}^{m}\begin{bmatrix} m\\ n_1\end{bmatrix}_q \frac{\prod_{i=1,i\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}} x^{n_1}a_l^{n_1}q^{n_1(n_1-1)}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+s-r} \times {}_r\phi_s(a_1q^{n_1},\dots,a_rq^{n_1},b_1q^{n_1},\dots,b_sq^{n_1},xq^{n_1(1+s-r)}).$$

Replacing $a_l \to a_l q$ in (17) and using the contiguous relation (10), we get

$$(18) \quad {}_r\phi_s(a_l\, q^{m+1})$$
$$= \sum_{n_1=0}^{m} \begin{bmatrix} m \\ n_1 \end{bmatrix}_q \frac{\prod_{i=1,i\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}} x^{n_1} a_l^{n_1} q^{n_1^2} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r}$$
$$\times \Big\{ {}_r\phi_s(a_1q^{n_1},\ldots,a_rq^{n_1},b_1q^{n_1},\ldots,b_sq^{n_1},xq^{n_1(1+s-r)})$$
$$+ (-1)^{1+s-r} a_l q^{n_1} x q^{n_1(1+s-r)} \frac{\prod_{i=1,i\neq l}^{r}(a_iq^{n_1};q)_1}{\prod_{i=1}^{s}(b_iq^{n_1};q)_1} {}_r\phi_s(a_1q^{n_1+1},\ldots,$$
$$a_rq^{n_1+1}, b_1q^{n_1+1},\ldots,b_sq^{n_1+1},xq^{(n_1+1)(1+s-r)})\Big\}.$$

Simplifying, (18) takes the form

$$(19) \quad {}_r\phi_s(a_l\, q^{m+1})$$
$$= \sum_{n_1=0}^{m+1} \left( \begin{bmatrix} m \\ n_1 \end{bmatrix}_q q^{n_1} + \begin{bmatrix} m \\ n_1-1 \end{bmatrix}_q \right) \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r} a_l^{n_1} x^{n_1} \frac{\prod_{i=1,\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}}$$
$$\times\, q^{n_1(n_1-1)} {}_r\phi_s(a_1q^{n_1},\ldots,a_rq^{n_1},b_1q^{n_1},\ldots,b_sq^{n_1},xq^{n_1(1+s-r)}).$$

Using $q$-Pascal identity [1]

$$\begin{bmatrix} m \\ n_1 \end{bmatrix}_q q^{n_1} + \begin{bmatrix} m \\ n_1-1 \end{bmatrix}_q = \begin{bmatrix} m+1 \\ n_1 \end{bmatrix}_q$$

in (19), we get

$$(20) \quad {}_r\phi_s(a_l\, q^{m+1})$$
$$= \sum_{n_1=0}^{m+1} \begin{bmatrix} m+1 \\ n_1 \end{bmatrix}_q \frac{\prod_{i=1,\neq l}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}} x^{n_1} a_l^{n_1} q^{n_1(n_1-1)} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r}$$
$$\times\, {}_r\phi_s(a_1q^{n_1},\ldots,a_rq^{n_1},b_1q^{n_1},\ldots,b_sq^{n_1},xq^{n_1(1+s-r)}).$$

This establishes (15) for $n = m+1$. Hence result (15) is true for all values of $n$. The second formula (16) is proved in a similar manner. □

**Corollary 2.4.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-hypergeometric series ${}_2\phi_1(a,b;c;q,x)$:*

$$(21) \quad {}_2\phi_1(aq^n,b;c;q,x)$$
$$= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(b;q)_{n_1} x^{n_1} a^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}} {}_2\phi_1(aq^{n_1},bq^{n_1};cq^{n_1};q,x),$$

$$(22) \quad {}_2\phi_1(aq^{-n},b;c;q,x)$$
$$= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1} (-x)^{n_1} a^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}} {}_2\phi_1(a,bq^{n_1};cq^{n_1};q,x).$$

As the $q$-series ${}_2\phi_1$ is symmetric for the numerator parameters $a$ and $b$, interchanging $a$ and $b$ of the above corollary, will give recursion formulas for numerator parameter $b$.

**Theorem 2.5.** *The following recursion formulas hold true for the denominator parameter $b_l$, $l = 1, \dots, s$, of the basic hypergeometric series ${}_r\phi_s$:*

$$\begin{aligned}(23)\quad & {}_r\phi_s(b_l q^{-n}) \\ & = {}_r\phi_s + (-1)^{1+s-r} x b_l \frac{\prod_{i=1}^{r}(1-a_i)}{\prod_{i=1,\neq l}^{s}(1-b_i)} \sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-b_l)(q^{n_1-1}-b_l)} \\ & \quad \times {}_r\phi_s(a_1q, \dots, a_rq, b_1q, \dots, b_{l-1}q, b_lq^{2-n_1}, b_{l+1}q, \dots, b_sq, xq^{1+s-r}).\end{aligned}$$

*Equivalently,*

$$\begin{aligned}{}_r\phi_s(b_lq^{-n}) = & \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{\prod_{i=1}^{r}(a_i;q)_{n_1} x^{n_1} b_l^{n_1} q^{-n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}(b_lq^{-n};q)_{n_1}} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r} \\ (24)\quad & \times {}_r\phi_s(a_1q^{n_1}, \dots, a_rq^{n_1}, b_1q^{n_1}, \dots, b_sq^{n_1}, xq^{n_1(1+s-r)}).\end{aligned}$$

*Proof.* Using the definition of basic hypergeometric series and the relation

$$\frac{1}{(b_lq^{-1};q)_m} = \left[1 + \frac{b_l(1-q^m)}{q-b_l}\right] \frac{1}{(b_l;q)_m},$$

we can easily obtain the following contiguous relation:

$$\begin{aligned}{}_r\phi_s(b_lq^{-1}) = {}_r\phi_s + (-1)^{1+s-r} x b_l & \frac{\prod_{i=1}^{r}(1-a_i)}{\prod_{i=1}^{s}(1-b_i)(q-b_l)} \\ (25)\quad & \times {}_r\phi_s(a_1q, \dots, a_rq, b_1q, \dots, b_sq, xq^{1+s-r}).\end{aligned}$$

Again replacing $b_l \to b_lq^{-1}$ in (25) and using (25), we get

$$\begin{aligned}{}_r\phi_s(b_lq^{-2}) = {}_r\phi_s + (-1)^{1+s-r} x b_l \frac{\prod_{i=1}^{r}(1-a_i)}{\prod_{i=1,i\neq l}^{s}(1-b_i)} \sum_{n_1=1}^{2} \frac{q^{n_1-1}}{(q^{n_1}-b_l)(q^{n_1-1}-b_l)} \\ (26)\quad \times {}_r\phi_s(a_1q, \dots, a_rq, b_1q, \dots, b_{l-1}q, b_lq^{2-n_1}, b_{l+1}q, \dots, b_sq, xq^{1+s-r}).\end{aligned}$$

Iterating this method on ${}_r\phi_s$ with $b_lq^{-n}$ for $n$ times, we get (23).

We give a proof of (24) using mathematical induction. For $n = 1$, the formula (24) is valid. Assume that the result (24) is true for $n = m$, that is,

$$\begin{aligned}{}_r\phi_s(b_lq^{-m}) = & \sum_{n_1=0}^{m} \begin{bmatrix} m \\ n_1 \end{bmatrix}_{q^{-1}} \frac{\prod_{i=1}^{r}(a_i;q)_{n_1} x^{n_1} b_l^{n_1} q^{-n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}(b_lq^{-m};q)_{n_1}} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r} \\ (27)\quad & \times {}_r\phi_s(a_1q^{n_1}, \dots, a_rq^{n_1}, b_1q^{n_1}, \dots, b_sq^{n_1}, xq^{n_1(1+s-r)}).\end{aligned}$$

Replacing $b_l$ by $b_lq^{-1}$ in the above relation and using (25), we get

$$(28)\quad {}_r\phi_s(b_lq^{-m-1})$$

$$= \sum_{n_1=0}^{m} \begin{bmatrix} m \\ n_1 \end{bmatrix}_{q^{-1}} \frac{\prod_{i=1}^{r}(a_i;q)_{n_1} x^{n_1} b_l^{n_1} q^{-2n_1}}{\prod_{i=1,i\neq l}^{s}(b_i;q)_{n_1}(b_l q^{-1};q)_{n_1}(b_l q^{-m-1};q)_{n_1}}$$
$$\times \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r} {}_r\phi_s(a_1 q^{n_1},\ldots,a_r q^{n_1}, b_1 q^{n_1},\ldots,b_s q^{n_1}, x q^{n_1(1+s-r)})$$
$$+ \sum_{n_1=0}^{m} \begin{bmatrix} m \\ n_1 \end{bmatrix}_{q^{-1}} \frac{\prod_{i=1}^{r}(a_i;q)_{n_1+1} x^{n_1+1} b_l^{n_1+1} q^{-n_1-1}}{\prod_{i=1,i\neq l}^{s}(b_i;q)_{n_1+1}(b_l q^{-1};q)_{n_1+2}(b_l q^{-m-1};q)_{n_1}}$$
$$\times \left[(-1)^{n_1+1} q^{\binom{n_1+1}{2}}\right]^{1+s-r} {}_r\phi_s(a_1 q^{n_1+1},\ldots,a_r q^{n_1+1}, b_1 q^{n_1+1},\ldots,b_s q^{n_1+1},$$
$$x q^{(n_1+1)(1+s-r)}).$$

This gives

(29) $${}_r\phi_s(b_l q^{-m-1})$$
$$= {}_r\phi_s + \sum_{k=1}^{m} \frac{(q^{-1};q^{-1})_m}{(q^{-1};q^{-1})_k (q^{-1};q^{-1})_{m+1-k}} \left(\frac{q^{-k}(1-q^{-m+k-1})}{1-b_l q^{-m+k-2}} + \frac{1-q^{-k}}{1-b_l q^{k-1}}\right)$$
$$\times \frac{\prod_{i=1}^{r}(a_i;q)_k\, x^k\, b_l^k\, q^{-k}}{\prod_{i=1,i\neq l}^{s}(b_i;q)_k (b_l q^{-1};q)_k (b_l q^{-m-1};q)_{k-1}} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}$$
$$\times {}_r\phi_s(a_1 q^k,\ldots,a_r q^k, b_1 q^k,\ldots,b_s q^k, x q^{k(1+s-r)})$$
$$+ \frac{\prod_{i=1}^{r}(a_i;q)_{m+1} x^{m+1} b_l^{m+1} q^{-m-1}}{\prod_{i=1}^{s}(b_i;q)_{m+1}(b_l q^{-m-1};q)_{m+1}} \left[(-1)^{m+1} q^{\binom{m+1}{2}}\right]^{1+s-r}$$
$$\times {}_r\phi_s(a_1 q^{m+1},\ldots,a_r q^{m+1}, b_1 q^{m+1},\ldots,b_s q^{m+1}, x q^{(m+1)(1+s-r)}).$$
$$= \sum_{n_1=0}^{m+1} \begin{bmatrix} m+1 \\ n_1 \end{bmatrix}_{q^{-1}} \frac{\prod_{i=1}^{r}(a_i;q)_{n_1} x^{n_1} b_l^{n_1} q^{-n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}(b_l q^{-m-1};q)_{n_1}} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r}$$
$$\times {}_r\phi_s(a_1 q^{n_1},\ldots,a_r q^{n_1}, b_1 q^{n_1},\ldots,b_s q^{n_1}, x q^{n_1(1+s-r)}).$$

This proves (24) for $n = m+1$. □

Note that the contiguous relation (25) was obtained by Swarttouw [11].

