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ON EULERIAN q-INTEGRALS FOR SINGLE AND MULTIPLE q-HYPERGEOMETRIC SERIES

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Abstract. In this paper we extend the two q-additions with powers in the umbrae, define a q-multinomial-coefficient, which implies a vector version of the q-binomial theorem, and an arbitrary complex power of a JHC power series is shown to be equivalent to a special case of the first q-Lauricella function. We then present several q-analogues of hypergeometric integral formulas from the two books by Exton and the paper by Choi and Rathie. We also find multiple q-analogues of hypergeometric integral formulas from the recent paper by Kim. Finally, we prove several multiple q-hypergeometric integral formulas emanating from a paper by Koschmieder, which are special cases of more general formulas by Exton.

1. Introduction

We will prove several Eulerian q-integrals using the q-beta function in ordinary and vector form. In each of the following references, as well as the present paper, we refer to the original papers by Koschmieder [9–11]. Earlier in [2] we proved multiple q-hypergeometric transformations by using the q-beta integral and the q-binomial theorem. Then, in [5], we started with a q-integral representation of the fourth q-Lauricella function. Then we found four q-hypergeometric integral transformations between the same q-Lauricella function with vector q-beta function coefficients.

The outline of the present article runs as follows: In this section we present the necessary definitions together with a q-integral representation of the first q-Lauricella function, which goes back to the original paper by Lauricella [12].

In Section 2 we start with general forms of integrands in the q-beta integral according to Exton and find special cases by using known q-summation formulas. An example by Rainville [13] is given, where powers of the JHC q-addition is used as function argument. In the latter part, multiple q-analogues occur because of the corresponding multiple q-hypergeometric formulas. They all

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have correct hypergeometric limits. In Subsection 2.1 we investigate products of two power series in the integrand.

In Section 3 we first find a general multiple q-integral formula and continue with q-analogues of some of Koschmieder's formulas [11]. The variety of these formulas is illustrated with many examples by Exton, an example by Rainville [13], where a multiple q-addition and powers of the JHC q-addition are used as function arguments. Finally, in Subsection 3.1 we show that the previous formulas can be generalized to multiple Eulerian q-integrals in an obvious way.

We only make the definitions which differ from [3], except from some special cases. For the following definition, compare with [3, p. 22].

Definition 1. Let S_r denote the additional poles of Γ_q, vertical if q is real and slanting if q is complex. Then the q-beta function, a function

$$(\mathbb{C}\setminus(\{\mathbb{Z}\leq 0\}\cup S_r))^2\times\mathbb{C}\mapsto\mathbb{C},$$

is defined as follows:

$$\mathrm{B}_q(x,y)\equiv\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1}$$

Definition 2. Similarly, if $\vec{x}$ and $\vec{y}$ have dimension n, the vector q-beta function, a function $(\mathbb{C}\setminus(\{\mathbb{Z}\leq 0\}\cup S_r))^{2n}\times\mathbb{C}\mapsto\mathbb{C}$, is defined as follows:

$$\mathrm{B}_q(\vec{x},\vec{y})\equiv\frac{\Gamma_q(\vec{x})\Gamma_q(\vec{y})}{\Gamma_q(\vec{x}+\vec{y})}. \tag{2}$$

Definition 3. We extend the two q-additions as follows: If we write a letter γ in the form

$$\gamma\equiv(\alpha\oplus_q\beta)^k\vee\gamma\equiv(\alpha\boxplus_q\beta)^k, \tag{3}$$

this means the two linear functionals

$$\gamma^n\equiv(\alpha\oplus_q\beta)^{nk}\vee\gamma^n\equiv(\alpha\boxplus_q\beta)^{nk}. \tag{4}$$

This definition will be used in formulas (27) and (69).

Definition 4 ([3, p. 387, p. 437])**.** The q-Lauricella functions are

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle a;q\rangle_m\langle\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}}, \tag{5}$$

$$\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle\vec{a},\vec{b};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle c;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}, \tag{6}$$

$$\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}|q;\vec{x})\equiv\sum_{\vec{m}}\frac{\langle a,b;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{c},\vec{1};q\rangle_{\vec{m}}}, \tag{7}$$

$$\Phi_{\mathrm{D}}^{(n)}(a,b_1,\ldots,b_n;c|q;x_1,\ldots,x_n)\equiv\sum_{\vec{m}}\frac{\langle a;q\rangle_m\prod_{j=1}^n\langle b_j;q\rangle_{m_j}x_j^{m_j}}{\langle c;q\rangle_m\prod_{j=1}^n\langle 1;q\rangle_{m_j}}. \tag{8}$$

The convergence regions for the above functions are [4] for

$$\Phi_{\mathrm{A}}^{(n)}(a,\vec{b};\vec{c}|\vec{x}):\ |x_1|\oplus_q\cdots\oplus_q|x_n|<1. \tag{9}$$

For

$$\Phi_{\mathrm{B}}^{(n)}(\vec{a},\vec{b};c|\vec{x}):\ \max(|x_1|,\ldots,|x_n|)<1. \tag{10}$$

For

$$\Phi_{\mathrm{C}}^{(n)}(a,b;\vec{c}|\vec{x}):\ |\sqrt{x_1}|\oplus_q\cdots\oplus_q|\sqrt{x_n}|<1. \tag{11}$$

For

$$\Phi_{\mathrm{D}}^{(n)}(a,\vec{b};c|\vec{x}):\ \max(|x_1|,\ldots,|x_n|)<1. \tag{12}$$

Definition 5 ([3, p. 367 f])**.** The vectors

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G_i,H_i,A',B',G'_i,H'_i.$$

Let

$$1+B+B'+H_i+H'_i-A-A'-G_i-G'_i\geq 0,\ i=1,\ldots,n.$$

Then the generalized q-Kampé de Fériet function q-Kampé de Fériet function is defined by

(13)

$$\Phi^{A+A':G_1+G'_1;\cdots;G_n+G'_n}_{B+B':H_1+H'_1;\cdots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\cdots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\cdots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\cdots;(g'_n)\\(b'):(h'_1);\cdots;(h'_n)\end{array}\right]$$

$$\equiv\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}$$
$$\times(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}$$
$$\times\operatorname{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\operatorname{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right).$$

It is assumed that there are no zero factors in the denominator. We assume that $(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$ contain factors of the form $\langle a(\hat{k});q\rangle_k,\ (s;q)_k,(s(k);q)_k$ or $\operatorname{QE}(f(\vec{m}))$.

Definition 6 ([3, p. 368 f])**.** The vectors

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G,H,A',B',G',H'.$$

Let

$$1+B+B'+H+H'-A-A'-G-G'\geq 0.$$

Then the generalized q-Kampé de Fériet function is defined by

(14)

$$\Phi^{A+A':G+G'}_{B+B':H+H'}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\dots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\dots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\dots;(g'_n)\\(b'):(h'_1);\dots;(h'_n)\end{array}\right]$$
$$\equiv\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle 1;q_j\rangle_{m_j})}$$
$$\times(-1)^{\sum_{j=1}^n m_j(1+H+H'-G-G'+B+B'-A-A')}$$
$$\times\,\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\mathrm{QE}\left((1+H+H'-G-G')\binom{m_j}{2},q_j\right),$$

where

$$\hat{a}\equiv a\vee\tilde{a}\vee\widetilde{\tfrac{m}{n}a}\vee_k\widetilde{a}\vee\triangle(q;l;\lambda). \tag{15}$$

We recall the following definition from [3, p. 110]:

Definition 7. Given an integer k, the formula

$$m_0+m_1+\cdots+m_j=k \tag{16}$$

determines a set $J_{m_0,\dots,m_j}\in\mathbb{N}^{j+1}$.

