

EXTENDED WRIGHT-BESSEL FUNCTION AND ITS PROPERTIES

MUHAMMAD ARSHAD, SHAHID MUBEEN, KOTTAKKARAN SOOPPY NISAR,
AND GAUHAR RAHMAN

ABSTRACT. In this present paper, our aim is to introduce an extended Wright-Bessel function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ which is established with the help of the extended beta function. Also, we investigate certain integral transforms and generalized integration formulas for the newly defined extended Wright-Bessel function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ and the obtained results are expressed in terms of Fox-Wright function. Some interesting special cases involving an extended Mittag-Leffler functions are deduced.

1. Introduction

The well known Wright-Bessel function (or Bessel-Maitland function, misnamed after the second name Maitland of E. M. Wright) defined in [24] as:

$$(1.1) \quad J_{\alpha}^{\lambda}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\alpha + \lambda n + 1)n!}; \lambda > -1, z \in \mathbb{C}.$$

Pathak [19] gave the generalization of (1.1) as:

$$(1.2) \quad J_{\alpha,\beta}^{\lambda}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{\alpha+2\beta+2n}}{\Gamma(\beta + n + 1)\Gamma(\alpha + \beta + \lambda n + 1)},$$

where $\lambda > 0$, $z \in \mathbb{C} \setminus (-\infty, 0)$, $\alpha, \beta \in \mathbb{C}$.

Singh et al. [22] defined another generalization of Wright-Bessel function in the following form:

$$(1.3) \quad J_{\alpha,q}^{\lambda,\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\alpha + \lambda n + 1)n!},$$

where $\alpha, \lambda, \gamma \in \mathbb{C}$, $\Re(\lambda) > 0$, $\Re(\alpha) > -1$, $\Re(\gamma) > 0$ and $q \in (0, 1) \cup \mathbb{N}$ and $(\gamma)_0 = 1$ and $(\gamma)_{nq} = \frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)}$ is the generalized Pochhammer symbol (see [13]).

Received February 8, 2017; Accepted March 14, 2017.

2010 *Mathematics Subject Classification.* 33B20, 33C20, 33C45, 33C60, 33B15, 26A33.

Key words and phrases. gamma function, generalized hypergeometric function, Oberhettinger formula, Wright-Bessel function, generalized Wright function, Mittag-Leffler function.

Clearly, if $q = 0$, then (1.3) reduces to (1.1).

We recall the generalized hypergeometric function ${}_pF_q(z)$ is defined in [5] as:

$$(1.4) \quad {}_pF_q(z) = {}_pF_q, \begin{bmatrix} (\alpha_1), (\alpha_2), \dots, (\alpha_p) \\ (\beta_1), (\beta_2), \dots, (\beta_q) \end{bmatrix}; z \\ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $\alpha_i, \beta_j \in \mathbb{C}$; $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ and $b_j \neq 0, -1, -2, \dots$ and $(z)_n$ is the Pochhammer symbols. The gamma function is defined as:

$$(1.5) \quad \Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt, \mu \in \mathbb{C},$$

$$(1.6) \quad \Gamma(z+1) = z\Gamma(z), z \in \mathbb{C},$$

and beta function is defined as:

$$(1.7) \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt.$$

For $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, the Wright type hypergeometric function is defined (see [24]-[26]) by the following series as:

$$(1.8) \quad {}_p\Psi_q(z) = {}_p\Psi_q, \begin{bmatrix} (\alpha_i, A_i)_{1,p} \\ (\beta_j, B_j)_{1,q} \end{bmatrix}; z \\ = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{z^n}{n!},$$

where β_r and μ_s are real positive numbers such that

$$1 + \sum_{s=1}^q \beta_s - \sum_{r=1}^p \alpha_r > 0.$$

Equation (4.1) differs from the generalized hypergeometric function ${}_pF_q(z)$ defined (3.3) only by a constant multiplier. The generalized hypergeometric function ${}_pF_q(z)$ is a special case of ${}_p\Psi_q(z)$ for $A_i = B_j = 1$, where $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$:

$$(1.9) \quad \frac{1}{\prod_{j=1}^q \Gamma(\beta_j)} {}_pF_q, \begin{bmatrix} (\alpha_1), \dots, (\alpha_p) \\ (\beta_1), \dots, (\beta_q) \end{bmatrix}; z = \frac{1}{\prod_{i=1}^p \Gamma(\alpha_i)} {}_p\Psi_q, \begin{bmatrix} (\alpha_i, 1)_{1,p} \\ (\beta_j, 1)_{1,q} \end{bmatrix}; z.$$

