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AUTOMATIC CONTINUITY OF n-JORDAN HOMOMORPHISMS ON BANACH ALGEBRAS

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Abstract. In this paper, we show that every unital n-Jordan homomorphism φ from a Banach algebra $\mathcal{A}$ onto a semisimple commutative Banach algebra $\mathcal{B}$ is continuous.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras and let $n \geq 2$ be an integer. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called an n-homomorphism [anti n-homomorphism] if, for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$,

$$\varphi(a_1a_2\cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)[= \varphi(a_n)\cdots\varphi(a_2)\varphi(a_1)].$$

A 2-homomorphism is then just a homomorphism in the usual sense. Furthermore, every homomorphism is clearly also an n-homomorphism for all $n \geq 2$, but the converse is false, in general. The concept of an n-homomorphism was studied for complex algebras by Hejazian et al. in [9].

Automatic continuity of n-homomorphisms considered for factorizable Banach algebras in [10], and it is extended for non factorizable Banach algebras in [8]. One may refer to [3] for automatic continuity of 3-homomorphism. It is due to Park and Trout that every $*$-preserving n-homomorphism between C^*-algebras is continuous [13].

In [7] Eshaghi Gordji introduced the concept of an n-Jordan homomorphism. A linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called an n-Jordan homomorphism if

$$\varphi(a^n) = \varphi(a)^n, \qquad (a \in \mathcal{A}).$$

A 2-Jordan homomorphism is called simply a Jordan homomorphism.

Obviously, each n-homomorphism is an n-Jordan homomorphism, but in general the converse is false.

Zelazko [14] has given a characterization of Jordan homomorphism, that we mention in the following. See [15] for another approach to the same result.

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Theorem 1.1. *Suppose that $\mathcal{A}$ is a Banach algebra, which need not be commutative, and suppose that $\mathcal{B}$ is a semisimple commutative Banach algebra. Then each Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism.*

Some results about 3-Jordan homomorphisms on Banach and C^*-algebras obtained by the author in [16].

In this paper we investigate automatic continuity of n-Jordan homomorphism, and prove that every unital n-Jordan homomorphism φ from a Banach algebra $\mathcal{A}$ into a semisimple commutative Banach algebra $\mathcal{B}$ is continuous.

We say that a linear map $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is a co-Jordan homomorphism if for all $a \in \mathcal{A}$, $\varphi(a^2) = -\varphi(a)^2$. For example, $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\varphi(a) = -a$ is a co-Jordan homomorphism.

2. Automatic continuity of n-Jordan homomorphisms

It is well known that every multiplicative linear functional φ on Banach algebra $\mathcal{A}$ is continuous and $\|\varphi\| \leq 1$, see [2] for example.

For Jordan (co-Jordan) homomorphism we have the following.

Proposition 2.1. *Let $\mathcal{A}$ be a Banach algebra and $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a Jordan or co-Jordan homomorphism. Then $\|\varphi\| \leq 1$.*

Proof. Let φ be a Jordan homomorphism and suppose that there exists $a \in \mathcal{A}$ with $\|a\| < 1$ and $|\varphi(a)| > 1$. Take $b = a/\varphi(a)$. Then $\|b\| < 1$ and $\varphi(b) = 1$, which is a contradiction by Theorem 6 of [15]. Thus, for all $a \in \mathcal{A}$ with $\|a\| < 1$, $|\varphi(a)| \leq 1$. The proof is similar, if φ is a co-Jordan homomorphism. □

Part (1) of the next theorem is due to Sinclair, and part (2) is due to Civin and Yood [4].

Theorem 2.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan homomorphism. Then φ is continuous if either:*

(1) *$\mathcal{B}$ is semisimple and $\varphi(\mathcal{A}) = \mathcal{B}$, or*
(2) *$\mathcal{B}$ is strongly semisimple and $\varphi(\mathcal{A})$ is dense in $\mathcal{B}$.*

Proof. See [12]. □

A Banach algebra $\mathcal{A}$ is called semiprime if $a\mathcal{A}a = \{0\}$ implies $a = 0$.

Proposition 2.3. *Let φ be a surjective Jordan homomorphism from Banach algebra $\mathcal{A}$ onto a semiprime Banach algebra $\mathcal{B}$. Then φ is continuous if either:*

(1) *$\mathcal{B}$ has a one-sided minimal ideals, or*
(2) *$\mathcal{B}$ is finite dimensional.*

Proof. If $\mathcal{B}$ has a one-sided minimal ideal, then it is semisimple by Corollary 4.1 of [6]. If $\mathcal{B}$ is finite dimensional, then it is semisimple by Corollary 8 of [5]. Thus, in both cases the result follows from Theorem 2.2. □

The next result follows from Zelazko's Theorem and Theorem 8, § 17 of [2].