**Corollary 2.6.** *The following recursion formulas hold true for the denominator parameter $c$ of the $q$-hypergeometric series ${}_2\phi_1(a,b;c;q,x)$:*

(30) $${}_2\phi_1(a,b;cq^{-n};q,x)$$
$$= {}_2\phi_1(a,b;c;q,x)$$
$$+ cx(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\, {}_2\phi_1(aq,bq;cq^{2-n_1};q,x).$$

*Equivalently,*

(31) $${}_2\phi_1(a,b;cq^{-n};q,x)$$

$$= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1}(b;q)_{n_1} x^{n_1} c^{n_1} q^{-n_1}}{(c;q)_{n_1}(cq^{-n};q)_{n_1}} \, {}_2\phi_1(aq^{n_1}, bq^{n_1}; cq^{n_1}; q, x).$$

## 3. Recursion formulas for $q$-Appell series

In this section, we obtain recursion formulas for $q$-Appell series. Following abbreviated notations are used. We, for example, write $\Phi^{(1)}$ for the series $\Phi^{(1)}(a;b,b';c;q;x,y)$, $\Phi^{(1)}(aq^{n_1})$ for the series $\Phi^{(1)}(aq^{n_1};b,b';c;q;x,y)$ and $\Phi^{(1)}(aq^{n_1},q^{n_2}x)$ stands for the series $\Phi^{(1)}(aq^{n_1};b,b';c;q;q^{n_2}x,y)$, etc. Throughout, $N_2$ stands for $n_1+n_2$.

**Recursion formulas for $\Phi^{(1)}$:**

**Theorem 3.1.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(1)}$:*

$$\Phi^{(1)}(aq^n) = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(1)}(aq^{n_1}, bq, cq, qy)$$

$$+ a\,y\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(1)}(aq^{n_1}, b'q, cq). \tag{32}$$

*Equivalently,*

$$\Phi^{(1)}(aq^n) = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(1)}(aq^{n_1}, bq, cq)$$

$$+ a\,y\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(1)}(aq^{n_1}, b'q, cq, qx). \tag{33}$$

*Also*

$$\Phi^{(1)}(aq^{-n}) = \Phi^{(1)} - a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(1)}(aq^{1-n_1}, bq, cq, qy)$$

$$- a\,y\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(1)}(aq^{1-n_1}, b'q, cq). \tag{34}$$

*Equivalently,*

$$\Phi^{(1)}(aq^{-n}) = \Phi^{(1)} - a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(1)}(aq^{1-n_1}, bq, cq)$$

$$- a\,y\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(1)}(aq^{1-n_1}, b'q, cq, qx). \tag{35}$$

*Proof.* From the definition of $q$-Appell function $\Phi^{(1)}$ and the relation

$$(aq;q)_{m_1+m_2} = \left[1 + \frac{aq^{m_2}(1-q^{m_1})}{1-a} + \frac{a(1-q^{m_2})}{1-a}\right](a;q)_{m_1+m_2} \tag{36}$$

we have the following contiguous relation:

$$\begin{aligned}\Phi^{(1)}(aq) \\ = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\Phi^{(1)}(aq, bq, cq, qy) + a\,y\frac{1-b'}{1-c}\Phi^{(1)}(aq, b'q, cq).\end{aligned} \tag{37}$$

To calculate $\Phi^{(1)}(aq^2)$, we replace $a \to aq$ in (37) and again use (37). This gives

$$\begin{aligned}\Phi^{(1)}(aq^2) = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\left[\Phi^{(1)}(aq, bq, cq, qy) + \ q\,\Phi^{(1)}(aq^2, bq, cq, qy)\right] \\ + a\,y\frac{1-b'}{1-c}\left[\Phi^{(1)}(aq, b'q, cq) + q\,\Phi^{(1)}(aq^2, b'q, cq)\right].\end{aligned} \tag{38}$$

Iterating this technique $n$ times on $\Phi^{(1)}$ with parameter $aq^n$, we obtain (32).

Next, we prove the result (33). Using the definition of $\Phi^{(1)}$ with the following result

$$(aq;q)_{m_1+m_2} = \left[1 + \frac{a(1-q^{m_1})}{1-a} + \frac{aq^{m_1}(1-q^{m_2})}{1-a}\right](a;q)_{m_1+m_2} \tag{39}$$

we get the following contiguous relation:

$$\begin{aligned}\Phi^{(1)}(aq) \\ = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\Phi^{(1)}(aq, bq, cq) + a\,y\frac{1-b'}{1-c}\Phi^{(1)}(aq, b'q, cq, qx).\end{aligned} \tag{40}$$

Now, replacing $a \to aq$ in (40) and using (40), we get the following result for $\Phi^{(1)}(aq^2)$:

$$\begin{aligned}\Phi^{(1)}(aq^2) = \Phi^{(1)} + a\,x\frac{1-b}{1-c}\left[\Phi^{(1)}(aq, bq, cq) + \ q\,\Phi^{(1)}(aq^2, bq, cq)\right] \\ + a\,y\frac{1-b'}{1-c}\left[\Phi^{(1)}(aq, b'q, cq, qx) + q\,\Phi^{(1)}(aq^2, b'q, cq, qx)\right].\end{aligned} \tag{41}$$

Iteratively, we get (33).

For the proof of (34), we replace $a \to aq^{-1}$ in (37). This gives the following contiguous relation:

$$\begin{aligned}\Phi^{(1)}(aq^{-1}) \\ = \Phi^{(1)} - aq^{-1}\,x\frac{1-b}{1-c}\Phi^{(1)}(bq, cq, qy) - \ aq^{-1}\,y\frac{1-b'}{1-c}\Phi^{(1)}(b'q, cq).\end{aligned} \tag{42}$$

Applying the above contiguous relation for $n$ times, we get (34).

Similarly, for the proof of (35), we replace $a \to aq^{-1}$ in (40) and obtain the following contiguous relation:

$$\begin{aligned}\Phi^{(1)}(aq^{-1}) \\ = \Phi^{(1)} - aq^{-1}\,x\frac{1-b}{1-c}\Phi^{(1)}(bq, cq) - \ aq^{-1}\,y\frac{1-b'}{1-c}\Phi^{(1)}(b'q, cq, qx).\end{aligned} \tag{43}$$

Iterating this technique $n$ times on $\Phi^{(1)}$ with parameter $aq^{-n}$, we obtain (35). □

Using contiguous relations (37) and (42), we get another form of recursion formulas for $\Phi^{(1)}$.

**Theorem 3.2.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(1)}$:*

$$\Phi^{(1)}(aq^n) = \sum_{N_2\le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{N_2}}\, q^{N_2(N_2-1)} \times \Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y), \tag{44}$$

$$\Phi^{(1)}(aq^{-n}) = \sum_{N_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1}(b';q)_{n_2}\,(-x)^{n_1}(-y)^{n_2}a^{N_2}}{(c;q)_{N_2}} \times q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(1)}(bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y), \tag{45}$$

*where summation runs over all $n_i \ge 0$, $i = 1, 2$, such that $n_1 + n_2 \le n$, and*

$$\begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n-N_2}}$$

*is the $q$-multinomial coefficient* [1].

*Proof.* The proof of (44) is by mathematical induction on $n \in \mathbb{N}$. For $n = 1$, (44) is true. Assuming (44) is true for $n = m$, that is,

$$\Phi^{(1)}(aq^m) = \sum_{N_2\le m} \begin{bmatrix} m \\ n_1, n_2 \end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{N_2}}\, q^{N_2(N_2-1)} \times \Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y), \tag{46}$$

where summation runs over all $n_i \ge 0$, $i = 1, 2$, such that $n_1 + n_2 \le m$.

Replacing $a \to aq$ in (46) and using the contiguous relation (37), we get

$$\begin{aligned}\Phi^{(1)}(aq^{m+1}) = \sum_{N_2\le m} &\begin{bmatrix} m \\ n_1, n_2 \end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{N_2}}\, q^{N_2^2} \\ &\times \Big\{ \Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y) + xaq^{N_2}\frac{(1-bq^{n_1})}{(1-cq^{N_2})} \\ &\times \Phi^{(1)}(aq^{N_2+1}, bq^{n_1+1}, b'q^{n_2}, cq^{N_2+1}, q^{n_1+1}y) + y\,a\,q^{N_2+n_1} \\ &\times \frac{(1-b'q^{n_2})}{(1-cq^{N_2})}\,\Phi^{(1)}(aq^{N_2+1}, bq^{n_1}, b'q^{n_2+1}, cq^{N_2+1}, q^{n_1}y)\Big\}.\end{aligned} \tag{47}$$

Simplifying, (47) takes the form,

$$\Phi^{(1)}(aq^{m+1}) = \sum_{n_1+n_2\le m+1} \frac{(q;q)_m(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{m+1-N_2}(c;q)_{N_2}}\,(1-q^{m+1-N_2})$$

$$\times\, q^{N_2^2}\Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y)$$
$$+ \sum_{n_1+n_2\le m+1} \frac{(q;q)_m (b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{m+1-N_2}(c;q)_{N_2}}(1-q^{n_1})$$
$$\times\, q^{N_2(N_2-1)}\Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y)$$
$$+ \sum_{n_1+n_2\le m+1} \frac{(q;q)_m (b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{m+1-N_2}(c;q)_{N_2}}(1-q^{n_2})$$
$$\times\, q^{N_2^2-n_2}\,\Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y). \tag{48}$$

After some manipulation, we get

$$\Phi^{(1)}(aq^{m+1}) = \sum_{n_1+n_2\le m+1} \begin{bmatrix} m+1 \\ n_1, n_2 \end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{N_2}}\, q^{N_2(N_2-1)}$$
$$\times\, \Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_1}y). \tag{49}$$

This establishes (44) for $n = m+1$. Hence by induction result given in (44) is true for all values of $n$. The second recursion formula (45) can be proved in a similar manner. □

Note that if we use contiguous relations (40) and (42), we get the following form of the recursion formulas for $\Phi^{(1)}$.