Recently Sharma and Devi [21] defined extended Wright type function in the following series:

$${}_{p+1}\Psi_{q+1}(z, p) = {}_{p+1}\Psi_{q+1}, \begin{bmatrix} (\alpha_i, A_i)_{1,p}, (\gamma, 1) \\ (\beta_j, B_j)_{1,q}, (c, 1) \end{bmatrix}; (z, p)$$

$$(1.10) \quad = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{B_p(\gamma + n, c - \gamma) z^n}{\Gamma(c - \gamma) n!},$$

where $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$, $\Re(p) > 0$, $\Re(c) > \Re(\gamma) > 0$ and $p \geq 0$. In this paper, we introduce an extended Wright-Bessel (Bessel-Maitland) function as defined in (1.3) as:

$$(1.11) \quad J_{\alpha,q}^{\lambda,\gamma,c}(z, p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq}(-z)^n}{\Gamma(\alpha + \lambda n + 1) n!},$$

where $p \geq 0$, $\alpha, \lambda, \gamma, c \in \mathbb{C}$, $\Re(p) > 0$, $\Re(\lambda) \geq 0$, $\Re(\alpha) > -1$, $\Re(\gamma) > 0$, $\Re(c) > 0$ and $B_p(x, y)$ is an extended beta function (see [2], [3]).

Remark 1.1. i) If we set $p = 0$ in (1.11), then extended Wright-Bessel function reduces to the (1.3).

ii) If we set $p, q = 0$ in (1.11), then extended Wright-Bessel function reduces to the (1.1).

iii) If α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to

$$J_{\alpha-1,q}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,q,c}(z)$$

which is an extended Mittag-Leffler function (see [14]).

iv) If $q = 1$ and α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to

$$J_{\alpha-1,1}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,c}(z)$$

which is an extended Mittag-Leffler function (see [18]).

For this continuation of our study, we recall the following result of Oberhettinger [17]

$$(1.12) \quad \begin{aligned} & \int_0^\infty z^{\alpha-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\beta} dz \\ &= 2\beta b^{-\beta} \left(\frac{b}{2} \right)^\alpha \frac{\Gamma(2\alpha)\Gamma(\beta - \alpha)}{\Gamma(1 + \alpha + \beta)}, \quad 0 < \Re(\alpha) < \Re(\beta). \end{aligned}$$

For various other investigation containing special function, the reader may refer to the recent work of researchers (see [1], [4], [6], [12], [15], [16]).

In the investigation of various properties and relations of the function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ as defined in (1.11), we need the following well known facts. Sneddon [23] defined Beta Euler and Laplace transforms in following forms:

$$(1.13) \quad B(f(z); \alpha, \beta) = \int_0^1 z^{\alpha-1} (1-z)^{\beta-1} f(z) dz,$$

where $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

$$(1.14) \quad L\{f(z)\} = \int_0^\infty e^{-sz} f(z) dz,$$

respectively.

Meijer [11] defined the following K -transform integral equation as:

$$(1.15) \quad R_v\{f(x); p\} = \int_0^\infty (px)^{\frac{1}{2}} K_v(px) f(x) dx,$$

where the parameter $p \in \mathbb{C}$ and $K_v(z)$ represent a modified Bessel function of third kind defined in [11].

Varma transform (see [20]) is defined by the following integral equation as:

$$(1.16) \quad V(f, k, m; s) = \int_0^\infty (sx)^{m-\frac{1}{2}} W_{k,m}(sx) f(x) dx,$$

where $W_{k,m}(z)$ represent Wittaker function (see [9, p. 55]).

2. Integral transforms of an extended Bessel function

In this section, we investigate some integral transforms such as Beta, Laplace, Varma and K -transforms of an extended Wright-Bessel function as defined in (1.11) and we express the obtained result in terms of an extended Wright-type hypergeometric function (1.10).

Theorem 2.1. *Let $\alpha, \lambda, \gamma, c, \mu, \delta, \nu \in \mathbb{C}$, $\Re(\alpha) \geq -1$, $\Re(\lambda) \geq 0$, $\Re(\gamma) \geq 0$, $\Re(\mu) \geq 0$, $\Re(c) \geq 0$, $\Re(\nu) \geq 0$, $p \geq 0$, $q \in (0, 1) \setminus \mathbb{N}$. Then the following integral transforms holds:*

$$(2.1) \quad \begin{aligned} & \int_0^1 z^{\mu-1} (1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \frac{\Gamma(\nu)}{\Gamma(\gamma)} {}_3\Psi_3 \left[\begin{matrix} (c, q), (\mu, \delta), (\gamma, 1) \\ (\alpha + 1, \lambda), (\mu + \nu, \delta), (c, 1) \end{matrix} ; |(-x, p) \right]. \end{aligned}$$

Proof. From (1.11) and (1.13), we obtain

$$\begin{aligned} & \int_0^1 z^{\mu-1} (1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \int_0^1 z^{\mu-1} (1-z)^{\nu-1} \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-1)^n (xz^\delta)^n}{\Gamma(\alpha + \lambda n + 1) n!} \right) dz. \end{aligned}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned} & \int_0^1 z^{\mu-1} (1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-1)^n (x)^n}{\Gamma(\alpha + \lambda n + 1) n!} \int_0^1 z^{\mu+\delta n-1} (1-z)^{\nu-1} dz \\ &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-1)^n (x)^n}{\Gamma(\alpha + \lambda n + 1) n!} \frac{\Gamma(\mu + \delta) \Gamma(\nu)}{\Gamma(\mu + \nu + \delta)} \end{aligned}$$

$$= \frac{\Gamma(\nu)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{\Gamma(c - \gamma)} \frac{\Gamma(c + nq)(-1)^n(x)^n}{\Gamma(\alpha + \lambda n + 1)n!} \frac{\Gamma(\mu + \delta)}{\Gamma(\mu + \nu + \delta)},$$

which upon using (1.10), we get the required result. \square

From above theorem, we observe the following two cases.

