Commun. Korean Math. Soc. **33** (2018), No. 1, pp. 171–177
https://doi.org/10.4134/CKMS.c170100
pISSN: 1225-1763 / eISSN: 2234-3024

UNIQUENESS OF SOLUTION FOR IMPULSIVE FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION

Sandeep Singhal and Pattani Samsudeen Sehik Uduman

Abstract. In this research paper considering a differential equation with impulsive effect and dependent delay and applied Banach fixed point theorem using the impulsive condition to the impulsive fractional functional differential equation of an order $\alpha \in (1, 2)$ to get an uniqueness solution. At last, theorem is verified by using a numerical example to illustrate the uniqueness solution.

1. Introduction

The differential equations with impulsive effects have been appeared as in natural description evolution processes. For more details of relevant development in fractional calculus and functional differential equations with state dependent delay in references [1–16]. Feckan et al. [3] present a counterexample to show an essence error in the formula of solutions to the impulsive Cauchy problems for differential equations with fractional order $q \in (0, 1)$. Feckan et al. find the correct formula and established the existence the by using the fixed point theorem. Author [16] established sufficient conditions for existence of the solutions by applying fixed point methods first for linear and nonlinear impulsive condition. Recently, Liu et al. [10] study the existence and uniqueness of solutions for nonlinear singular multiterm impulsive fractional differential equations.

Our present work motivated by the papers [3,10,16]. In this work, we study linear impulsive fractional functional integro-differential equation of the form:

$$D_t^\alpha y(t) = J_t^{2-\alpha} f(t, y_{\rho(t,y_t)}, B(y_{\rho(t,y_t)})), t \in J = [0, T],\ t \neq t_k, \tag{1}$$

$$\Delta y(t_k) = x_k,\ \ \Delta y'(t_k) = z_k, \quad k = 1, 2, \ldots, m, \tag{2}$$

$$y(t) = \phi(t),\ y'(t) = \varphi(t),\ t \in [-d, 0], \tag{3}$$

where y' denotes the derivative of y with respect to t and D_t^α is Caputo's derivative of order $\alpha \in (1, 2)$. $f : J \times PC_0 \times PC_0 \to X$ is given continuous

Received March 14, 2017; Accepted April 3, 2017.
2010 *Mathematics Subject Classification.* 26A33, 34K05, 34A12, 34A37.
Key words and phrases. fractional order differential equation, functional differential equations, impulsive conditions, fixed point theorem.

function and PC_0 is an abstract phase space with y_t the element of PC_0 defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-d, 0]$. We have impulse point $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, and $x_k, z_k \in \mathbb{R}$. We have $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ and $\Delta y'(t_k) = y'(t_k^+) - y'(t_k^-)$. The term $B(y_{\rho(t,y_t)})$ is given by $B(y_{\rho(t,y_t)}) = \int_0^t K(t,s)(y_{\rho(s,y_s)})ds$, where $K \in C(D, \mathbb{R}^+)$, is the set of all positive functions which are continuous on $D = \{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t < T\}$ and $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds < \infty$, respectively.

This paper is concerned with the existence results for impulsive fractional functional integro-differential equations with state dependent delay subject to initial conditions. To the best of the author's knowledge the concept of existence result of the considered problem in this paper is not treated in the literature. This work has four sections, second section provides some preliminaries. Third section is quipped with main results for the considered problem and in fourth section has an example.

2. Preliminary

Let $(X, \|\cdot\|_X)$ be a complex Banach space of functions with the norm $\|u\|_X = \sup_{t\in J}\{|u(t)| : u \in X\}$.

To avoid the repetitions of some definitions used in this paper we refer the paper [2] such as Riemann–Liouville fractional integral operator, Caputo's derivative, Phase space PC_0, PC_0' and others preliminary.

Definition (Podulbuny [12])**.** Caputo's derivative of order $\alpha > 0$ with lower limit a, for a function $f : [a, \infty) \to \mathbb{R}$ such that $f \in C^n([a, \infty), X)$, is defined as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = {}_a J_t^{n-\alpha} f^{(n)}(t), \ t > a, $$

where $a \geq 0,\ n-1 < \alpha < n,\ n \in \mathbb{N}$.