**Theorem 3.3.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(1)}$:*

$$\Phi^{(1)}(aq^{n}) = \sum_{N_2\le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{N_2}}\, q^{N_2(N_2-1)}$$
$$\times\, \Phi^{(1)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_2}x), \tag{50}$$

$$\Phi^{(1)}(aq^{-n}) = \sum_{N_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1}(b';q)_{n_2}\,(-x)^{n_1}(-y)^{n_2}a^{N_2}}{(c;q)_{N_2}}$$
$$\times\, q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(1)}(bq^{n_1}, b'q^{n_2}, cq^{N_2}, q^{n_2}x), \tag{51}$$

*where summation runs over all $n_i \ge 0$, $i = 1, 2$, such that $n_1 + n_2 \le n$.*

We now present recursion formulas for the remaining numerator parameters $b$, $b'$ of the $q$-Appell function $\Phi^{(1)}$. We omit the proof of the given below theorems.

**Theorem 3.4.** *The following recursion formulas hold true for the numerator parameters $b$, $b'$ of the $q$-Appell function $\Phi^{(1)}$:*

$$\Phi^{(1)}(bq^{n}) = \Phi^{(1)} + bx\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(1)}(aq, bq^{n_1}, cq), \tag{52}$$

$$\Phi^{(1)}(bq^{-n}) = \Phi^{(1)} - bx\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(1)}(aq, bq^{-n_1+1}, cq), \tag{53}$$

$$\Phi^{(1)}(b'q^{n}) = \Phi^{(1)} + b'y\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(1)}(aq, b'q^{n_1}, cq), \tag{54}$$

$$\Phi^{(1)}(b'q^{-n}) = \Phi^{(1)} - b'y\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(1)}(aq, b'q^{-n_1+1}, cq). \tag{55}$$

Here we establish another form of the above theorem.

**Theorem 3.5.** *The following recursion formulas hold true for the numerator parameters $b$, $b'$ of the $q$-Appell function $\Phi^{(1)}$:*

$$\Phi^{(1)}(bq^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(a;q)_{n_1} x^{n_1} b^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}}\,\Phi^{(1)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{56}$$

$$\Phi^{(1)}(bq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1} (-x)^{n_1} b^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}}\,\Phi^{(1)}(aq^{n_1}, cq^{n_1}), \tag{57}$$

$$\Phi^{(1)}(b'q^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(a;q)_{n_1} y^{n_1} b'^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}}\,\Phi^{(1)}(aq^{n_1}, b'q^{n_1}, cq^{n_1}), \tag{58}$$

$$\Phi^{(1)}(b'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1} (-y)^{n_1} b'^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}}\,\Phi^{(1)}(aq^{n_1}, cq^{n_1}). \tag{59}$$

**Theorem 3.6.** *The following recursion formulas hold true for the denominator parameter $c$ of the $q$-Appell function $\Phi^{(1)}$:*

$$\begin{aligned} &\Phi^{(1)}(cq^{-n}) \\ &= \Phi^{(1)} + cx(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(1)}(aq, bq, cq^{2-n_1}) \\ &\quad + cy(1-a)(1-b')\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(1)}(aq, b'q, cq^{2-n_1}, q^{n_1}x). \end{aligned} \tag{60}$$

*Equivalently,*

$$\begin{aligned} &\Phi^{(1)}(cq^{-n}) \\ &= \Phi^{(1)} + cx(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(1)}(aq, bq, cq^{2-n_1}, q^{n_1}y) \\ &\quad + cy(1-a)(1-b')\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(1)}(aq, b'q, cq^{2-n_1}). \end{aligned} \tag{61}$$

*Proof.* Using the definition of $q$-Appell function $\Phi^{(1)}$ and the relation

$$\frac{1}{(cq^{-1};q)_{m_1+m_2}} = \left[1+\frac{c(1-q^{m_1})}{q-c}+\frac{cq^{m_1}(1-q^{m_2})}{q-c}\right]\frac{1}{(c;q)_{m_1+m_2}}$$

we get the following contiguous relation:

$$\begin{aligned}\Phi^{(1)}(cq^{-1}) = \Phi^{(1)} &+ cx(1-a)(1-b)\frac{1}{(q-c)(1-c)}\Phi^{(1)}(aq,bq,cq)\\ &+ cy(1-a)(1-b')\frac{1}{(q-c)(1-c)}\Phi^{(1)}(aq,b'q,cq,qx).\end{aligned} \tag{62}$$

Iteratively, we get (60).

By using the definition of $q$-Appell function $\Phi^{(1)}$ and the relation

$$\frac{1}{(cq^{-1};q)_{m_1+m_2}} = \left[1+\frac{c(1-q^{m_2})}{q-c}+\frac{cq^{m_2}(1-q^{m_1})}{q-c}\right]\frac{1}{(c;q)_{m_1+m_2}},$$

we get (61). □

**Recursion formulas for $\boldsymbol{\Phi^{(2)}}$:** Now, we present the recursion formulas for the numerator parameters $a$, $b$ and $b'$ of the $q$-Appell function $\Phi^{(2)}$. We omit the proofs of the following theorems.

**Theorem 3.7.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(2)}$:*

$$\begin{aligned}\Phi^{(2)}(aq^n) = \Phi^{(2)} &+ a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(2)}(aq^{n_1},bq,cq,qy)\\ &+ a\,y\frac{1-b'}{1-c'}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(2)}(aq^{n_1},b'q,c'q).\end{aligned} \tag{63}$$

*Equivalently,*

$$\begin{aligned}\Phi^{(2)}(aq^n) = \Phi^{(2)} &+ a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(2)}(aq^{n_1},bq,cq)\\ &+ a\,y\frac{1-b'}{1-c'}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(2)}(aq^{n_1},b'q,c'q,qx).\end{aligned} \tag{64}$$

*Also*

$$\begin{aligned}\Phi^{(2)}(aq^{-n}) = \Phi^{(2)} &- a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(2)}(aq^{1-n_1},bq,cq,qy)\\ &- a\,y\frac{1-b'}{1-c'}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(2)}(aq^{1-n_1},b'q,c'q).\end{aligned} \tag{65}$$

*Equivalently,*

$$\Phi^{(2)}(aq^{-n}) = \Phi^{(2)} - a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(2)}(aq^{1-n_1}, bq, cq)$$

$$- a\,y\frac{1-b'}{1-c'}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(2)}(aq^{1-n_1}, b'q, c'q, qx). \tag{66}$$

**Theorem 3.8.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(2)}$:*

$$\Phi^{(2)}(aq^{n}) = \sum_{n_1+n_2\leq n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}\, q^{N_2(N_2-1)}$$

$$\times\, \Phi^{(2)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{n_1}, c'q^{n_2}, q^{n_1}y). \tag{67}$$

*Equivalently,*

$$\Phi^{(2)}(aq^{n}) = \sum_{n_1+n_2\leq n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_q \frac{(b;q)_{n_1}(b';q)_{n_2}\, x^{n_1}y^{n_2}a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}\, q^{N_2(N_2-1)}$$

$$\times\, \Phi^{(2)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, cq^{n_1}, c'q^{n_2}, q^{n_2}x). \tag{68}$$

*Also*

$$\Phi^{(2)}(aq^{-n}) = \sum_{n_1+n_2\leq n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1}(b';q)_{n_2}\,(-x)^{n_1}(-y)^{n_2}a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}$$

$$\times\, q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(2)}(bq^{n_1}, b'q^{n_2}, cq^{n_1}, c'q^{n_2}, q^{n_1}y). \tag{69}$$

*Equivalently,*

$$\Phi^{(2)}(aq^{-n}) = \sum_{n_1+n_2\leq n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1}(b';q)_{n_2}\,(-x)^{n_1}(-y)^{n_2}a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}$$

$$\times\, q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(2)}(bq^{n_1}, b'q^{n_2}, cq^{n_1}, c'q^{n_2}, q^{n_2}x), \tag{70}$$

*where summation runs over all $n_i \geq 0$, $i = 1, 2$, such that $N_2 \leq n$.*

**Theorem 3.9.** *The following recursion formulas hold true for the numerator parameters $b$, $b'$ of the $q$-Appell function $\Phi^{(2)}$:*

$$\Phi^{(2)}(bq^{n}) = \Phi^{(2)} + bx\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(2)}(aq, bq^{n_1}, cq), \tag{71}$$

$$\Phi^{(2)}(bq^{-n}) = \Phi^{(2)} - bx\,\frac{1-a}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(2)}(aq, bq^{-n_1+1}, cq), \tag{72}$$

$$\Phi^{(2)}(b'q^{n}) = \Phi^{(2)} + b'y\,\frac{1-a}{1-c'}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(2)}(aq, b'q^{n_1}, c'q), \tag{73}$$

$$\Phi^{(2)}(b'q^{-n}) = \Phi^{(2)} - b'y\,\frac{1-a}{1-c'} \sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(2)}(aq, b'q^{-n_1+1}, c'q). \tag{74}$$

Here we establish another form of the above theorem.