Then if $f(x)$ is the formal power series $\sum_{l=0}^{\infty}a_lx^l$, it's k'th JHC-power is given by

$$\begin{aligned}(\boxplus_{q,l=0}^{\infty}a_lx^l)^k&\equiv(a_0\boxplus_q a_1x\boxplus_q\cdots)^k\\&\equiv\sum_{|\vec{m}|=k}\prod_{m_l\in J_{m_0,\dots,m_j}}(a_lx^l)^{m_l}\binom{k}{\vec{m}}_q q^{\binom{\vec{n}}{2}},\end{aligned} \tag{17}$$

where $\vec{n}=(m_1,\dots,m_n)$.

In an equivalent way we can define a q-analogue of the function

$$F(u_1,x_1,\dots,u_n,x_n)\equiv(1-u_1x_1-\cdots-u_nx_n)^{-a}. \tag{18}$$

The function $F(u_1,x_1,\dots,u_n,x_n)$ is just a special Lauricella function $\Phi_{\mathrm{A}}^{(n)}$ in its region of convergence as the following reasoning shows.

Definition 8. Assume that $\vec{m}\equiv(m_1,\dots,m_n),\ m\equiv m_1+\cdots+m_n$ and $a\in\mathbb{C}^{\star}$. The q-multinomial-coefficient $\binom{a}{\vec{m}}_q$ is defined by

$$\binom{a}{\vec{m}}_q\equiv\frac{\langle -a;q\rangle_m(-1)^mq^{-\binom{\vec{m}}{2}+am}}{\langle 1;q\rangle_{m_1}\langle 1;q\rangle_{m_2}\cdots\langle 1;q\rangle_{m_n}}. \tag{19}$$

In accordance with (17) and (19), we can now define the following q-analogue of (18), a vector version of the q-binomial theorem.

Definition 9.

(20)

$$(1 \boxminus_q q^{am_1} x_1 \boxminus_q \cdots \boxminus_q q^{am_n} x_n)^{-a} \equiv \sum_{m_1,\ldots,m_n=0}^{\infty} \prod_{j=1}^{n} (-x_j)^{m_j} \binom{-a}{\vec{m}}_q q^{\binom{\vec{m}}{2}+am}.$$

Corollary 1.1.

$$(1 \boxminus_q q^{am_1} x_1 \boxminus_q \cdots \boxminus_q q^{am_n} x_n)^{-a} = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle a; q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}}. \tag{21}$$

Remark 1. There are several q-Taylor formulas, some of them very similar, and some with q-integral remainder term. All of these formulas can be generalized to n variables, where the summation indices and the variables are written in the same form, but with vectors. The formula (21) is a very simple example of such a vector q-Taylor formula.

All the integral representations of Lauricella functions from the original paper by Lauricella [12, pp. 145–147] are given without proofs in Exton [6]. We will now find a q-analogue of the first one by using the previous definition for a power of a JHC sum. The other integral representations, which use another beta integral due to Dirichlet, are not suitable for q-deformation, since they require a more complicated multiple integral.

Theorem 1.2 (A q-analogue of [12, p. 145], [6, p. 48 2.3.3], a q-integral representation of the first q-Lauricella function)**.**

(22)

$$\begin{aligned} &\mathrm{B}_q(\vec{b}, c \vec{-} b)\Phi_{\mathrm{A}}^{(n)}(a, \vec{b}; \vec{c}|q; \vec{x}) \\ = &\int_0^1 \cdots (n) \cdots \int_0^1 {u_1}^{b_1-1} \cdots {u_n}^{b_n-1} (qu_1; q)_{c_1-b_1-1} \\ &\cdots (qu_n; q)_{c_n-b_n-1} (1 \boxminus_q q^{am_1} u_1 x_1 \boxminus_q \cdots \boxminus_q q^{am_n} u_n x_n)^{-a} d_q(u_1) \cdots d_q(u_n). \end{aligned}$$

Proof.

(23)

$$\begin{aligned} \mathrm{LHS} &= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\langle a; q\rangle_m \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}} \Gamma_q \left[\begin{matrix} b_1+m_1, c_1-b_1, \ldots, b_n+m_n, c_n-b_n \\ c_1+m_1, \ldots, c_n+m_n \end{matrix} \right] \\ &= \int_{\vec{0}}^{\vec{1}} (q\vec{u}; q)_{\vec{c}-\vec{b}-\vec{1}} \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\langle a; q\rangle_m \vec{u}^{\vec{m}} \vec{x}^{\vec{m}}}{\langle \vec{1}; q\rangle_{\vec{m}}} \vec{u}^{\vec{b-1}} \, d_q(\vec{u}) \overset{\text{by } (21)}{=} \mathrm{RHS}. \end{aligned}$$

□

2. Eulerian q-integrals for q-hypergeometric series

We start with q-analogues of some general integrals from Exton [7, p. 32 ff]:

Theorem 2.1 (A q-analogue of [7, p. 32 2.1.6])). *Assume that $f(x)$ can be written as the power series*

$$f(x)=\sum_{n=0}^{\infty}c_nx^n. \tag{24}$$

Then we have

$$\int_0^1 x^{\alpha-1}(qx;q)_{\beta-1}f(x)d_q(x)=\mathrm{B}_q(\alpha,\beta)\sum_{n=0}^{\infty}\frac{\langle\alpha;q\rangle_n c_n}{\langle\alpha+\beta;q\rangle_n}. \tag{25}$$

Proof. Use the q-beta integral formula and the fundamental properties of the Γ_q function. □

Theorem 2.2 (A q-analogue of [7, p. 33 2.1.1.2])).

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\beta-1}\ {}_2\phi_1(\gamma,\delta;\lambda|q;yx^k)d_q(x)\\&=\mathrm{B}_q(\alpha,\beta)\ {}_{2+2k}\phi_{1+2k}\left[\begin{array}{c}\gamma,\delta,\triangle(q;k;\alpha)\\\lambda,\triangle(q;k;\alpha+\beta)\end{array}|q;y\right].\end{aligned} \tag{26}$$

Proof. Put $f(x)={}_2\phi_1(\gamma,\delta;\lambda|q;yx^k)$ in (25). □

Theorem 2.3 (Almost a q-analogue of [13, p. 104 (4)]). *Let $k,s\in\mathbb{N}$*

(27)

$$\begin{aligned}&\mathrm{B}_q(\alpha,\beta)\ {}_{p+2(k+2s)}\phi_{r+2(k+s)}\left[\begin{array}{c}c_1,\ldots,c_p,\triangle(q;k;\alpha),\triangle(q;s;\beta),2s\infty\\d_1,\ldots,d_r,\triangle(q;k+s;\alpha+\beta)\end{array}|q;t\right]\\&=\int_0^1 x^{\alpha-1}(xq;q)_{\beta-1}\ {}_{p+2s}\phi_r(c_1,\ldots,c_p;d_1,\ldots,d_r|q;tx^k(1\boxminus_q xq^{\beta})^s)d_q(x).\end{aligned}$$

Proof.