Theorem 2.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Jordan homomorphism. If $\mathcal{B}$ is commutative and semisimple, then φ is continuous.*

Theorem 2.5. *Let φ be a 3-Jordan homomorphism between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{B}$ is commutative and semisimple, then φ is continuous.*

Proof. Let $\psi : \mathcal{B} \longrightarrow \mathbb{C}$ be a 3-Jordan homomorphism. Thus, ψ is either a Jordan homomorphism or a co-Jordan homomorphism by Lemma 2.3 of [16]. Hence, ψ is bounded by Proposition 2.1, and

$$\psi \circ \varphi(a^3) = \psi(\varphi(a^3)) = \psi(\varphi(a)^3) = \psi(\varphi(a))^3 = \psi \circ \varphi(a)^3.$$

Therefore $\psi \circ \varphi$ is a 3-Jordan homomorphism from $\mathcal{A}$ into $\mathbb{C}$, so it is bounded. Now suppose that (a_m) is a sequence in $\mathcal{A}$ such that $\lim_m a_m = a$ and $\lim_m \varphi(a_m) = b$. Then

$$\psi(b) = \psi(\lim_m \varphi(a_m)) = \lim_m \psi \circ \varphi(a_m) = \psi \circ \varphi(a),$$

thus, $\psi(b - \varphi(a)) = 0$. Since $\mathcal{B}$ is semisimple, we get $\varphi(a) = b$, and the close graph Theorem applies. □

The next lemma proves that each Jordan homomorphism is n-Jordan.

Lemma 2.6. *Every Jordan homomorphism φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is n-Jordan homomorphism, for $n \geqslant 2$.*

Proof. Assume that φ is a Jordan homomorphism, then $\varphi(a^2) = \varphi(a)^2$ for all $a \in \mathcal{A}$. Replacing a by $a + b$, we get

$$\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a). \tag{2.1}$$

Replacing b by a^2 in (2.1), gives

$$\varphi(a^3) = \varphi(a)^3 \tag{2.2}$$

for all $a \in \mathcal{A}$, and so φ is 3-Jordan homomorphism. Replacing b by a^3 in (2.1), we get

$$2\varphi(a^4) = \varphi(a)\varphi(a^3) + \varphi(a^3)\varphi(a), \quad (a \in \mathcal{A}). \tag{2.3}$$

By (2.2) and (2.3), we get $\varphi(a^4) = \varphi(a)^4$. Thus, φ is 4-Jordan homomorphism. An easy induction argument now finishes the proof. □

A linear map φ between unital Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is called unital if $\varphi(e) = e$, where e is the unit for both $\mathcal{A}$ and $\mathcal{B}$.

Theorem 2.7. *For $n \geq 2$, every unital $(n + 1)$-Jordan homomorphism φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is n-Jordan homomorphism.*

Proof. Let $n = 2$ and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a unital 3-Jordan homomorphism. Then for all $a \in \mathcal{A}$,

$$\varphi(a^3) = \varphi(a)^3. \tag{2.4}$$

Replacing a by $a+e$ in (2.4), we get $\varphi(a^2) = \varphi(a)^2$. So φ is a Jordan homomorphism.

Now let $n = 3$, and φ be a unital 4-Jordan homomorphism. Then $\varphi(a^4) = \varphi(a)^4$ for all $a \in \mathcal{A}$. Replacing a by $a+e$, we get

$$\varphi(3a^2 + 2a^3) = 3\varphi(a)^2 + 2\varphi(a)^3. \tag{2.5}$$

Replacing a by $a+e$ in (2.5), we get

$$\varphi(a^2) = \varphi(a)^2. \tag{2.6}$$

It follows from (2.5) and (2.6) that $\varphi(a^3) = \varphi(a)^3$ for all $a \in \mathcal{A}$. A similar discussion reveals that the result will be established for $n \geqslant 4$. □

The next corollary follows from Lemma 2.6 and Theorem 2.7.

Corollary 2.8. *A unital linear map φ between Banach algebras $\mathcal{A}$ and $\mathcal{B}$ is Jordan if and only if it is an n-Jordan homomorphism for $n \geq 2$.*

From Corollary 2.8 and Theorem 2.4, we deduce the following result.

Corollary 2.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, where $\mathcal{B}$ is semisimple and commutative. Then every unital n-Jordan homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is automatically continuous.*

The below result follows from Theorem 2.7 and Theorem 2.2.

Corollary 2.10. *Every unital n-Jordan homomorphism φ from Banach algebra $\mathcal{A}$ onto a semisimple Banach algebra $\mathcal{B}$ is automatically continuous.*

Since every C^*-algebra is semisimple, we get the next result.