**Theorem 3.10.** *The following recursion formulas hold true for the numerator parameters $b$, $b'$ of the $q$-Appell function $\Phi^{(2)}$:*

$$\Phi^{(2)}(bq^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(a;q)_{n_1} x^{n_1} b^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{75}$$

$$\Phi^{(2)}(bq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1} (-x)^{n_1} b^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, cq^{n_1}), \tag{76}$$

$$\Phi^{(2)}(b'q^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(a;q)_{n_1} y^{n_1} b'^{n_1} q^{n_1(n_1-1)}}{(c';q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, b'q^{n_1}, c'q^{n_1}), \tag{77}$$

$$\Phi^{(2)}(b'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1} (-y)^{n_1} b'^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c';q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, c'q^{n_1}). \tag{78}$$

**Theorem 3.11.** *The following recursion formulas hold true for the denominator parameters $c$ and $c'$ of the $q$-Appell function $\Phi^{(2)}$:*

$$\begin{aligned}\Phi^{(2)}(cq^{-n}) &\\ = \Phi^{(2)} &+ cx(1-a)(1-b) \sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\,\Phi^{(2)}(aq, bq, cq^{2-n_1}),\end{aligned} \tag{79}$$

$$\begin{aligned}\Phi^{(2)}(c'q^{-n}) &\\ = \Phi^{(2)} &+ c'y(1-a)(1-b') \sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c')(q^{n_1-1}-c')}\,\Phi^{(2)}(aq, b'q, c'q^{2-n_1}).\end{aligned} \tag{80}$$

*Equivalently,*

$$\begin{aligned}&\Phi^{(2)}(cq^{-n})\\ &= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1}(b;q)_{n_1} x^{n_1} c^{n_1} q^{-n_1}}{(c;q)_{n_1}(cq^{-n};q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, bq^{n_1}, cq^{n_1}),\end{aligned} \tag{81}$$

$$\begin{aligned}&\Phi^{(2)}(c'q^{-n})\\ &= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1}(b';q)_{n_1} y^{n_1} c'^{n_1} q^{-n_1}}{(c';q)_{n_1}(c'q^{-n};q)_{n_1}}\,\Phi^{(2)}(aq^{n_1}, b'q^{n_1}, c'q^{n_1}).\end{aligned} \tag{82}$$

**Recursion formulas for $\Phi^{(3)}$:**

**Theorem 3.12.** *The following recursion formulas hold true for the numerator parameters $a$, $a'$ of the $q$-Appell function $\Phi^{(3)}$:*

$$\Phi^{(3)}(aq^n) = \Phi^{(3)} + ax\,\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(3)}(aq^{n_1}, bq, cq), \tag{83}$$

$$\Phi^{(3)}(aq^{-n}) = \Phi^{(3)} - ax\,\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(3)}(aq^{-n_1+1}, bq, cq), \tag{84}$$

$$\Phi^{(3)}(a'q^n) = \Phi^{(3)} + a'y\,\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\,\Phi^{(3)}(a'q^{n_1}, b'q, cq), \tag{85}$$

$$\Phi^{(3)}(a'q^{-n}) = \Phi^{(3)} - a'y\,\frac{1-b'}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\,\Phi^{(3)}(a'q^{-n_1+1}, b'q, cq). \tag{86}$$

The above theorem can also be written in an another form.

**Theorem 3.13.** *The following recursion formulas hold true for the numerator parameters $a$, $a'$ of the $q$-Appell function $\Phi^{(3)}$:*

$$\Phi^{(3)}(aq^n) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(b;q)_{n_1} x^{n_1} a^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}} \Phi^{(3)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{87}$$

$$\Phi^{(3)}(aq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{n_1} (-x)^{n_1} a^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}} \Phi^{(3)}(bq^{n_1}, cq^{n_1}), \tag{88}$$

$$\Phi^{(3)}(a'q^n) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{(b';q)_{n_1} y^{n_1} a'^{n_1} q^{n_1(n_1-1)}}{(c;q)_{n_1}} \Phi^{(3)}(a'q^{n_1}, b'q^{n_1}, cq^{n_1}), \tag{89}$$

$$\Phi^{(3)}(a'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(b';q)_{n_1} (-y)^{n_1} a'^{n_1} q^{\frac{-n_1(n_1+1)}{2}}}{(c;q)_{n_1}} \Phi^{(3)}(b'q^{n_1}, cq^{n_1}). \tag{90}$$

The formulas for $\Phi^{(3)}(bq^{\pm n})$ and $\Phi^{(3)}(b'q^{\pm n})$ are obtained by replacing $a \leftrightarrow b$ and $a' \leftrightarrow b'$ in Theorems 3.12–3.13, respectively.

**Theorem 3.14.** *The following recursion formulas hold true for the denominator parameter $c$ of the $q$-Appell function $\Phi^{(3)}$:*

$$\begin{aligned} &\Phi^{(3)}(cq^{-n}) \\ &= \Phi^{(3)} + cx(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)} \Phi^{(3)}(aq, bq, cq^{2-n_1}) \\ &\quad + cy(1-a')(1-b')\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)} \Phi^{(3)}(a'q, b'q, cq^{2-n_1}, q^{n_1}x). \end{aligned} \tag{91}$$

*Equivalently,*

$$(92)\quad \Phi^{(3)}(cq^{-n})$$
$$= \Phi^{(3)} + cx(1-a)(1-b)\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(3)}(aq, bq, cq^{2-n_1}, q^{n_1}y)$$
$$+ cy(1-a')(1-b')\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(3)}(a'q, b'q, cq^{2-n_1}).$$

**Recursion formulas for $\Phi^{(4)}$:**

**Theorem 3.15.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(4)}$:*

$$\Phi^{(4)}(aq^n) = \Phi^{(4)} + a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(4)}(aq^{n_1}, bq, cq, qy)$$
$$(93)\qquad + a\,y\frac{1-b}{1-c'}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(4)}(aq^{n_1}, bq, c'q).$$

*Equivalently,*

$$\Phi^{(4)}(aq^n) = \Phi^{(4)} + a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(4)}(aq^{n_1}, bq, cq)$$
$$(94)\qquad + a\,y\frac{1-b}{1-c'}\sum_{n_1=1}^{n} q^{n_1-1}\Phi^{(4)}(aq^{n_1}, bq, c'q, qx).$$

*Also*

$$\Phi^{(4)}(aq^{-n}) = \Phi^{(4)} - a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(4)}(aq^{1-n_1}, bq, cq, qy)$$
$$(95)\qquad - a\,y\frac{1-b}{1-c'}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(4)}(aq^{1-n_1}, bq, c'q).$$

*Equivalently,*

$$\Phi^{(4)}(aq^{-n}) = \Phi^{(4)} - a\,x\frac{1-b}{1-c}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(4)}(aq^{1-n_1}, bq, cq)$$
$$(96)\qquad - a\,y\frac{1-b}{1-c'}\sum_{n_1=1}^{n} q^{-n_1}\Phi^{(4)}(aq^{1-n_1}, bq, c'q, qx).$$

**Theorem 3.16.** *The following recursion formulas hold true for the numerator parameter $a$ of the $q$-Appell series $\Phi^{(4)}$:*

$$\Phi^{(4)}(aq^n) = \sum_{n_1+n_2\leq n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_q \frac{(b;q)_{N_2}\, x^{n_1} y^{n_2} a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}\, q^{N_2(N_2-1)}$$

$$\times\,\Phi^{(4)}(aq^{N_2}, bq^{N_2}, cq^{n_1}, c'q^{n_2}, q^{n_1}y). \tag{97}$$

*Equivalently,*

$$\Phi^{(4)}(aq^n) = \sum_{n_1+n_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_q \frac{(b;q)_{N_2}\, x^{n_1} y^{n_2} a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}\, q^{N_2(N_2-1)}$$
$$\times\,\Phi^{(4)}(aq^{N_2}, bq^{N_2}, cq^{n_1}, c'q^{n_2}, q^{n_2}x). \tag{98}$$

*Also*

$$\Phi^{(4)}(aq^{-n}) = \sum_{n_1+n_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{N_2}\,(-x)^{n_1}(-y)^{n_2} a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}$$
$$\times\, q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(4)}(bq^{N_2}, cq^{n_1}, c'q^{n_2}, q^{n_1}y). \tag{99}$$

*Equivalently,*

$$\Phi^{(4)}(aq^{-n}) = \sum_{n_1+n_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(b;q)_{N_2}\,(-x)^{n_1}(-y)^{n_2} a^{N_2}}{(c;q)_{n_1}(c';q)_{n_2}}$$
$$\times\, q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}\,\Phi^{(4)}(bq^{N_2}, cq^{n_1}, c'q^{n_2}, q^{n_2}x). \tag{100}$$

The formulas for $\Phi^{(4)}(bq^{\pm n})$ are obtained by replacing $a \leftrightarrow b$ in Theorems 3.15–3.16.

**Theorem 3.17.** *The following recursion formulas hold true for the denominator parameters $c$ and $c'$ of the $q$-Appell function $\Phi^{(4)}$:*

$$\Phi^{(4)}(cq^{-n}) \tag{101}$$
$$= \Phi^{(4)} + cx(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c)(q^{n_1-1}-c)}\Phi^{(4)}(aq, bq, cq^{2-n_1}),$$

$$\Phi^{(4)}(c'q^{-n}) \tag{102}$$
$$= \Phi^{(4)} + c'y(1-a)(1-b)\sum_{n_1=1}^{n} \frac{q^{n_1-1}}{(q^{n_1}-c')(q^{n_1-1}-c')}\Phi^{(4)}(aq, bq, c'q^{2-n_1}).$$

*Equivalently,*

$$\Phi^{(4)}(cq^{-n}) \tag{103}$$
$$= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1}(b;q)_{n_1} x^{n_1} c^{n_1} q^{-n_1}}{(c;q)_{n_1}(cq^{-n};q)_{n_1}}\,\Phi^{(4)}(aq^{n_1}, bq^{n_1}, cq^{n_1}),$$

$$\Phi^{(4)}(c'q^{-n}) \tag{104}$$
$$= \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{(a;q)_{n_1}(b;q)_{n_1} y^{n_1} c'^{n_1} q^{-n_1}}{(c';q)_{n_1}(c'q^{-n};q)_{n_1}}\,\Phi^{(4)}(aq^{n_1}, bq^{n_1}, c'q^{n_1}).$$

## 4. Recursion formulas for general double $q$-hypergeometric series

In this section, we present recursion formulas for general double $q$-hypergeometric series. Following abbreviated notations are used. We, for example, write

$$q^{n_1}a_A = q^{n_1}a_1, \dots, q^{n_1}a_A,$$
$$q^{n_1}a_A^i = q^{n_1}a_1, \dots, q^{n_1}a_{i-1}, q^{n_1}a_{i+1}, \dots, q^{n_1}a_A, \;\; i = 1, \dots, A, \text{ etc.}$$