(28)

$$\begin{aligned}\text{RHS}=&\sum_{m=0}^{\infty}\frac{\langle c_1,\ldots,c_p;q\rangle_m t^m}{\langle 1,d_1,\ldots,d_r;q\rangle_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+r-p-2s}\\&\times\int_0^1 x^{\alpha+mk-1}(xq;q)_{\beta+sm-1}d_q(x)\\=&\sum_{m=0}^{\infty}\frac{\langle c_1,\ldots,c_p;q\rangle_m t^m}{\langle 1,d_1,\ldots,d_r;q\rangle_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+r-p-2s}\Gamma_q\left[\begin{array}{c}\alpha+km,\beta+sm\\\alpha+\beta+m(k+s)\end{array}\right]\\=&\sum_{m=0}^{\infty}\frac{\langle c_1,\ldots,c_p,\triangle(q;k;\alpha),\triangle(q;s;\beta);q\rangle_m t^m}{\langle 1,d_1,\ldots,d_r,\triangle(q;k+s;\alpha+\beta);q\rangle_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+r-p-2s}\\&\times\Gamma_q\left[\begin{array}{c}\alpha,\beta\\\alpha+\beta\end{array}\right]=\text{LHS}.\end{aligned}$$

□

Corollary 2.4 (A q-analogue of [7, p. 33 2.1.1.5])).

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\beta-1}\ {}_2\phi_1(\gamma,\delta;\lambda|q;yx)d_q(x)\\ &= \mathrm{B}_q(\alpha,\beta)\ {}_3\phi_2\left[\begin{array}{c}\gamma,\delta,\alpha\\ \lambda,\alpha+\beta\end{array}|q;y\right].\end{aligned} \tag{29}$$

Proof. Put $k=1$ in (26). □

We now give a couple of examples of when the q-Clausenian function in (29) can be explicitly computed.

Corollary 2.5 (A q-analogue of [7, p. 33 2.1.1.4])).

$$\int_0^1 x^{\alpha-1}\frac{(qx;q)_{\beta-1}}{(yx;q)_\delta}d_q(x) = \mathrm{B}_q(\alpha,\beta)\ {}_2\phi_1\left[\begin{array}{c}\delta,\alpha\\ \alpha+\beta\end{array}|q;y\right]. \tag{30}$$

Proof. Put $\gamma=\lambda$ in (29). □

Corollary 2.6 (A q-analogue of the corrected version of [7, p. 33 2.1.1.6]).

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\delta-\alpha-1}\ {}_2\phi_1(\gamma,\delta;\alpha|q;yx)d_q(x)\\ &= \mathrm{B}_q(\alpha,\delta-\alpha)\frac{1}{(y;q)_\gamma},\ |y|<1.\end{aligned} \tag{31}$$

Proof. Put $\lambda=\alpha,\ \beta=\delta-\alpha$ in (29). □

Corollary 2.7.

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\beta-1}\ {}_2\phi_1(\gamma,\delta;\alpha|q;xq^{\alpha+\beta-\gamma-\delta})d_q(x)\\ &= \Gamma_q\left[\begin{array}{c}\alpha,\beta,\alpha+\beta-\gamma-\delta\\ \alpha+\beta-\gamma,\alpha+\beta-\delta\end{array}\right].\end{aligned} \tag{32}$$

Proof. Put $\alpha=\lambda,\ y=q^{\alpha+\beta-\gamma-\delta}$ in (29). □

Corollary 2.8.

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\delta-\lambda-n}\ {}_2\phi_1(-n,\delta;\lambda|q;qx)d_q(x)\\ &= \mathrm{B}_q(\alpha,1+\delta-\lambda-n)\frac{\langle\lambda-\alpha,\lambda-\delta;q\rangle_n}{\langle\lambda,\lambda-\alpha-\delta;q\rangle_n}.\end{aligned} \tag{33}$$

Proof. Put $\gamma=-n,\ \beta=1+\delta-\lambda-n,\ y=q$ in (29). □

Corollary 2.9 (A q-analogue of [1, p. 287 (3.1)]).

$$\begin{aligned}(34)\quad &\int_0^1 x^{\beta-1}(qx;q)_{\frac{1}{2}(-N-1-\beta)}\ {}_3\phi_2\left[\begin{array}{c}\widetilde{\frac{\lambda}{2},\frac{\lambda}{2}},-N\\ \frac{-N+1+\beta}{2},\lambda\end{array}|q;qx\right]d_q(x)\\ &=\mathrm{B}_q(\beta,\frac{1}{2}(-N+1-\beta))\begin{cases}\frac{\langle\frac{1}{2},\frac{1+\lambda-\beta}{2};q^2\rangle_{\frac{N}{2}}}{\langle\frac{1-\beta}{2},\frac{1+\lambda}{2};q^2\rangle_{\frac{N}{2}}}, & \text{if } N \text{ even;}\\ 0, & \text{if } N \text{ uneven.}\end{cases}\end{aligned}$$

Proof. Use the q-analogue of the Watson-Schafheitlin summation formula [3, p. 281 (8.9)]. □

Corollary 2.10 (A q-analogue of [1, p. 285 (2.6)]).

$$\begin{aligned}(35)\quad &\int_0^1 x^{\gamma-1}(qx;q)_{\gamma-1}\ {}_3\phi_2\left[\begin{array}{c}\alpha,\widetilde{\gamma},-N\\ \frac{-N+1+\alpha}{2},\widetilde{\frac{-N+1+\alpha}{2}}\end{array}|q;qx\right]d_q(x)\\ &=\mathrm{B}_q(\gamma,\gamma)\begin{cases}\frac{\langle\frac{1}{2},\frac{1-\alpha}{2}+\gamma;q^2\rangle_{\frac{N}{2}}}{\langle\frac{1-\alpha}{2},\frac{1}{2}+\gamma;q^2\rangle_{\frac{N}{2}}}, & \text{if } N \text{ even;}\\ 0, & \text{if } N \text{ uneven.}\end{cases}\end{aligned}$$

Proof. Use the q-analogue of the Watson-Schafheitlin summation formula [3, p. 281 (8.9)]. □

We will now find nine q-analogues of [8, p. 332 (2.1)-(2.3)].

Corollary 2.11 (Three q-analogues of [8, p. 332 (2.1)]).

(36)

$$\begin{aligned}&\int_0^1 x^{\beta-1}(qx;q)_{\alpha-2\beta}\ {}_4\phi_4\left[\begin{array}{c}\alpha,1+\frac{1}{2}\alpha,\widetilde{1+\frac{1}{2}a},\gamma\\ \frac{1}{2}\alpha,\widetilde{\frac{1}{2}a},1+\alpha-\gamma,\infty\end{array}|q;xq^{1+\alpha-\beta-\gamma}\right]d_q(x)\\ &=\Gamma_q\left[\begin{array}{c}\beta,\alpha-2\beta+1,1+\alpha-\gamma\\ 1+\alpha,1+\alpha-\beta-\gamma\end{array}\right],\end{aligned}$$

$$\begin{aligned}(37)\quad &\int_0^1 x^{\beta-1}(qx;q)_{\alpha-2\beta}\ {}_5\phi_3\left[\begin{array}{c}\alpha,1+\frac{1}{2}\alpha,\widetilde{1+\frac{1}{2}a},-n,\infty\\ \frac{1}{2}\alpha,\widetilde{\frac{1}{2}a},1+\alpha+n\end{array}|q;xq^{n-\beta}\right]d_q(x)\\ &=\Gamma_q\left[\begin{array}{c}\beta,\alpha-2\beta+1,1+\alpha+n\\ 1+\alpha,1+\alpha-\beta+n\end{array}\right]q^{-n\beta},\end{aligned}$$

$$\begin{aligned}(38)\quad &\int_0^1 x^{\beta-1}(qx;q)_{\alpha-2\beta}\ {}_3\phi_2\left[\begin{array}{c}\alpha,1+\frac{1}{2}\alpha,\gamma\\ \frac{1}{2}\alpha,1+\alpha-\gamma\end{array}|q;-xq^{1+\frac{1}{2}\alpha-\beta-\gamma}\right]d_q(x)\\ &=\Gamma_q\left[\begin{array}{c}\beta,\alpha-2\beta+1,1+\alpha-\gamma,1+\frac{\alpha}{2}+b,1+\frac{\alpha}{2}-\beta-\gamma+b\\ 1+\alpha,1+\alpha-\beta-\gamma,1+\frac{\alpha}{2}-\beta+b,1+\frac{\alpha}{2}-\gamma+b\end{array}\right],\end{aligned}$$

where $b\equiv\frac{\log(-1)}{\log q}$.