Corollary 2.11. *Every unital n-Jordan homomorphism φ from Banach algebra $\mathcal{A}$ onto a C^*-algebra $\mathcal{B}$ is automatically continuous.*

In general, the kernel of an n-Jordan homomorphism may not be an ideal. A counter-example has been given in [15].

For n-Jordan homomorphism, we have the following.

Corollary 2.12. *Let φ be a unital n-Jordan homomorphism from Banach algebra $\mathcal{A}$ into a semiprime Banach algebra $\mathcal{B}$. Then $\ker\varphi$ is an ideal if either φ is surjective, or $\mathcal{B}$ is commutative.*

Proof. The result follows from Theorem 2.7 and Corollary 6.3.8 of [12], if φ is surjective, and it follows from Proposition 2 of [15], if $\mathcal{B}$ is commutative. □

3. Almost n-Jordan homomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a linear map. Then φ is called almost n-multiplicative, if there exists $\xi > 0$ such that for all $a_1, a_2, \ldots, a_n \in \mathcal{A}$,

$$\|\varphi(a_1 a_2 \cdots a_n) - \varphi(a_1) \cdots \varphi(a_n)\| \leq \xi \, \|a_1\| \|a_2\| \cdots \|a_n\|.$$

Moreover, φ is said to be almost n-Jordan homomorphism if there exists $\delta > 0$ such that

$$\|\varphi(a^n) - \varphi(a)^n\| \leq \delta \; \|a\|^n, \qquad (a \in \mathcal{A}).$$

It is obvious that if φ is almost n-multiplicative, then it is almost n-Jordan, but in general the converse is not true.

In [11], Jarosz proved the following theorem.

Theorem 3.1. *Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be an almost multiplicative linear functional. Then φ is continuous and $\|\varphi\| \leq 1 + \xi$.*

The next result, which is a generalization of Jarosz's theorem, obtained in [1].

Theorem 3.2. *Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an almost multiplicative linear map. If $\mathcal{B}$ is semisimple, then φ is continuous.*

In the next result we prove that Theorem 3.1 is valid for almost n-Jordan homomorphism.

Theorem 3.3. *Let φ be an almost n-Jordan homomorphism from Banach algebra $\mathcal{A}$ into $\mathbb{C}$. Then $\|\varphi\| \leq 1 + \delta$.*

Proof. By definition we have

$$\|\varphi\| = \sup\{|\varphi(a)| : a \in \mathcal{A}, \; \|a\| = 1\}.$$

Thus, for $0 < \lambda < \sqrt{\delta}$, there exists $a \in \mathcal{A}$ with $\|a\| = 1$ and $\|\varphi\| - \lambda < |\varphi(a)|$. So

$$|\varphi(a)|^n - |\varphi(a^n)| = |\varphi(a)^n| - |\varphi(a^n)| \leq |\varphi(a^n) - \varphi(a)^n| \leq \delta.$$

Hence

$$|\varphi(a)|^n \leq |\varphi(a^n)| + \delta.$$

Since $\|a\| = 1$, we have

$$(\|\varphi\| - \lambda)^n < |\varphi(a)|^n \leq |\varphi(a^n)| + \delta \leq \|\varphi\| + \delta.$$

Letting $\lambda \longrightarrow 0$, we get $\|\varphi\|^n - \|\varphi\| \leq \delta$. Now let $\|\varphi\| > 1 + \delta$. Then

$$(1+\delta)^{n-1} < \|\varphi\|^{n-1} \leq 1 + \frac{\delta}{\|\varphi\|} \leq 1 + \delta,$$

which is a contradiction. So, $\|\varphi\| \leq 1 + \delta$. □

Theorem 3.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two Banach algebras and $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be an almost n-Jordan homomorphism. If $\mathcal{B}$ is commutative and semisimple, then φ is continuous.*

Proof. Let $\psi : \mathcal{B} \longrightarrow \mathbb{C}$ be an n-Jordan homomorphism. Then ψ is bounded by Theorem 3.3, so we get

$$|\psi \circ \varphi(a^n) - (\psi \circ \varphi(a))^n| \leq \|\psi\| \; \|\varphi(a^n) - \varphi(a)^n\| \leq (1+\delta) \; \delta \; \|a\|^n.$$

Therefore $\psi \circ \varphi$ is an almost n-Jordan homomorphism, and hence it is continuous by above theorem. Suppose $(a_m) \subseteq \mathcal{A}$ is a sequence converging to zero and $\varphi(a_m)$ converges to $b \in \mathcal{B}$. Then

$$\psi(b) = \psi(\lim_m \varphi(a_m)) = \lim_m \psi \circ \varphi(a_m) = 0,$$

thus, $\psi(b) = 0$. Since $\mathcal{B}$ is semisimple, it follows that $b = 0$. Hence from the close graph Theorem, φ is continuous. □

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