Also, we denote

$$\begin{aligned} &[a_A] = \prod_{j=1}^{A}(a_j;q)_1, \;\; [a_A^i] = \prod_{j=1,\neq i}^{A}(a_j;q)_1, \;\; [a_A]_{n_1} = \prod_{j=1}^{A}(a_j;q)_{n_1}, \\ &[a_A^i]_{n_1} = \prod_{j=1,\neq i}^{A}(a_j;q)_{n_1}, \;\; [q^k a_A]_{n_1} = \prod_{j=1}^{A}(q^k a_j;q)_{n_1}, \\ &[q^k a_A^i]_{n_1} = \prod_{j=1,\neq i}^{A}(q^k a_j;q)_{n_1}, \; i = 1, \dots, A, \text{ etc.,} \end{aligned} \tag{105}$$

where $n_1$ and $k$ are non-negative integers. Note that for $n_1 = 1$, we have $[a_A]_1 = [a_A]$, $[a_A^i]_1 = [a_A^i]$. Further, for simplicity, we write $\Phi$ in place of general double $q$-hypergeometric series (6) and $\Phi(a_i q^{n_1})$ for the series

$$\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} a_A^i, a_i q^{n_1} : b_B; c_C \\ d_D \;\;\; : e_E\,; f_F \end{matrix}; q; x, y\right].$$

**Theorem 4.1.** *The following recursion formulas hold true for the numerator parameter $a_i$, $i = 1, \dots, A$, of $\Phi$:*

$$\begin{aligned} \Phi(a_i q^n) = \;& \Phi + a_i x(-1)^{K_1}\frac{[a_A^i][b_B]}{[d_D][e_E]}\sum_{n_1=1}^{n} q^{n_1-1} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} a_A^i q, a_i q^{n_1} : b_B q; c_C \\ d_D q \;\;\; : e_E q\,; f_F \end{matrix}; q; xq^{K_1}, yq^{1+D-A}\right] \\ & + a_i y(-1)^{K_2}\frac{[a_A^i][c_C]}{[d_D][f_F]}\sum_{n_1=1}^{n} q^{n_1-1} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} a_A^i q, a_i q^{n_1} : b_B; c_C q \\ d_D q \;\;\; : e_E\,; f_F q \end{matrix}; q; xq^{D-A}, yq^{K_2}\right]. \end{aligned} \tag{106}$$

*Equivalently,*

$$\begin{aligned} \Phi(a_i q^n) = \;& \Phi + a_i x(-1)^{K_1}\frac{[a_A^i][b_B]}{[d_D][e_E]}\sum_{n_1=1}^{n} q^{n_1-1} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q a_A^i, q^{n_1} a_i : q b_B; c_C \\ q d_D \;\;\; : q e_E\,; f_F \end{matrix}; q; xq^{K_1}, yq^{D-A}\right] \end{aligned}$$

$$+ a_i y(-1)^{K_2} \frac{[a_A^i][c_C]}{[d_D][f_F]} \sum_{n_1=1}^{n} q^{n_1-1}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A^i, q^{n_1}a_i : b_B; qc_C \\ qd_D \quad : e_E\, ; qf_F \end{matrix}; q; xq^{1+D-A}, yq^{K_2}\right]. \tag{107}$$

*Also*

$$\Phi(a_i q^{-n}) = \Phi - a_i x(-1)^{K_1} \frac{[a_A^i][b_B]}{[d_D][e_E]} \sum_{n_1=1}^{n} q^{-n_1}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A^i, q^{1-n_1}a_i : qb_B; c_C \\ qd_D \quad : qe_E\, ; f_F \end{matrix}; q; x_1 q^{K_1}, x_2 q^{1+D-A}\right]$$

$$- a_i y(-1)^{K_2} \frac{[a_A^i][c_C]}{[d_D][f_F]} \sum_{n_1=1}^{n} q^{-n_1}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A^i, q^{1-n_1}a_i : b_B; qc_C \\ qd_D \quad : e_E\, ; qf_F \end{matrix}; q; xq^{D-A}, yq^{K_2}\right]. \tag{108}$$

*Equivalently,*

$$\Phi(a_i q^{-n}) = \Phi - a_i x(-1)^{K_1} \frac{[a_A^i][b_B]}{[d_D][e_E]} \sum_{n_1=1}^{n} q^{-n_1}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A^i, q^{1-n_1}a_i : qb_B; c_C \\ qd_D \quad : qe_E\, ; f_F \end{matrix}; q; xq^{K_1}, yq^{D-A}\right]$$

$$- a_i y(-1)^{K_2} \frac{[a_A^i][c_C]}{[d_D][f_F]} \sum_{n_1=1}^{n} q^{-n_1}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A^i, q^{1-n_1}a_i : b_B; qc_C \\ qd_D \quad : e_E\, ; qf_F \end{matrix}; q; xq^{1+D-A}, yq^{K_2}\right], \tag{109}$$

*where* $K_1 = 1 + D - A + E - B$, *and* $K_2 = 1 + D - A + F - C$.

**Theorem 4.2.** *The following recursion formulas hold true for the numerator parameter* $a_i$, $i = 1, \ldots, A$, *of* $\Phi$:

$$\Phi(a_i q^n) = \sum_{n_1+n_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_q \left[(-1)^{n_1+n_2} q^{\binom{n_1+n_2}{2}}\right]^{D-A} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+E-B}$$

$$\times \left[(-1)^{n_2} q^{\binom{n_2}{2}}\right]^{1+F-C} x^{n_1} y^{n_2} q^{N_2(N_2-1)} a_i^{N_2} \frac{[a_A^i]_{N_2}[b_B]_{n_1}[c_C]_{n_2}}{[d_D]_{N_2}[e_E]_{n_1}[f_F]_{n_2}}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{N_2}a_A : q^{n_1}b_B; q^{n_2}c_C \\ q^{N_2}d_D : q^{n_1}e_E\, ; q^{n_2}f_F \end{matrix}; q; xq^{n_1K_1+n_2(D-A)}, yq^{n_1(1+D-A)+n_2K_2}\right]. \tag{110}$$

*Equivalently,*

$$\Phi(a_i q^n) = \sum_{n_1+n_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_q \left[(-1)^{n_1+n_2} q^{\binom{n_1+n_2}{2}}\right]^{D-A} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+E-B}$$

$$\times\left[(-1)^{n_2}q^{\binom{n_2}{2}}\right]^{1+F-C}x^{n_1}y^{n_2}q^{N_2(N_2-1)}a_i^{N_2}\frac{[a_A^i]_{N_2}[b_B]_{n_1}[c_C]_{n_2}}{[d_D]_{N_2}[e_E]_{n_1}[f_F]_{n_2}}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}q^{N_2}a_A:q^{n_1}b_B;q^{n_2}c_C\\q^{N_2}d_D:q^{n_1}e_E;q^{n_2}f_F\end{matrix};q;xq^{n_1K_1+n_2(1+D-A)},yq^{n_1(D-A)+n_2K_2}\right].\tag{111}$$

*Also*

$$\Phi(a_iq^{-n})=\sum_{n_1+n_2\le n}\begin{bmatrix}n\\n_1,n_2\end{bmatrix}_{q^{-1}}\left[(-1)^{n_1+n_2}q^{\binom{n_1+n_2}{2}}\right]^{D-A}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+E-B}$$

$$\times\left[(-1)^{n_2}q^{\binom{n_2}{2}}\right]^{1+F-C}q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}(-x)^{n_1}(-y)^{n_2}a_i^{N_2}\frac{[a_A^i]_{N_2}[b_B]_{n_1}[c_C]_{n_2}}{[d_D]_{N_2}[e_E]_{n_1}[f_F]_{n_2}}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}q^{N_2}a_A^i,a_i:q^{n_1}b_B;q^{n_2}c_C\\q^{N_2}d_D\quad\ :q^{n_1}e_E;q^{n_2}f_F\end{matrix};q;xq^{n_1K_1+n_2(D-A)},yq^{n_1(1+D-A)+n_2K_2}\right].\tag{112}$$

*Equivalently,*

$$\Phi(a_iq^{-n})=\sum_{n_1+n_2\le n}\begin{bmatrix}n\\n_1,n_2\end{bmatrix}_{q^{-1}}\left[(-1)^{n_1+n_2}q^{\binom{n_1+n_2}{2}}\right]^{D-A}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+E-B}$$

$$\times\left[(-1)^{n_2}q^{\binom{n_2}{2}}\right]^{1+F-C}q^{-\frac{n_1(n_1+1)}{2}-\frac{n_2(n_2+1)}{2}}(-x)^{n_1}(-y)^{n_2}a_i^{N_2}\frac{[a_A^i]_{N_2}[b_B]_{n_1}[c_C]_{n_2}}{[d_D]_{N_2}[e_E]_{n_1}[f_F]_{n_2}}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}q^{N_2}a_A^i,a_i:q^{n_1}b_B;q^{n_2}c_C\\q^{N_2}d_D\quad\ :q^{n_1}e_E;q^{n_2}f_F\end{matrix};q;xq^{n_1K_1+n_2(1+D-A)},yq^{n_1(D-A)+n_2K_2}\right].\tag{113}$$

**Theorem 4.3.** *The following recursion formulas hold true for the numerator parameter $b_i$, $i=1,\dots,B$, of $\Phi$:*

$$\Phi(b_iq^n)=\Phi+b_ix(-1)^{K_1}\frac{[a_A][b_B^i]}{[d_D][e_E]}\sum_{n_1=1}^{n}q^{n_1-1}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}qa_A:qb_B^i,q^{n_1}b_i;c_C\\qd_D:\quad qe_E\quad;f_F\end{matrix};q;xq^{K_1},yq^{D-A}\right],\tag{114}$$

$$\Phi(b_iq^{-n})=\Phi-b_ix(-1)^{K_1}\frac{[a_A][b_B^i]}{[d_D][e_E]}\sum_{n_1=1}^{n}q^{-n_1}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}qa_A:qb_B^i,q^{1-n_1}b_i;c_C\\qd_D:\quad qe_E\quad;f_F\end{matrix};q;xq^{K_1},yq^{D-A}\right].\tag{115}$$