(Three q-analogues of [8, p. 332 (2.2)])

(39)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\frac{1}{2}\alpha-\gamma-1}\ {}_4\phi_4\left[\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, 1+\widetilde{\tfrac{1}{2}a}, \beta \\ 1+\alpha-\beta, 1+\alpha-\gamma, \widetilde{\tfrac{1}{2}a}, \infty\end{matrix}\ |q; xq^{1+\alpha-\beta-\gamma}\right] d_q(x)$$
$$= \Gamma_q\left[\begin{matrix}\gamma, \frac{1}{2}\alpha-\gamma, 1+\alpha-\beta, 1+\alpha-\gamma \\ \frac{1}{2}\alpha, 1+\alpha, 1+\alpha-\beta-\gamma\end{matrix}\right],$$

(40)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\frac{1}{2}\alpha-\gamma-1}\ {}_5\phi_3\left[\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, -n, \widetilde{1+\tfrac{1}{2}a}, \infty \\ 1+\alpha+n, 1+\alpha-\gamma, \widetilde{\tfrac{1}{2}a}\end{matrix}\ |q; xq^{n-\gamma}\right] d_q(x)$$
$$= \Gamma_q\left[\begin{matrix}\gamma, \frac{1}{2}\alpha-\gamma, 1+\alpha-\gamma, 1+\alpha+n \\ \frac{1}{2}\alpha, 1+\alpha, 1+\alpha-\gamma+n\end{matrix}\right] q^{-n\gamma},$$

(41)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\frac{1}{2}\alpha-\gamma-1}\ {}_3\phi_2\left[\begin{matrix}\alpha, 1+\frac{1}{2}\alpha, \beta \\ 1+\alpha-\beta, 1+\alpha-\gamma\end{matrix}\ |q; -xq^{1+\frac{1}{2}\alpha-\beta-\gamma}\right] d_q(x)$$
$$= \Gamma_q\left[\begin{matrix}\gamma, \frac{1}{2}\alpha-\gamma, 1+\alpha-\beta, 1+\alpha-\gamma, 1+\frac{\alpha}{2}+b, 1+\frac{\alpha}{2}-\beta-\gamma+b \\ \frac{1}{2}\alpha, 1+\alpha, 1+\alpha-\beta-\gamma, 1+\frac{\alpha}{2}-\beta+b, 1+\frac{\alpha}{2}-\gamma+b\end{matrix}\right],$$

where $b \equiv \frac{\log(-1)}{\log q}$.

(Three q-analogues of [8, p. 332 (2.3)])

(42)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\alpha-\beta-\gamma}\ {}_4\phi_4\left[\begin{matrix}\alpha, \beta, 1+\frac{1}{2}\alpha, 1+\widetilde{\tfrac{1}{2}a} \\ 1+\alpha-\gamma, \frac{1}{2}\alpha, \widetilde{\tfrac{1}{2}a}, \infty\end{matrix}\ |q; xq^{1+\alpha-\beta-\gamma}\right] d_q(x)$$
$$= \mathrm{B}_q(\gamma, 1+\alpha-\gamma),$$

(43)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\alpha+n-\gamma}\ {}_5\phi_3\left[\begin{matrix}\alpha, -n, 1+\frac{1}{2}\alpha, \widetilde{1+\tfrac{1}{2}a}, \infty \\ 1+\alpha-\gamma, \frac{1}{2}\alpha, \widetilde{\tfrac{1}{2}a}\end{matrix}\ |q; xq^{n-\gamma}\right] d_q(x)$$
$$= \mathrm{B}_q(\gamma, 1+\alpha-\gamma)q^{-n\gamma},$$

(44)
$$\int_0^1 x^{\gamma-1}(qx;q)_{\alpha-\beta-\gamma}\ {}_3\phi_2\left[\begin{matrix}\alpha, \beta, 1+\frac{1}{2}\alpha \\ 1+\alpha-\gamma, \frac{1}{2}\alpha\end{matrix}\ |q; -xq^{1+\frac{1}{2}\alpha-\beta-\gamma}\right] d_q(x)$$
$$= \Gamma_q\left[\begin{matrix}\gamma, 1+\alpha-\gamma, 1+\frac{\alpha}{2}+b, 1+\frac{\alpha}{2}-\beta-\gamma+b \\ 1+\alpha, 1+\frac{\alpha}{2}-\beta+b, 1+\frac{\alpha}{2}-\gamma+b\end{matrix}\right],$$

where $b \equiv \frac{\log(-1)}{\log q}$.

Proof. Use [3, p. 269 (7.114)-(7.116)]. □

Corollary 2.12 (A q-analogue of [1, p. 289 (4.1)])**.**

$$\begin{aligned}&\int_0^1 x^{\beta-1}(qx;q)_{\alpha-2\beta}\ {}_3\phi_2\left[\begin{matrix}\alpha,\gamma,\widetilde{1+\frac{1}{2}\alpha}\\ 1+\alpha-\gamma,\widetilde{\frac{1}{2}\alpha}\end{matrix}|q;xq^{1+\frac{1}{2}\alpha-\beta-\gamma}\right]d_q(x)\\ &=\Gamma_q\left[\begin{matrix}\beta,\alpha-2\beta+1,1+\frac{\alpha}{2},1+\alpha-\gamma,1+\frac{\alpha}{2}-\beta-\gamma\\ 1+\alpha,1+\frac{\alpha}{2}-\beta,1+\frac{\alpha}{2}-\gamma,1+\alpha-\beta-\gamma\end{matrix}\right].\end{aligned}\tag{45}$$

Proof. Use [3, p. 268 (7.113)]. □

We will now find q-analogues of [8, p. 333 (2.5)-(2.7)].