Definition (Podulbuny [12])**.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$ with lower limit a, for a continuous function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$ {}_aJ_t^0 f(t) = f(t),\ {}_aJ_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad t > a, $$

where $a \geq 0$ and $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.1. *A piecewise continuous differential function $y(t) : [-d, T] \to X$ is a solution of the system* (1)-(3) *if and only if it satisfied the following integral* *equation*

$$(4)\quad y(t)=\begin{cases}\phi(0)+\varphi(0)t+\int_0^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds, & t\in(0,t_1]\\ \phi(0)+\varphi(0)t+\sum_{0<t_k<t}x_k+\sum_{0<t_k<t}(t-t_k)z_k\\ +\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds\\ +\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds\\ +\int_{t_k}^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds, & t\in(t_k,t_{k+1}].\end{cases}$$

Proof. If $t\in(0,t_1]$, then by a standard procedure the solution of (1) we get

$$y(t)=a_0+b_0t+\int_0^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds, \tag{5}$$

using the initial condition $y(0)=\phi(0)$, we get $a_0=\phi(0)$, then the equation (5) become

$$y(t)=\phi(0)+b_0t+\int_0^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds. \tag{6}$$

On differentiating (5) with respect to t and by initial condition $y'(0)=\varphi(0)$, we get $b_0=\varphi(0)$, then the equation (6) become

$$y(t)=\phi(0)+\varphi(0)t+\int_0^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds. \tag{7}$$

If $t\in(t_1,t_2]$, then the solution of equation (1) we have

$$y(t)=a_1+b_1(t-t_1)+\int_{t_1}^t(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds, \tag{8}$$

by impulsive condition $y(t_1^+)-y(t_1^-)=x_1$ the equation (8) written as

$$y(t_1^+)=a_1, \tag{9}$$

similarly by impulsive condition $y(t_1^+)-y(t_1^-)=x_1$ the equation (7) written as

$$y(t_1^-)=\phi(0)+\varphi(0)t_1+\int_0^{t_1}(t_1-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds, \tag{10}$$

by the equations (9) and (10) we get

$$a_1=\phi(0)+\varphi(0)t_1+x_1+\int_0^{t_1}(t_1-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds. \tag{11}$$

Hence (8) can be written as

$$\begin{aligned}y(t)=\phi(0)+\varphi(0)t_1+x_1+b_1(t-t_1)+\int_0^{t_1}&(t_1-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})\\ +\int_{t_1}^t&(t-s)f(s,y_{\rho(s,y_s)},B(y_{\rho(s,y_s)})ds.\end{aligned} \tag{12}$$

On differentiating equation (12) with respect to t and by impulsive condition $y'(t_1^+) - y'(t_1^-) = z_1$, we get

$$y'(t_1^+) = b_1. \tag{13}$$

On differentiating equation (7) with respect to t and by impulsive condition $y'(t_1^+) - y'(t_1^-) = z_1$, we get

$$y'(t_1^-) = \varphi(0) + \int_0^{t_1} f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds. \tag{14}$$

From equation (13) and (14) we have

$$b_1 = \varphi(0) + z_1 + \int_0^{t_1} f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds. \tag{15}$$

Hence equation (12) can be written as

$$\begin{aligned} y(t) = \phi(0) + x_1 + \varphi(0)t + (t - t_1)z_1 + (t - t_1)\int_0^{t_1} f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds \\ + \int_0^{t_1} (t_1 - s)f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds \\ + \int_{t_1}^{t} (t - s)f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds. \end{aligned}$$

Repeating the process in this way, the solution $y(t)$ for $t \in (t_k, t_{k+1}]$ can be written as

$$\begin{aligned} y(t) = \phi(0) + \varphi(0)t + \sum_{0<t_k<t} x_k + \sum_{0<t_k<t} (t - t_k)z_k \\ + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds \\ + \sum_{0<t_k<t} (t - t_k)\int_{t_{k-1}}^{t_k} f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds \\ + \int_{t_k}^{t} (t - s)f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)})ds. \end{aligned}$$

Summarizing the result we get equation (4). It clear that the solution given in (4) satisfies the system (1)-(3). This completes the proof of the lemma. □

3. Existence results

To prove our results we shall assume the function $\rho : [0, T] \times PC_0 \to (-\infty, T]$ is continuous. For the forthcoming analysis, we need the following assumption:

(H_1) $f : J \times PC_0 \times PC_0 \to X$ is jointly continuous function and there exist positive constants L_{f1}, L_{f1} such that

$$\|f(t, \psi, \varphi) - f(t, \xi, \chi)\|_X \le L_{f1}\|\psi - \xi\|_{PC_0} + L_{f2}\|\varphi - \chi\|_{PC_0},\ \forall\ \psi, \varphi, \chi, \xi \in PC_0.$$

The proof of present theorem is based on contraction principal.

Theorem 3.1. *Suppose that the assumption* (H_1) *satisfied and*

$$\delta = \frac{T^2}{2}(2+3m)(L_{f1} + B^* L_{f2}) < 1,$$

where $B^* = \sup_{t\in[0,t]} \int_0^t K(t,s)ds$. *Then the problem* (1)-(3) *has a unique solution on* J.