**Theorem 4.4.** *The following recursion formulas hold true for the numerator parameter $b_i$, $i=1,\dots,B$, of $\Phi$:*

$$\Phi(b_iq^n)=\sum_{n_1=0}^{n}\begin{bmatrix}n\\n_1\end{bmatrix}_q\frac{[a_A]_{n_1}[b_B^i]_{n_1}}{[d_D]_{n_1}[e_E]_{n_1}}b_i^{n_1}x^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_1}q^{n_1(n_1-1)}$$

$$\times\Phi^{A:B;C}_{D:E;F}\left[\begin{matrix}q^{n_1}a_A:q^{n_1}b_B;c_C\\q^{n_1}d_D:q^{n_1}e_E;f_F\end{matrix};q;xq^{n_1K_1},yq^{n_1(D-A)}\right],\tag{116}$$

$$\Phi(b_iq^{-n})=\sum_{n_1=0}^{n}\begin{bmatrix}n\\n_1\end{bmatrix}_{q^{-1}}\frac{[a_A]_{n_1}[b_B^i]_{n_1}}{[d_D]_{n_1}[e_E]_{n_1}}b_i^{n_1}(-x)^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_1}q^{\frac{-n_1(n_1+1)}{2}}$$

$$\times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{n_1}a_A : q^{n_1}b^i_B, b_i; c_C \\ q^{n_1}d_D : q^{n_1}e_E; f_F \end{matrix}; q; xq^{n_1K_1}, yq^{n_1(D-A)}\right]. \tag{117}$$

**Theorem 4.5.** *The following recursion formulas hold true for the numerator parameter $c_i$, $i = 1, \dots, C$, of $\Phi$:*

$$\begin{aligned}\Phi(c_iq^n) = {}& \Phi + c_iy(-1)^{K_2}\frac{[a_A][c^i_C]}{[d_D][f_F]}\sum_{n_1=1}^{n} q^{n_1-1} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : b_B, ; qc^i_C, q^{n_1}c_i \\ qd_D : e_E; \quad qf_F \end{matrix}; q; xq^{D-A}, yq^{K_2}\right],\end{aligned} \tag{118}$$

$$\begin{aligned}\Phi(c_iq^{-n}) = {}& \Phi - c_iy(-1)^{K_2}\frac{[a_A][c^i_C]}{[d_D][f_F]}\sum_{n_1=1}^{n} q^{-n_1} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : b_B; qc^i_C, q^{1-n_1}c_i \\ qd_D : e_E; qf_F \end{matrix}; q; xq^{D-A}, yq^{K_2}\right].\end{aligned} \tag{119}$$

**Theorem 4.6.** *The following recursion formulas hold true for the numerator parameter $c_i$, $i = 1, \dots, C$, of $\Phi$:*

$$\begin{aligned}\Phi(c_iq^n) = {}& \sum_{n_1=0}^{n}\begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{[a_A]_{n_1}[c^i_C]_{n_1}}{[d_D]_{n_1}[f_F]_{n_1}} c_i^{n_1}y^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_2} q^{n_1(n_1-1)} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{n_1}a_A : b_B; q^{n_1}c_C \\ q^{n_1}d_D : e_E; q^{n_1}f_F \end{matrix}; q; xq^{n_1(D-A)}, yq^{n_1K_2}\right],\end{aligned} \tag{120}$$

$$\begin{aligned}\Phi(c_iq^{-n}) = {}& \sum_{n_1=0}^{n}\begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{[a_A]_{n_1}[c^i_C]_{n_1}}{[d_D]_{n_1}[f_F]_{n_1}} c_i^{n_1}(-y)^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_2} q^{\frac{-n_1(n_1+1)}{2}} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{n_1}a_A : b_B; q^{n_1}c^i_C, c_i \\ q^{n_1}d_D : e_E; q^{n_1}f_F \end{matrix}; q; xq^{n_1(D-A)}, yq^{n_1K_2}\right].\end{aligned} \tag{121}$$

**Theorem 4.7.** *The following recursion formulas hold true for the denominator parameter $d_i$, $i = 1, \dots, D$, of $\Phi$:*

$$\begin{aligned}\Phi(d_iq^{-n}) = {}& \Phi + d_ix(-1)^{K_1}\frac{[a_A][b_B]}{[d^i_D][e_E]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-d_i)(q^{n_1-1}-d_i)} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : qb_B, ; c_C \\ qd^i_D, q^{2-n_1}d_i : qe_E; f_F \end{matrix}; q; xq^{K_1}, yq^{1+D-A}\right] \\ & + d_iy(-1)^{K_2}\frac{[a_A][c_C]}{[d^i_D][f_F]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-d_i)(q^{n_1-1}-d_i)} \\ & \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : b_B, ; qc_C \\ qd^i_D, q^{2-n_1}d_i : e_E; qf_F \end{matrix}; q; xq^{D-A}, yq^{K_2}\right].\end{aligned} \tag{122}$$

*Equivalently,*

$$\Phi(d_iq^{-n}) = \Phi + d_i x(-1)^{K_1}\frac{[a_A][b_B]}{[d_D^i][e_E]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-d_i)(q^{n_1-1}-d_i)} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : qb_B,; c_C \\ qd_D^i, q^{2-n_1}d_i : qe_E; f_F\end{matrix}; q; xq^{K_1}, yq^{D-A}\right] + d_i y(-1)^{K_2}\frac{[a_A][c_C]}{[d_D^i][f_F]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-d_i)(q^{n_1-1}-d_i)} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : b_B,; qc_C \\ qd_D^i, q^{2-n_1}d_i : e_E; qf_F\end{matrix}; q; xq^{1+D-A}, yq^{K_2}\right]. \tag{123}$$

**Theorem 4.8.** *The following recursion formulas hold true for the denominator parameter $e_i$, $i = 1, \ldots, E$, of $\Phi$:*

$$\Phi(e_iq^{-n}) = \Phi + (-1)^{K_1}e_i x\frac{[a_A][b_B]}{[d_D][e_E^i]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-e_i)(q^{n_1-1}-e_i)} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : qb_B,; c_C \\ qd_D : qe_E^i, q^{2-n_1}e_i; f_F\end{matrix}; q; xq^{K_1}, yq^{D-A}\right]. \tag{124}$$

*Equivalently,*

$$\Phi(e_iq^{-n}) = \sum_{n_1=0}^{n}\begin{bmatrix} n \\ n_1\end{bmatrix}_{q^{-1}}\frac{[a_A]_{n_1}[b_B]_{n_1}}{[d_D]_{n_1}[e_E]_{n_1}(e_iq^{-n};q)_{n_1}}x^{n_1}e_i^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_1}q^{-n_1} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{n_1}a_A : q^{n_1}b_B; c_C \\ q^{n_1}d_D : q^{n_1}e_E; f_F\end{matrix}; q; xq^{n_1K_1}, yq^{n_1(D-A)}\right]. \tag{125}$$

**Theorem 4.9.** *The following recursion formulas hold true for the denominator parameter $f_i$, $i = 1, \ldots, F$, of $\Phi$:*

$$\Phi(f_iq^{-n}) = \Phi + (-1)^{K_2}f_i y\frac{[a_A][c_C]}{[d_D][f_F^i]}\sum_{n_1=1}^{n}\frac{q^{n_1-1}}{(q^{n_1}-f_i)(q^{n_1-1}-f_i)} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} qa_A : b_B; qc_C \\ qd_D : e_E; qf_F^i, q^{2-n_1}f_i\end{matrix}; q; xq^{D-A}, yq^{K_2}\right]. \tag{126}$$

*Equivalently,*

$$\Phi(f_iq^{-n}) = \sum_{n_1=0}^{n}\begin{bmatrix} n \\ n_1\end{bmatrix}_{q^{-1}}\frac{[a_A]_{n_1}[c_C]_{n_1}}{[d_D]_{n_1}[f_F]_{n_1}(f_iq^{-n};q)_{n_1}}y^{n_1}f_i^{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_2}q^{-n_1} \times \Phi^{A:B;C}_{D:E;F}\left[\begin{matrix} q^{n_1}a_A : b_B; q^{n_1}c_C \\ q^{n_1}d_D : e_E; q^{n_1}f_F\end{matrix}; q; xq^{n_1(D-A)}, yq^{n_1K_2}\right]. \tag{127}$$

## 5. $q$-differential recursion formulas

We can express recursion formulas of $q$-hypergeometric series obtained in the previous sections in terms of $q$-derivatives of respective $q$-hypergeometric series. Note that the $q$-derivative operator $D_{x,q}$ is defined for fixed $q$ by [1]

$$D_{x,q}f(x) = \frac{f(x) - f(xq)}{(1-q)x}, \quad q \neq 1. \tag{128}$$

Indeed, as $q \to 1$, the $q$-derivative operator becomes ordinary differential operator, provided $f$ is differentiable at $x$.

It can be verified that the $n_1$th-order $q$-derivative of generalized basic hypergeometric series is given by

$$\begin{aligned} D^{n_1}_{x,q}\, {}_r\phi_s &= \frac{\prod_{i=1}^{r}(a_i;q)_{n_1}}{\prod_{i=1}^{s}(b_i;q)_{n_1}(1-q)^{n_1}} \left[(-1)^{n_1} q^{\binom{n_1}{2}}\right]^{1+s-r} \\ &\times {}_r\phi_s(a_1q^{n_1},\dots,a_rq^{n_1},b_1q^{n_1},\dots,b_sq^{n_1},xq^{n_1(1+s-r)}). \end{aligned} \tag{129}$$

So (15), (16) and (24) can be expressed as:

$$ {}_r\phi_s(a_l\,q^n) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{x^{n_1}a_l^{n_1}(1-q)^{n_1}}{(a_l;q)_{n_1}}\, q^{n_1(n_1-1)}\, D^{n_1}_{x,q}\, {}_r\phi_s, \tag{130}$$

$$ {}_r\phi_s(a_l\,q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}(1-q)^{n_1}}{(a_l^{-1}q;q)_{n_1}} q^{-\frac{n_1(n_1+1)}{2}} D^{n_1}_{x,q}\, {}_r\phi_s(a_lq^{-n_1}) \tag{131}$$

and

$$ {}_r\phi_s(b_l\,q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}b_l^{n_1}(1-q)^{n_1}}{(b_lq^{-n};q)_{n_1}}\, q^{-n_1}\, D^{n_1}_{x,q}\, {}_r\phi_s, \tag{132}$$

respectively.