Corollary 2.13 (A q-analogue of [7, p. 35 (2.1.2.5)] and of [8, p. 333 (2.5)])**.**
(46)

$$\begin{aligned}&\int_0^1 x^{\beta-1}(qx;q)_{\alpha-2\beta}\ {}_5\phi_4\left[\begin{matrix}\alpha,1+\frac{1}{2}\alpha,\widetilde{1+\frac{1}{2}a},\gamma,\delta\\ \frac{1}{2}\alpha,\widetilde{\frac{1}{2}a},1+\alpha-\gamma,1+\alpha-\delta\end{matrix}|q;xq^{1+\alpha-\beta-\gamma-\delta}\right]d_q(x)\\ &=\Gamma_q\left[\begin{matrix}\beta,\alpha-2\beta+1,1+\alpha-\gamma,1+\alpha-\delta,1+\alpha-\beta-\gamma-\delta\\ 1+\alpha,1+\alpha-\beta-\gamma,1+\alpha-\beta-\delta,1+\alpha-\gamma-\delta\end{matrix}\right].\end{aligned}$$

(A q-analogue of [8, p. 333 (2.6)])
(47)

$$\begin{aligned}&\int_0^1 x^{\gamma-1}(qx;q)_{\frac{1}{2}\alpha-\gamma-1}\ {}_5\phi_4\left[\begin{matrix}\alpha,1+\frac{1}{2}a,\widetilde{1+\frac{1}{2}a},\beta,\delta\\ 1+\alpha-\beta,1+\alpha-\gamma,\widetilde{\frac{1}{2}a},1+\alpha-\delta\end{matrix}|q;xq^{1+\alpha-\beta-\gamma-\delta}\right]d_q(x)\\ &=\Gamma_q\left[\begin{matrix}\gamma,\frac{1}{2}\alpha-\gamma,1+\alpha-\beta,1+\alpha-\gamma,1+\alpha-\delta,1+\alpha-\beta-\gamma-\delta\\ \frac{1}{2}\alpha,1+\alpha,1+\alpha-\beta-\gamma,1+\alpha-\beta-\delta,1+\alpha-\gamma-\delta\end{matrix}\right].\end{aligned}$$

(A q-analogue of [8, p. 333 (2.7)])
(48)

$$\begin{aligned}&\int_0^1 x^{\delta-1}(qx;q)_{\alpha-2\delta}\ {}_5\phi_4\left[\begin{matrix}\alpha,\beta,1+\frac{1}{2}\alpha,\widetilde{1+\frac{1}{2}a},\gamma\\ 1+\alpha-\gamma,\frac{1}{2}\alpha,\widetilde{\frac{1}{2}a},1+\alpha-\beta\end{matrix}|q;xq^{1+\alpha-\beta-\gamma-\delta}\right]d_q(x)\\ &=\Gamma_q\left[\begin{matrix}\delta,1+\alpha-2\delta,1+\alpha-\beta,1+\alpha-\gamma,1+\alpha-\beta-\gamma-\delta\\ 1+\alpha,1+\alpha-\beta-\delta,1+\alpha-\beta-\gamma,1+\alpha-\gamma-\delta\end{matrix}\right].\end{aligned}$$

Proof. Use [3, p. 268 (7.112)]. □

Corollary 2.14 (A q-analogue of [1, p. 291 (5.1)])**.**

$$\begin{aligned}&\int_0^1 x^{c-1}(qx;q)_{c-e}\ {}_3\phi_2\left[\begin{matrix}\widetilde{c},1+n,-n\\ \widetilde{1},e\end{matrix}|q;qx\right]d_q(x)\\ &=\mathrm{B}_q(c,c-e+1)\frac{\langle\widetilde{e},e-2c;q\rangle_\infty}{\langle\widetilde{e-n},-n+e-2c;q\rangle_\infty}\\ &\times\frac{\langle\frac{e+1+n}{2},\frac{-n+e+1}{2}-2c,-n+e;q^2\rangle_\infty}{\langle\frac{-n+e+1}{2},\frac{n+1+e}{2}-2c,e;q^2\rangle_\infty}.\end{aligned}\tag{49}$$

Proof. Use [3, p. 284 (8.22)]. □

2.1. Eulerian q-integrals with product argument

We will now investigate some q-integrals with product of two power series in the integrand.

Theorem 2.15 (A q-analogue of [7, p. 48 2.3.4]). *Assume that $f(x)$ can be written as the power series*

$$f(x) = \sum_{m=0}^{\infty} c_m x^m. \tag{50}$$

Then we have

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\beta-1}\ {}_2\phi_1(\gamma,\delta;\lambda|q;yx^k)f(x)d_q(x)\\ &= \mathrm{B}_q(\alpha,\beta)\sum_{m=0}^{\infty}\frac{\langle\alpha;q\rangle_m c_m}{\langle\alpha+\beta;q\rangle_m}\ {}_{2+2k}\phi_{1+2k}\left[\begin{matrix}\gamma,\delta,\triangle(q;k;\alpha+m)\\ \lambda,\triangle(q;k;\alpha+\beta+m)\end{matrix}|q;y\right].\end{aligned}\tag{51}$$

Corollary 2.16 (A q-analogue of the corrected version of [7, p. 48 2.3.6]). *Assume* (50). *Then we have*

$$\begin{aligned}&\int_0^1 x^{\alpha-1}(qx;q)_{\delta-\lambda-n}\ {}_2\phi_1(-n,\delta;\lambda|q;qx)f(x)d_q(x)\\ &= \mathrm{B}_q(\alpha,1+\delta-\lambda-n)\frac{\langle\lambda-\delta;q\rangle_n}{\langle\lambda;q\rangle_n}\\ &\sum_{m=0}^{\infty}\frac{\langle\alpha,1+\alpha-\lambda;q\rangle_m\langle 1+\alpha-\lambda+\delta;q\rangle_{m-n}c_m q^{\delta n}}{\langle\alpha+\delta+1-\lambda-n,1+\alpha-\lambda+\delta;q\rangle_m\langle 1+\alpha-\lambda;q\rangle_{m-n}}.\end{aligned}\tag{52}$$

Proof. Use formula (51) and

$$\frac{\langle\lambda-\alpha-m;q\rangle_n}{\langle\lambda-\alpha-\delta-m;q\rangle_n} = \frac{\langle 1+\alpha-\lambda;q\rangle_m\langle 1+\alpha-\lambda+\delta;q\rangle_{m-n}q^{\delta n}}{\langle 1+\alpha-\lambda+\delta;q\rangle_m\langle 1+\alpha-\lambda;q\rangle_{m-n}}. \tag{53}$$ □

3. Eulerian q-integrals for multiple q-hypergeometric series

We continue our investigations of the q-beta integral and find a vector version of our previous formula.

Theorem 3.1 (Almost a q-analogue of Exton [7, p. 121 6.1.6])). *Assume that $f(\vec{x})$ can be written as the multiple power series*

$$f(\vec{x}) = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} c_{\vec{m}}\vec{x}^{\vec{m}}. \tag{54}$$

Then we have

$$\int_{\vec{0}}^{\vec{1}} \vec{x}^{\vec{\alpha}-\vec{1}}(q\vec{x};q)_{\vec{\beta}-\vec{\alpha}-\vec{1}} f(\vec{x}\vec{s})d_q(\vec{x}) = \mathrm{B}_q(\vec{\alpha},\vec{\beta}-\vec{\alpha})\sum_{\vec{m}=\vec{0}}^{\vec{\infty}}\frac{\langle\vec{\alpha};q\rangle_{\vec{m}}}{\langle\vec{\beta};q\rangle_{\vec{m}}}c_{\vec{m}}\vec{s}^{\vec{m}}. \tag{55}$$

Proof. Similar to above. □

Remark 2. Formula (55) also applies when s is a scalar like in formulas (56), (59) and (62).

We next prove some special Eulerian q-integrals by using the q-binomial theorem. Observe that the q-Lauricella functions are the same in each equation.