Proof. Consider the space $PC_0'' = \{y \in PC_0' : y(0) = \phi(0)\}$ and $y(t) = \phi(t) - g(y)$ for $t \in [-d, 0]$ endowed with the uniform convergence topology.

Now let us define the operator $P : PC_0'' \to PC_0''$ by

(16)

$$Py(t) = \begin{cases} \phi(0) + \varphi(0)t + \int_0^t (t-s) f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) ds, \ t \in (0, t_1] \\ \phi(0) + \varphi(0)t + \sum_{0<t_k<t} x_k + \sum_{0<t_k<t} (t-t_k) z_k \\ + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s) f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) ds \\ + \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) ds \\ + \int_{t_k}^t (t-s) f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) ds, \ t \in (t_k, t_{k+1}]. \end{cases}$$

It is clear that the operator P is well defined. We will show that the operator $P : PC_0'' \to PC_0''$ has a fixed point. To show this, let us consider $y, y^* \in PC_0''$ for $t \in (0, t_1]$ then

$$\begin{aligned} &\|P(y)(t) - P(y^*)(t)\|_X \\ &\le \int_0^t (t-s) \|f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)}, B(y^*_{\rho(s,y^*_s)})\|_X ds \\ &\le \frac{T^2}{2}(L_{f1} + B^* L_{f2}) \|y - y^*\|_X. \end{aligned}$$

For $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &\|P(y)(t) - P(y^*)(t)\|_X \\ \le &\sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s) \|f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)}, B(y^*_{\rho(s,y^*_s)})\| ds \\ &+ \sum_{0<t_k<t} (t-t_k) \int_{t_{k-1}}^{t_k} \|f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)}, B(y^*_{\rho(s,y^*_s)})\| ds \\ &+ \int_{t_k}^t (t-s) \|f(s, y_{\rho(s,y_s)}, B(y_{\rho(s,y_s)}) - f(s, y^*_{\rho(s,y^*_s)}, B(y^*_{\rho(s,y^*_s)})\| ds \\ \le &\frac{T^2}{2}(2+3m)(L_{f1} + B^* L_{f2}) \|y - y^*\|_X. \end{aligned}$$

For all $t \in [0, T]$

$$\|P(y)(t) - P(y^*)(t)\|_X \le \delta \|y - y^*\|_X.$$

Since $\delta < 1$, its implies that P is a contraction mapping and P has a unique fixed point $y \in PC_0''$. Its means that the system (1)-(3) has an unique solution on J. This complete the proof of the theorem. □

4. Example

Consider the following nonlinear impulsive fractional functional integral boundary value problem

$$D_t^{\frac{3}{2}} y(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} [\frac{e^{-t} y(t-\sigma(\|y\|))}{(9+e^t)(1+y(t-\sigma(\|y\|)))}$$

$$+ \int_0^t \cos(t-s) \frac{y(t-\sigma(\|y\|))}{25} ds] ds, \; t \neq \frac{1}{2}, \tag{17}$$

$$y(t) = \phi(t), \; y'(t) = \psi(t), t \in [-d, 0], \tag{18}$$

$$\Delta y|_{t=\frac{1}{2}} = \frac{x_{\frac{1}{2}}}{5}, \; \Delta y'|_{t=\frac{1}{2}} = \frac{z_{\frac{1}{2}}}{9}. \tag{19}$$

For the phase space. By setting

$$f(t, \phi, B\phi) = \frac{e^{-t}\phi}{(9+e^t)(1+\phi)} + \int_0^t \cos(t-s) \frac{\phi}{25} ds,$$

the equations (17)-(19) can be written in the abstract form as (1)-(3). We can easily verified all assumptions of (H_1). Therefore system (17)-(19) has a unique solution.

5. Conclusion

The conclusion is based on the results derived from this paper. The Banach fixed point theorem has been interpolated to obtain the sufficient condition of uniqueness solution for an impulsive fractional functional differential equation order $\alpha \in (1, 2)$, with the state dependent delay and an impulsive effect and also the numerical example has been illustrated to verify the theorem. This result is having vast application in the field of science, engineering, astrophysics, classical mechanics and harmonic motion function. Further this paper may be carried forward to higher fractional order of differential equation.

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Sandeep Singhal
Department of Mathematics
B. S. Abuar Rahman University
Vandalur, Chennai-600048, India
Email address: ss221085@gmail.com

Pattani Samsudeen Sehik Uduman
Department of Mathematics
B. S. Abuar Rahman University
Vandalur, Chennai-600048, India
Email address: sheikuduman@bsauniv.ac.in