Next, we obtain $q$-derivatives of $\Phi^{(1)}$ with respect to $x$ and $y$ as follows:

$$D^{n_1}_{x,q}\Phi^{(1)} = \frac{(a;q)_{n_1}(b;q)_{n_1}}{(c;q)_{n_1}(1-q)^{n_1}}\Phi^{(1)}(aq^{n_1},bq^{n_1},cq^{n_1}), \tag{133}$$

$$D^{n_1}_{y,q}\Phi^{(1)} = \frac{(a;q)_{n_1}(b';q)_{n_1}}{(c;q)_{n_1}(1-q)^{n_1}}\Phi^{(1)}(aq^{n_1},b'q^{n_1},cq^{n_1}), \tag{134}$$

$$D^{n_1}_{x,q}D^{n_2}_{y,q}\Phi^{(1)} = \frac{(a;q)_{N_2}(b;q)_{n_1}(b';q)_{n_2}}{(c;q)_{N_2}(1-q)^{N_2}}\Phi^{(1)}(aq^{N_2},bq^{n_1},b'q^{n_2},cq^{N_2}). \tag{135}$$

Using these $q$-derivatives of $\Phi^{(1)}$, we write recursion formulas (44)-(45), (50)-(51), (56)-(59) of $\Phi^{(1)}$ as follows:

$$\begin{aligned} \Phi^{(1)}(aq^n) &= \sum_{N_2\le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1}y^{n_2}(1-q)^{N_2}a^{N_2}}{(a;q)_{N_2}} \\ &\times q^{N_2(N_2-1)}D^{n_1}_{x,q}D^{n_2}_{y,q}\Phi^{(1)}(q^{n_1}y), \end{aligned} \tag{136}$$

$$\begin{aligned} \Phi^{(1)}(aq^{-n}) &= \sum_{N_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\,x^{n_1}(q^{n_1}y)^{n_2}}{(a^{-1}q;q)_{N_2}} \\ &\times D^{n_1}_{x,q}D^{n_2}_{y,q}\Phi^{(1)}(aq^{-N_2},q^{n_1}y), \end{aligned} \tag{137}$$

$$\Phi^{(1)}(aq^n) = \sum_{N_2 \le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1} y^{n_2} (1-q)^{N_2} a^{N_2}}{(a;q)_{N_2}} \times q^{N_2(N_2-1)} D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(1)}(q^{n_2}x), \tag{138}$$

$$\Phi^{(1)}(aq^{-n}) = \sum_{N_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\,(q^{n_2}x)^{n_1} y^{n_2}}{(a^{-1}q;q)_{N_2}} \times D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(1)}(aq^{-N_2}, q^{n_2}x), \tag{139}$$

$$\Phi^{(1)}(bq^n) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{x^{n_1} b^{n_1} (1-q)^{n_1} q^{n_1(n_1-1)}}{(b;q)_{n_1}} D_{x,q}^{n_1} \Phi^{(1)}, \tag{140}$$

$$\Phi^{(1)}(bq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}(1-q)^{n_1}}{(b^{-1}q;q)_{n_1}} D_{x,q}^{n_1} \Phi^{(1)}(bq^{-n_1}), \tag{141}$$

$$\Phi^{(1)}(b'q^n) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \frac{y^{n_1} {b'}^{n_1} (1-q)^{n_1} q^{n_1(n_1-1)}}{(b';q)_{n_1}} D_{y,q}^{n_1} \Phi^{(1)}, \tag{142}$$

$$\Phi^{(1)}(b'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}(1-q)^{n_1}}{({b'}^{-1}q;q)_{n_1}} D_{y,q}^{n_1} \Phi^{(1)}(b'q^{-n_1}). \tag{143}$$

Similarly, using the $q$-derivatives of $\Phi^{(2)}$

$$D_{x,q}^{n_1}\Phi^{(2)} = \frac{(a;q)_{n_1}(b;q)_{n_1}}{(c;q)_{n_1}(1-q)^{n_1}} \Phi^{(2)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{144}$$

$$D_{y,q}^{n_1}\Phi^{(2)} = \frac{(a;q)_{n_1}(b';q)_{n_1}}{(c';q)_{n_1}(1-q)^{n_1}} \Phi^{(2)}(aq^{n_1}, b'q^{n_1}, c'q^{n_1}), \tag{145}$$

$$D_{x,q}^{n_1} D_{y,q}^{n_2}\Phi^{(2)} = \frac{(a;q)_{N_2}(b;q)_{n_1}(b';q)_{n_2}}{(c;q)_{n_1}(c';q)_{n_2}(1-q)^{N_2}} \Phi^{(2)}(aq^{N_2}, bq^{n_1}, b'q^{n_2}, c'q^{n_1}, c'q^{n_2}), \tag{146}$$

we write the recursion formulas (67)-(70), (75)-(78), (81)-(82) as follows:

$$\Phi^{(2)}(aq^n) = \sum_{N_2 \le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1} y^{n_2} (1-q)^{N_2} a^{N_2}}{(a;q)_{N_2}} \times q^{N_2(N_2-1)} D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(2)}(q^{n_1}y), \tag{147}$$

$$\Phi^{(2)}(aq^{-n}) = \sum_{N_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\, x^{n_1} (q^{n_1}y)^{n_2}}{(a^{-1}q;q)_{N_2}} \times D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(2)}(aq^{-N_2}, q^{n_1}y), \tag{148}$$

$$\Phi^{(2)}(aq^n) = \sum_{N_2 \le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1} y^{n_2} (1-q)^{N_2} a^{N_2}}{(a;q)_{N_2}} \times q^{N_2(N_2-1)} D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(2)}(q^{n_2}x), \tag{149}$$

$$\Phi^{(1)}(aq^{-n}) = \sum_{N_2\le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\,(q^{n_2}x)^{n_1}y^{n_2}}{(a^{-1}q;q)_{N_2}} \times D_{x,q}^{n_1} D_{y,q}^{n_2} \Phi^{(1)}(aq^{-N_2}, q^{n_2}x), \tag{150}$$

$$\Phi^{(2)}(bq^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q} \frac{x^{n_1}b^{n_1}(1-q)^{n_1}q^{n_1(n_1-1)}}{(b;q)_{n_1}} D_{x,q}^{n_1}\Phi^{(2)}, \tag{151}$$

$$\Phi^{(2)}(bq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}(1-q)^{n_1}}{(b^{-1}q;q)_{n_1}} D_{x,q}^{n_1}\Phi^{(2)}(bq^{-n_1}), \tag{152}$$

$$\Phi^{(2)}(b'q^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q} \frac{y^{n_1}b'^{n_1}(1-q)^{n_1}q^{n_1(n_1-1)}}{(b';q)_{n_1}} D_{y,q}^{n_1}\Phi^{(2)}, \tag{153}$$

$$\Phi^{(2)}(b'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}(1-q)^{n_1}}{(b'^{-1}q;q)_{n_1}} D_{y,q}^{n_1}\Phi^{(2)}(b'q^{-n_1}), \tag{154}$$

$$\Phi^{(2)}(cq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}c^{n_1}(1-q)^{n_1}q^{-n_1}}{(cq^{-n};q)_{n_1}} D_{x,q}^{n_1}\Phi^{(2)}, \tag{155}$$

$$\Phi^{(2)}(c'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}c'^{n_1}(1-q)^{n_1}q^{-n_1}}{(c'q^{-n};q)_{n_1}} D_{y,q}^{n_1}\Phi^{(2)}. \tag{156}$$

The $q$-derivatives of $\Phi^{(3)}$ are given by

$$D_{x,q}^{n_1}\Phi^{(3)} = \frac{(a;q)_{n_1}(b;q)_{n_1}}{(c;q)_{n_1}(1-q)^{n_1}}\Phi^{(3)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{157}$$

$$D_{y,q}^{n_1}\Phi^{(3)} = \frac{(a';q)_{n_1}(b';q)_{n_1}}{(c';q)_{n_1}(1-q)^{n_1}}\Phi^{(3)}(a'q^{n_1}, b'q^{n_1}, c'q^{n_1}), \tag{158}$$

$$D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(3)} = \frac{(a;q)_{n_1}(a';q)_{n_2}(b;q)_{n_1}(b';q)_{n_2}}{(c;q)_{N_2}(1-q)^{N_2}} \times \Phi^{(3)}(aq^{n_1}, a'q^{n_2}, bq^{n_1}, b'q^{n_2}, cq^{n_1}, c'q^{n_2}). \tag{159}$$

Using these $q$-derivatives of $\Phi^{(3)}$, we can write recursion formulas (87)-(90) for $\Phi^{(3)}$ as follows:

$$\Phi^{(3)}(aq^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q} \frac{x^{n_1}a^{n_1}(1-q)^{n_1}q^{n_1(n_1-1)}}{(a;q)_{n_1}} D_{x,q}^{n_1}\Phi^{(3)}, \tag{160}$$

$$\Phi^{(3)}(aq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}(1-q)^{n_1}}{(a^{-1}q;q)_{n_1}} D_{x,q}^{n_1}\Phi^{(3)}(aq^{-n_1}), \tag{161}$$

$$\Phi^{(3)}(a'q^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q} \frac{y^{n_1}a'^{n_1}(1-q)^{n_1}q^{n_1(n_1-1)}}{(a';q)_{n_1}} D_{y,q}^{n_1}\Phi^{(3)}, \tag{162}$$

$$\Phi^{(3)}(a'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}(1-q)^{n_1}}{(a'^{-1}q;q)_{n_1}} D_{y,q}^{n_1}\Phi^{(3)}(a'q^{-n_1}). \tag{163}$$

The recursion formulas for $\Phi^{(3)}(bq^{\pm n})$ and $\Phi^{(3)}(b'q^{\pm n})$ are obtained by replacing $a \leftrightarrow b$ and $a' \leftrightarrow b'$ in (160)-(163), respectively.