Theorem 3.2 (A q-analogue of Koschmieder [11, 2.5 p. 65])**.**

$$\begin{aligned} &\mathrm{B}_q(\alpha,\lambda-\alpha)\Phi_{\mathrm{D}}^{(n)}(\alpha,\vec{\beta};\gamma|q;\vec{x}) \\ &\cong \int_0^1 s^{\alpha-1}(qs;q)_{\lambda-\alpha-1}\Phi_{\mathrm{D}}^{(n)}(\lambda,\vec{\beta};\gamma|q;s\vec{x})\ d_q(s). \end{aligned} \tag{56}$$

Proof. Compute the RHS:

$$\begin{aligned} &\sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\lambda;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\gamma;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\alpha+m)}\langle 1+k;q\rangle_{\lambda-\alpha-1} \\ &= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\lambda;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\gamma;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\alpha+m)}\frac{\langle\lambda-\alpha;q\rangle_k\langle 1;q\rangle_\infty}{\langle 1;q\rangle_k\langle\lambda-\alpha;q\rangle_\infty} \\ &= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\lambda;q\rangle_m\langle\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\gamma;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\frac{\langle m+\lambda,1;q\rangle_\infty}{\langle\lambda-\alpha,\alpha+m;q\rangle_\infty} = \text{LHS}. \end{aligned} \tag{57}$$

□

Corollary 3.3.

$$\begin{aligned} &\mathrm{B}_q(\alpha,\lambda-\alpha)\Phi_1(\alpha,\beta_1,\beta_2;\gamma|q;x_1,x_2) \\ &\cong \int_0^1 s^{\alpha-1}(qs;q)_{\lambda-\alpha-1}\Phi_1(\lambda,\beta_1,\beta_2;\gamma|q;sx_1,sx_2)\ d_q(s). \end{aligned} \tag{58}$$

Theorem 3.4 (A q-analogue of [11, 3.2 p. 66]))**.**

$$\begin{aligned} &\mathrm{B}_q(\nu,\gamma-\nu)\Phi_{\mathrm{B}}^{(n)}(\vec{\alpha},\vec{\beta};\gamma|q;\vec{x}) \\ &\cong \int_0^1 s^{\nu-1}(qs;q)_{\gamma-\nu-1}\Phi_{\mathrm{B}}^{(n)}(\vec{\alpha},\vec{\beta};\nu|q;s\vec{x})\ d_q(s). \end{aligned} \tag{59}$$

Proof. We compute the right hand side:

$$\begin{aligned} &\sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\vec{\alpha},\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\nu;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\nu+m)}\langle 1+k;q\rangle_{\gamma-\nu-1} \\ &= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\vec{\alpha},\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\nu;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\sum_{k=0}^{\infty}q^{k(\nu+m)}\frac{\langle\gamma-\nu;q\rangle_k\langle 1;q\rangle_\infty}{\langle 1;q\rangle_k\langle\gamma-\nu;q\rangle_\infty} \\ &= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle\vec{\alpha},\vec{\beta};q\rangle_{\vec{m}}\vec{x}^{\vec{m}}}{\langle\nu;q\rangle_m\langle\vec{1};q\rangle_{\vec{m}}}(1-q)\frac{\langle m+\gamma,1;q\rangle_\infty}{\langle\gamma-\nu,\nu+m;q\rangle_\infty} = \text{LHS}. \end{aligned} \tag{60}$$

□

Corollary 3.5.

$$\begin{aligned} &\mathrm{B}_q(\nu,\gamma-\nu)\Phi_3(\alpha_1,\alpha_2;\beta_1,\beta_2;\gamma|q;x_1,x_2) \\ &\cong \int_0^1 s^{\nu-1}(qs;q)_{\gamma-\nu-1}\Phi_3(\alpha_1,\alpha_2;\beta_1,\beta_2;\nu|q;sx_1,sx_2)\ d_q(s). \end{aligned} \tag{61}$$

Theorem 3.6 (A q-analogue of [11, 3.4 p. 66])**.**

$$\begin{aligned} &\mathrm{B}_q(\alpha,\lambda-\alpha)\Phi_{\mathrm{C}}^{(n)}(\alpha,\beta;\vec{\gamma}|q;\vec{x}) \\ &\cong \int_0^1 s^{\alpha-1}(qs;q)_{\lambda-\alpha-1}\Phi_{\mathrm{C}}^{(n)}(\lambda,\beta;\vec{\gamma}|q;s\vec{x})\ d_q(s). \end{aligned} \tag{62}$$

Corollary 3.7.

$$\begin{aligned} &\mathrm{B}_q(\alpha,\lambda-\alpha)\Phi_4(\alpha,\beta;\gamma_1,\gamma_2|q;x_1,x_2) \\ &\cong \int_0^1 s^{\alpha-1}(qs;q)_{\lambda-\alpha-1}\Phi_4(\lambda,\beta;\gamma_1,\gamma_2|q;sx_1,sx_2)\ d_q(s). \end{aligned} \tag{63}$$

Theorem 3.8 (A q-analogue of [11, 3.5 p. 66])**.**

$$\begin{aligned} &\Phi_{\mathrm{C}}^{(n)}(\alpha,\beta;\vec{\gamma}|q;\vec{x}) \\ &\cong \Gamma_q\left[\begin{matrix}\lambda,\mu \\ \alpha,\lambda-\alpha,\beta,\mu-\beta\end{matrix}\right]\int_{\vec{0}}^{\vec{1}} s^{\alpha-1} \\ &\times(qs;q)_{\lambda-\alpha-1}t^{\beta-1}(qt;q)_{\mu-\beta-1}\Phi_{\mathrm{C}}^{(n)}(\lambda,\mu;\vec{\gamma}|q;st\vec{x})\ d_q(s)d_q(t). \end{aligned} \tag{64}$$

Proof. We compute the right hand side ($\vec{k}$ has dimension 2):

$$\begin{aligned} &\Gamma_q\sum_{\vec{m},\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\lambda,\mu;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{\gamma},\vec{1};q\rangle_{\vec{m}}}(1-q)^2q^{\overline{k((\alpha,\beta)+\vec{m})}}\langle\vec{1}+\vec{k};q\rangle_{\lambda,\vec{\mu-\alpha},\beta-\vec{1}} \\ &=\Gamma_q\sum_{\vec{m},\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\lambda,\mu;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{\gamma},\vec{1};q\rangle_{\vec{m}}}(1-q)^2q^{\overline{k((\alpha,\beta)+\vec{m})}}\frac{\langle(\lambda-\alpha,\mu-\beta);q\rangle_{\vec{k}}\langle\vec{1};q\rangle_{\vec{\infty}}}{\langle\vec{1};q\rangle_{\vec{k}}\langle(\lambda-\alpha,\mu-\beta);q\rangle_{\vec{\infty}}} \\ &=\Gamma_q\sum_{\vec{m},\vec{k}=\vec{0}}^{\vec{\infty}}\frac{\langle\lambda,\mu;q\rangle_m\vec{x}^{\vec{m}}}{\langle\vec{\gamma},\vec{1};q\rangle_{\vec{m}}}(1-q)^2\frac{\langle\lambda+m,\mu+m,\vec{m}+\vec{\gamma},\vec{1};q\rangle_{\vec{\infty}}}{\langle(\lambda-\alpha,\mu-\beta,\lambda+m,\mu+m);q\rangle_{\vec{\infty}}}=\mathrm{LHS}. \end{aligned} \tag{65}$$

□

Corollary 3.9.