Using the $q$-derivatives of $\Phi^{(4)}$ given by

$$D_{x,q}^{n_1}\Phi^{(4)} = \frac{(a;q)_{n_1}(b;q)_{n_1}}{(c;q)_{n_1}(1-q)^{n_1}}\Phi^{(4)}(aq^{n_1}, bq^{n_1}, cq^{n_1}), \tag{164}$$

$$D_{y,q}^{n_1}\Phi^{(4)} = \frac{(a;q)_{n_1}(b;q)_{n_1}}{(c';q)_{n_1}(1-q)^{n_1}}\Phi^{(4)}(aq^{n_1}, bq^{n_1}, c'q^{n_1}), \tag{165}$$

$$D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(4)} = \frac{(a;q)_{N_2}(b;q)_{N_2}}{(c;q)_{n_1}(c';q)_{n_2}(1-q)^{N_2}}\Phi^{(4)}(aq^{N_2}, bq^{N_2}, cq^{n_1}, c'q^{n_2}), \tag{166}$$

we can write the recursion formulas (97)-(100), (103)-(104) as

$$\begin{aligned}\Phi^{(4)}(aq^n) = &\sum_{N_2 \le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1}y^{n_2}(1-q)^{N_2}a^{N_2}}{(a;q)_{N_2}} \\ &\times q^{N_2(N_2-1)} D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(4)}(q^{n_1}y),\end{aligned} \tag{167}$$

$$\begin{aligned}\Phi^{(4)}(aq^n) = &\sum_{N_2 \le n} \begin{bmatrix} n \\ n_1,\, n_2 \end{bmatrix}_q \frac{x^{n_1}y^{n_2}(1-q)^{N_2}a^{N_2}}{(a;q)_{N_2}} \\ &\times q^{N_2(N_2-1)} D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(4)}(q^{n_2}x),\end{aligned} \tag{168}$$

$$\begin{aligned}\Phi^{(4)}(aq^{-n}) = &\sum_{N_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\, x^{n_1}(q^{n_1}y)^{n_2}}{(a^{-1}q;q)_{N_2}} \\ &\times D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(4)}(aq^{-N_2}, q^{n_1}y),\end{aligned} \tag{169}$$

$$\begin{aligned}\Phi^{(4)}(aq^{-n}) = &\sum_{N_2 \le n} \begin{bmatrix} n \\ n_1, n_2 \end{bmatrix}_{q^{-1}} \frac{(1-q)^{N_2}\, y^{n_2}(q^{n_2}x)^{n_1}}{(a^{-1}q;q)_{N_2}} \\ &\times D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi^{(4)}(aq^{-N_2}, q^{n_2}x),\end{aligned} \tag{170}$$

respectively.

The recursion formulas for $\Phi^{(4)}(bq^{\pm n})$ are obtained by replacing $a \leftrightarrow b$ in (167)–(170), respectively.

$$\Phi^{(4)}(cq^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}c^{n_1}(1-q)^{n_1}q^{-n_1}}{(cq^{-n};q)_{n_1}} D_{x,q}^{n_1}\Phi^{(4)}, \tag{171}$$

$$\Phi^{(4)}(c'q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}c'^{n_1}(1-q)^{n_1}q^{-n_1}}{(c'q^{-n};q)_{n_1}} D_{y,q}^{n_1}\Phi^{(4)}. \tag{172}$$

Finally, to obtain $q$-differential recursion formulas for $\Phi$, we use

$$D_{x,q}^{n_1}\Phi = \frac{[a_A]_{n_1}[b_B]_{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_1}}{[d_D]_{n_1}[e_E]_{n_1}(1-q)^{n_1}} \times \Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} q^{n_1}a_A : q^{n_1}b_B; c_C \\ q^{n_1}d_D : q^{n_1}e_E ; f_F\end{matrix}; q; xq^{n_1K_1}, yq^{n_1(D-A)}\right], \tag{173}$$

$$D_{y,q}^{n_1}\Phi = \frac{[a_A]_{n_1}[c_C]_{n_1}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{K_2}}{[d_D]_{n_1}[f_F]_{n_1}(1-q)^{n_1}} \times \Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} q^{n_1}a_A : b_B; q^{n_1}c_C \\ q^{n_1}d_D : e_E ; q^{n_1}f_F\end{matrix}; q; xq^{n_1(D-A)}, yq^{n_1K_2}\right], \tag{174}$$

$$\begin{aligned} &D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi \\ &= \frac{[a_A]_{N_2}[b_B]_{n_1}[c_C]_{n_2}}{[d_D]_{N_2}[e_E]_{n_1}[f_F]_{n_2}(1-q)^{N_2}}\left[(-1)^{N_2}q^{\binom{N_2}{2}}\right]^{D-A}\left[(-1)^{n_1}q^{\binom{n_1}{2}}\right]^{1+E-B}\left[(-1)^{n_2}q^{\binom{n_2}{2}}\right]^{1+F-C} \\ &\times \Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} q^{N_2}a_A : q^{n_1}b_B; q^{n_2}c_C \\ q^{N_2}d_D : q^{n_1}e_E ; q^{n_2}f_F\end{matrix}; q; xq^{n_1K_1+n_2(D-A)}, yq^{n_1(D-A)+n_2K_2}\right] \end{aligned} \tag{175}$$

to write recursion formulas (110)-(113), (116)-(117), (120)-(121), (125) and (127) as

$$\begin{aligned}\Phi(a_iq^n) = &\sum_{n_1+n_2\le n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_q q^{N_2(N_2-1)}a_i^{N_2}\frac{(1-q)^{N_2}}{(a_i;q)_{N_2}} \\ &\times x^{n_1}y^{n_2}D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi(q^{n_1}y),\end{aligned} \tag{176}$$

$$\begin{aligned}\Phi(a_iq^n) = &\sum_{n_1+n_2\le n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_q q^{N_2(N_2-1)}a_i^{N_2}\frac{(1-q)^{N_2}}{(a_i;q)_{N_2}} \\ &\times x^{n_1}y^{n_2}D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi(q^{n_2}x),\end{aligned} \tag{177}$$

$$\begin{aligned}\Phi(a_iq^{-n}) = &\sum_{n_1+n_2\le n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_{q^{-1}} x^{n_1}(q^{n_1}y)^{n_2} \\ &\times \frac{(1-q)^{N_2}}{(a_i^{-1}q;q)_{N_2}}D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} a_A^i, q^{-N_2}a_i : b_B; c_C \\ d_D : e_E ; f_F\end{matrix}; q; x, yq^{n_1}\right],\end{aligned} \tag{178}$$

$$\begin{aligned}\Phi(a_iq^{-n}) = &\sum_{n_1+n_2\le n}\begin{bmatrix} n \\ n_1, n_2\end{bmatrix}_{q^{-1}} (q^{n_2}x)^{n_1}y^{n_2} \\ &\times \frac{(1-q)^{N_2}}{(a_i^{-1}q;q)_{N_2}}D_{x,q}^{n_1}D_{y,q}^{n_2}\Phi_{D:E;F}^{A:B;C}\left[\begin{matrix} a_A^i, q^{-N_2}a_i : b_B; c_C \\ d_D : e_E ; f_F\end{matrix}; q; xq^{n_2}, y\right],\end{aligned} \tag{179}$$

$$\Phi(b_iq^n) = \sum_{n_1=0}^{n}\begin{bmatrix} n \\ n_1\end{bmatrix}_q \frac{x^{n_1}\,b_i^{n_1}(1-q)^{n_1}}{(b_i;q)_{n_1}}\,q^{n_1(n_1-1)}D_{x,q}^{n_1}\Phi, \tag{180}$$

$$\Phi(b_i q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}(1-q)^{n_1}}{(b_i^{-1}q;q)_{n_1}} D_{x,q}^{n_1}\Phi(q^{-n_1}b_i), \tag{181}$$

$$\Phi(c_i q^{n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q} \frac{y^{n_1}\, c_i^{n_1}(1-q)^{n_1}}{(c_i;q)_{n_1}}\, q^{n_1(n_1-1)} D_{y,q}^{n_1}\Phi, \tag{182}$$

$$\Phi(c_i q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}(1-q)^{n_1}}{(c_i^{-1}q;q)_{n_1}} D_{y,q}^{n_1}\Phi(q^{-n_1}c_i), \tag{183}$$

$$\Phi(e_i q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{x^{n_1}e_i^{n_1}(1-q)^{n_1}q^{-n_1}}{(e_i q^{-n};q)_{n_1}} D_{x,q}^{n_1}\Phi, \tag{184}$$

$$\Phi(f_i q^{-n}) = \sum_{n_1=0}^{n} \begin{bmatrix} n \\ n_1 \end{bmatrix}_{q^{-1}} \frac{y^{n_1}f_i^{n_1}(1-q)^{n_1}q^{-n_1}}{(f_i q^{-n};q)_{n_1}} D_{y,q}^{n_1}\Phi. \tag{185}$$

## References


[1] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Second edition, Encyclopedia of Mathematics and its Applications, 96. Cambridge University Press, Cambridge, 2004.

[2] F. H. Jackson, *On basic double hypergeometric functions*, Quart. J. Math. (Oxford) **13** (1942), 69–82.

[3] S. B. Opps, N. Saad, and H. M. Srivastava, *Recursion formulas for Appell's hypergeometric function $F_2$ with some applications to radiation field problems*, Appl. Math. Comput. **207** (2009), no. 2, 545–558.

[4] E. D. Rainville, *Special Functions*, Chelsea Publishing Company, New York, 1960.

[5] V. Sahai and A. Verma, *Recursion formulas for multivariable hypergeometric functions*, Asian-Eur. J. Math. **8** (2015), no. 4, 1550082, 50 pp.

[6] ———, *Recursion formulas for Exton's triple hypergeometric functions*, Kyungpook Math. J. **56** (2016), no. 2, 473–506.

[7] ———, *Recursion formulas for Srivastava's general triple hypergeometric function*, Asian-Eur. J. Math. **9** (2016), no. 3, 1650063, 17 pp.

[8] ———, *Recursion formulas for the Srivastava-Daoust and related multivariable hypergeometric functions*, Asian-Eur. J. Math. **9** (2016), no. 4, 1650081, 35 pp.

[9] H. M. Srivastava and M. C. Daoust, *Certain generalized Neumann expansions associated with the Kampé de Fériet function*, Indag. Math. **31** (1969), 449–457.

[10] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications. Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1985.

[11] R. F. Swarttouw, *The contiguous function relations for basic hypergeometric function*, J. Math. Anal. Appl. **149** (1990), no. 1, 151–159.

[12] X. Wang, *Recursion formulas for Appell functions*, Integral Transforms Spec. Funct. **23** (2012), no. 6, 421–433.

Vivek Sahai
Department of Mathematics and Astronomy
Lucknow University
Lucknow 226007, India
*Email address*: sahai_vivek@hotmail.com

Ashish Verma
Department of Mathematical and Statistical Sciences
Shri Ramswaroop Memorial University
Deva Road Lucknow 225003, India
*Email address*: vashish.lu@gmail.com