$$\begin{aligned} &\Phi_4(\alpha,\beta;\gamma_1,\gamma_2|q;x_1,x_2) \\ &\cong \Gamma_q\left[\begin{matrix}\lambda,\mu \\ \alpha,\lambda-\alpha,\beta,\mu-\beta\end{matrix}\right]\int_{\vec{0}}^{\vec{1}} s^{\alpha-1} \\ &\times(qs;q)_{\lambda-\alpha-1}t^{\beta-1}(qt;q)_{\mu-\beta-1}\Phi_4(\lambda,\mu;\gamma_1,\gamma_2|q;stx_1,stx_2)\ d_q(s)d_q(t). \end{aligned} \tag{66}$$

All of these q-integral formulas are similar to some general q-analogues of Exton [7, p. 35 ff]:

Theorem 3.10 (A q-analogue of [7, p. 35 2.1.3.5]).

$$\begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D;D'}_{C':G;G'}\left[\begin{matrix}(c):(d);(d')\\(c'):(g);(g')\end{matrix}|q;rx^k,sx^k\right]d_q(x)\\&=\mathrm{B}_q(a,b)\Phi^{C+2k:D;D'}_{C'+2k:G;G'}\left[\begin{matrix}(c),\triangle(q;k;a):(d);(d')\\(c'),\triangle(q;k;a+b):(g);(g')\end{matrix}|q;r,s\right].\end{aligned}\tag{67}$$

(A q-analogue of [7, p. 36 2.1.3.6])

$$\begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D;D'}_{C':G;G'}\left[\begin{matrix}(c):(d);(d')\\(c'):(g);(g')\end{matrix}|q;r,sx^k\right]d_q(x)\\&=\mathrm{B}_q(a,b)\Phi^{C:D;D'+2k}_{C':G;G'+2k}\left[\begin{matrix}(c):(d);(d'),\triangle(q;k;a)\\(c'):(g);(g'),\triangle(q;k;a+b)\end{matrix}|q;r,s\right].\end{aligned}\tag{68}$$

(A q-analogue of [7, p. 36 2.1.3.7])

$$\begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D;D'}_{C:D-1;D-1}\left[\begin{matrix}(c):(d);(d')\\(c'):(g);(g')\end{matrix}|q;rx^k,s(1\boxminus_q q^bx)^k\right]d_q(x)\\&=\mathrm{B}_q(a,b)\Phi^{C+2k:D+2k;D'+2k}_{C+2k:D+2k-1;D'+2k-1}\left[\begin{matrix}(c),2k\infty:(d),\triangle(q;k;a);(d'),\triangle(q;k;b)\\(c'),\triangle(q;k;a+b):(g),2k\infty;(g'),2k\infty\end{matrix}|q;r,s\right].\end{aligned}\tag{69}$$

Proof. We only prove the last formula, (69). Start with the left hand side and change the order between summation and q-integration.

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{\langle(c);q\rangle_{m+n}\langle(d);q\rangle_m\langle(d');q\rangle_n r^m s^n}{\langle(c');q\rangle_{m+n}\langle1,(g);q\rangle_m\langle1,(g)');q\rangle_n}\int_0^1 x^{a+mk-1}(qx;q)_{b+kn-1}\\&=\sum_{m,n=0}^{\infty}\frac{\langle(c);q\rangle_{m+n}\langle(d);q\rangle_m\langle(d');q\rangle_n r^m s^n}{\langle(c');q\rangle_{m+n}\langle1,(g);q\rangle_m\langle1,(g');q\rangle_n}\Gamma_q\left[\begin{matrix}a+km,b+kn\\a+b+k(m+n)\end{matrix}\right]\\&=\sum_{m,n=0}^{\infty}\frac{\langle(c);q\rangle_{m+n}\langle(d),\triangle(q;k;a);q\rangle_m\langle(d'),\triangle(q;k;b);q\rangle_n r^m s^n}{\langle(c'),\triangle(q;k;a+b);q\rangle_{m+n}\langle1,(g);q\rangle_m\langle1,(g');q\rangle_n}\\&\times\Gamma_q\left[\begin{matrix}a,b\\a+b\end{matrix}\right]=\text{RHS}.\end{aligned}\tag{70}$$

□

Corollary 3.11 (A q-analogue of [7, p. 36 2.1.3.9]).

$$\begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{2k:D;D'}_{2k:G;G'}\left[\begin{matrix}\triangle(q;k;a+b):(d);(d')\\(\triangle(q;k;a):(g);(g')\end{matrix}|q;rx^k,sx^k\right]d_q(x)\\&=\mathrm{B}_q(a,b)\ {}_D\Phi_G\left[\begin{matrix}(d)\\(g)\end{matrix}|q;r\right]\ {}_{D'}\Phi_{G'}\left[\begin{matrix}(d')\\(g')\end{matrix}|q;s\right].\end{aligned}\tag{71}$$

Proof. Put

$$C=C'=2k,\ \prod_{k=1}^{C}\langle c_k;q\rangle_m=\langle\triangle(q;k;a+b)\rangle_m,\ \prod_{k=1}^{C}\langle c'_k;q\rangle_m=\langle\triangle(q;k;a)\rangle_m$$

in (67). □

Corollary 3.12 (A q-analogue of the corrected version of [7, p. 36 2.1.3.10])**.**

(72)

$$\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:1;2k+1}_{C':0;2k}\left[\begin{array}{c}(c):d;d',\triangle(q;k;a+b)\\(c'):-;\triangle(q;k;a)\end{array}|q;s,sx^kq^{-d'}\right]d_q(x)$$
$$=\mathrm{B}_q(a,b)_{C+1}\Phi_{C'}\left((c),d+d';(c')|q;r|q;sq^{-d'}\right).$$

Proof. Put

$$\text{(73)}\quad \begin{aligned}&r=s,\ D=1,\ D'=2k+1,\ G=0,\ G'=2k,\ (d')=d',\triangle(q;k;a+b),\\&g'=\triangle(q;k;a)\end{aligned}$$

in (68) and use [3, 10.143, p. 390]. □

Corollary 3.13 (A q-analogue of [7, p. 36 2.1.3.11])**.**

(74)

$$\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:1;1}_{C:0;0}\left[\begin{array}{c}(c):\infty;\infty\\(c'):\triangle(q;k;a);\triangle(q;k;b)\end{array}|q;rx^k,s(1\boxminus_q q^bx)^k\right]d_q(x)$$
$$=\mathrm{B}_q(a,b)_{C+2k+1}\Phi_{C+2k}\left((c),(2k+1)\infty;(c'),\triangle(q;k;a+b)|q;r\oplus_q s\right).$$

Proof. Put

$$D=D'=1,\ (g)=\triangle(q;k;a),\ (g')=\triangle(q;k;b)$$

in (69) and use the fundamental property of NWA q-addition. □

We will now compute two Eulerian q-integrals with the Karlsson definition (14).

Theorem 3.14 (A q-analogue of [7, p. 37 2.1.4.4])**.**

$$\text{(75)}\quad \begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D}_{C':G}\left[\begin{array}{c}(c):\vec{d}\\(c'):\vec{g}\end{array}|q;x^k\vec{s}\right]d_q(x)\\&=\mathrm{B}_q(a,b)\Phi^{C+2k:D}_{C'+2k:G}\left[\begin{array}{c}(c),\triangle(q;k;a):\vec{d}\\(c'),\triangle(q;k;a+b):\vec{g}\end{array}|q;\vec{s}\right].\end{aligned}$$

Theorem 3.15 (Compare with [7, p. 37 2.1.4.2])**.**

$$\text{(76)}\quad \begin{aligned}&\int_0^1 x^{a-1}(qx;q)_{b-1}\Phi^{C:D}_{C:D-1}\left[\begin{array}{c}(c):\vec{d}\\(c'):\vec{g}\end{array}|q;x^k\vec{s}||(xq^b;q)_{mk}\right]d_q(x)\\&=\mathrm{B}_q(a,b)\sum_{\vec{m}}\frac{\langle(c),\triangle(q;k;a,b);q\rangle_m\langle\vec{d};q\rangle_{\vec{m}}\vec{s}^{\vec{m}}}{\langle(c'),\triangle(q;2k;a+b);q\rangle_m\langle\vec{1},\vec{g};q\rangle_{\vec{m}}}.\end{aligned}$$

3.1. Multiple Eulerian q-integrals

We first observe that formula (25) can be generalized in the following way:

Theorem 3.16 (A q-analogue of [7, p. 121 6.1.6]). *Assume that $f(\vec{x})$ can be written as the multiple power series*

$$f(\vec{x}) = \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} A_{\vec{m}} \vec{x}^{\vec{m}}. \tag{77}$$

Then we have

$$\int_{\vec{0}}^{\vec{1}} \vec{u}^{\vec{\alpha}-\vec{1}} (q\vec{u};q)_{\vec{\beta}-\vec{1}} f(\vec{x}\vec{u}) d_q(\vec{u}) = \mathrm{B}_q(\vec{\alpha},\vec{\beta}) \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle \vec{\alpha};q\rangle_{\vec{m}} A_{\vec{m}}}{\langle \vec{\alpha}+\vec{\beta};q\rangle_{\vec{m}}} \vec{x}^{\vec{m}}. \tag{78}$$

Theorem 3.17 (A q-analogue of [7, p. 121 6.1.1.1] written in vector form).

(79)

$$\int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha\vec{-}1} (q\vec{u};q)_{\beta-\vec{\alpha}-1}\ {}_C\phi_D((c),(d)|q;x_1u_1 \oplus_q x_2u_2 \oplus_q \cdots \oplus_q x_nu_n) d_q(\vec{u})$$
$$= \mathrm{B}_q(\vec{\alpha},\beta\vec{-}\alpha)\Phi^{C:2}_{D+1:1}\left[\begin{array}{c} (c):\vec{\infty},\vec{\alpha} \\ (d),\infty:\vec{\beta} \end{array} |q;\vec{x}\right].$$

Proof.

(80)

$$\text{LHS} = \int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha\vec{-}1} (q\vec{u};q)_{\beta-\vec{\alpha}-1} \sum_{k=0}^{\infty} \frac{\langle (c);q\rangle_k}{\langle 1,(d);q\rangle_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+D-C}$$
$$\sum_{|\vec{m}|=k} \prod_{l=1}^{n} (x_l u_l)^{m_l} \frac{\langle 1;q\rangle_k}{\langle 1;q\rangle_{\vec{m}}} d_q(\vec{u})$$
$$= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha+\vec{m}-1} (q\vec{u};q)_{\beta-\vec{\alpha}-1} d_q(\vec{u}) \frac{\langle (c);q\rangle_m}{\langle (d);q\rangle_m} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+D-C} \frac{\vec{x}^{\vec{m}}}{\langle \vec{1};q\rangle_{\vec{m}}}$$
$$= \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle (c);q\rangle_m \vec{x}^{\vec{m}}}{\langle (d);q\rangle_m \langle \vec{1};q\rangle_{\vec{m}}} \Gamma_q\left[\begin{array}{c} \alpha\vec{+}m,\beta\vec{-}\alpha \\ \beta\vec{+}m \end{array}\right]$$
$$= \Gamma_q\left[\begin{array}{c} \vec{\alpha},\beta\vec{-}\alpha \\ \vec{\beta} \end{array}\right] \sum_{\vec{m}=\vec{0}}^{\vec{\infty}} \frac{\langle (c);q\rangle_m \langle \vec{\alpha};q\rangle_{\vec{m}} \vec{x}^{\vec{m}}}{\langle (d);q\rangle_m \langle \vec{1},\vec{\beta};q\rangle_{\vec{m}}} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+D-C}$$
$$= \text{RHS}.$$

□

Theorem 3.18 (A q-analogue of [7, p. 122 6.1.2.1])**.**

$$\begin{aligned}(81)\quad &\int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha \vec{-} 1}(q\vec{u};q)_{\beta \vec{-} 1}\Phi_{F:G}^{C:D}\left[\begin{array}{c}(c):(\vec{\delta})\\(\varphi):(\vec{\gamma})\end{array}|q;x_1u_1,x_2u_2,\ldots,x_nu_n\right]d_q(\vec{u})\\ &=\mathrm{B}_q(\vec{\alpha},\vec{\beta})\Phi_{F:G+1}^{C:D+1}\left[\begin{array}{c}(c):(\vec{\delta}),\vec{\alpha},\\(\varphi):(\vec{\gamma}),\alpha \vec{+} \beta\end{array}|q;\vec{x}\right].\end{aligned}$$

Proof. Use formula (78). □

Corollary 3.19 (A q-analogue of [7, p. 123 6.1.3.2])**.** *A formula for the q-Laguerre polynomial.*

$$\begin{aligned}(82)\quad &\int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha \vec{-} 1}(q\vec{u};q)_{\beta-\vec{\alpha}-1}\mathrm{L}_{m,q}^{(\gamma)}(x_1u_1\oplus_q x_2u_2\oplus_q\cdots\oplus_q x_nu_n)d_q(\vec{u})\\ &=\mathrm{B}_q(\vec{\alpha},\beta \vec{-} \alpha)\frac{\langle\gamma+1;q\rangle_m}{\langle 1;q\rangle_m}\Phi_{2:1}^{1:2}\left[\begin{array}{c}-m:\vec{\infty},\vec{\alpha}\\\gamma+1,\infty:\vec{\beta}\end{array}|q;\vec{x}(-(1-q)q^{m+\gamma+1})\right].\end{aligned}$$

Proof. Use (79) and $\mathrm{L}_{n,q}^{(\alpha)}(x) = \frac{\langle\alpha+1;q\rangle_n}{\langle 1;q\rangle_n}{}_1\phi_1\left(-n;\alpha+1|q;-x(1-q)q^{n+\alpha+1}\right)$. □

Corollary 3.20 (A q-analogue of [7, p. 123 6.1.3.4])**.**

$$\begin{aligned}(83)\quad &\int_{\vec{0}}^{\vec{1}} \vec{u}^{\alpha \vec{-} 1}(q\vec{u};q)_{\beta-\vec{\alpha}-1}\Phi_{\mathrm{A}}^{(n)}(\gamma,\vec{\delta};\vec{\varphi}|q;x_1u_1,x_2u_2,\ldots,x_nu_n)d_q(\vec{u})\\ &=\mathrm{B}_q(\vec{\alpha},\beta \vec{-} \alpha)\Phi_{1:2}^{1:3}\left[\begin{array}{c}\gamma:\vec{\infty},\vec{\alpha},\vec{\delta}\\\infty:\vec{\beta},\vec{\varphi}\end{array}|q;\vec{x}\right].\end{aligned}$$

Proof. Use (81). □

4. Conclusion

All proofs use the same q-beta integral technique and the $\triangle$ notation by Srivastava, as well as the (multiple) NWA q-addition, which is used in the formulas. We have saved the old Koschmieder articles from oblivion and maybe some other multiple hypergeometric function might have a similar formula. In a forthcoming article we will consider the corresponding Laplace integrals, which are q-analogues of the Laplace transform.

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