Case (i). If $\lambda = \delta$ and $\mu = \alpha + 1$, then (2.1) reduces to the following result:

$$(2.2) \quad \int_0^1 z^\alpha (1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\lambda, p) dz = \Gamma(\nu) J_{\alpha+\nu,q}^{\delta,\gamma,c}(x)$$

and

Case (ii). If $\lambda = \delta$ and $\mu = \alpha + 1$ and $z = (1-z)$, then (2.1) reduces to the following result:

$$(2.3) \quad \int_0^1 z^{\mu-1} (1-z)^\alpha J_{\alpha,q}^{\lambda,\gamma,c}(x(1-z)^\lambda, p) dz = \Gamma(\mu) J_{\alpha+\mu,q}^{\delta,\gamma,c}(x).$$

Theorem 2.2. Let $\alpha, \lambda, \gamma, c, \mu, \delta \in \mathbb{C}$, $\Re(\alpha) \geq -1$, $\Re(\lambda) \geq 0$, $\Re(\gamma) \geq 0$, $\Re(\mu) \geq 0$, $\Re(c) \geq 0$, $\Re(s) \geq 0$, $p \geq 0$, $q \in (0, 1) \setminus \mathbb{N}$. Then the following integral transforms holds:

$$(2.4) \quad \int_0^\infty z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz = \frac{s^{-\mu}}{\Gamma(\gamma)} {}_3\Psi_2 \left[\begin{matrix} (c, q), (\mu, \delta), (\gamma, 1) & ; \\ (\alpha + 1, \lambda), (c, 1) & ; \end{matrix} \middle| (-\frac{x}{s^\delta}, p) \right].$$

Proof. From (1.11) and (1.14), we have

$$\begin{aligned} & \int_0^\infty z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \int_0^\infty z^{\mu-1} e^{-sz} \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq}(-1)^n (xz^\delta)^n}{\Gamma(\alpha + \lambda n + 1)n!} \right) dz. \end{aligned}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned} & \int_0^\infty z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha + \lambda n + 1)n!} \int_0^\infty z^{\mu+\delta n-1} e^{-sz} dz \\ &= \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha + \lambda n + 1)n!} s^{-\mu-\delta n} \Gamma(\mu + \delta n) \\ &= \frac{s^{-\mu}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{\Gamma(c - \gamma)} \frac{\Gamma(c + nq)(-1)^n (x)^n}{\Gamma(\alpha + \lambda n + 1)n!} \Gamma(\mu + \delta n), \end{aligned}$$

which by using (1.10), we get the required result. \square

From above theorem, we observe the following two particular cases:

(i) If $p = 0$, $q = 1$, $\mu = \alpha + 1$, $\lambda = \delta$, then (2.4) reduces to

$$(2.5) \quad \int_0^\infty z^\alpha e^{-sz} J_{\alpha,1}^{\lambda,\gamma}(xz^\lambda) dz = s^{-\alpha-1} (1 + xs^{-\lambda})^{-\gamma},$$

by using the relation $\sum_{n=0}^\infty \frac{(a)_n z^n}{n!} = (1 - z)^{-a}$ (see [20]), and

(ii) If $p = 0$, $\gamma = q = 1$, $\mu = \alpha + 1$, $\lambda = \delta$ and $x = \pm t$, then (2.4) reduces to

$$\begin{aligned} (2.6) \quad \int_0^\infty z^\alpha e^{-sz} J_{\alpha,1}^{\lambda,1}(\mp z^\lambda) dz &= s^{-\alpha-1} \sum_{n=0}^\infty \left(\frac{\mp t}{s^\lambda} \right)^n \\ &= \frac{s^{-\alpha-1}}{1 - \frac{\mp t}{s^\lambda}}; \left| \frac{\mp t}{s^\lambda} \right| < 1 \\ &= \frac{s^{\lambda-\alpha-1}}{s^\lambda \pm t}. \end{aligned}$$

Theorem 2.3. Let $\alpha, \lambda, \delta, \rho, \gamma, c \in \mathbb{C}$ and $\Re(\alpha) \geq -1$, $\Re(\gamma) \geq 0$, $\Re(\delta) \geq 0$, $\Re(\rho) \geq 0$, $\Re(c) \geq 0$, $p \geq 0$ and $q \in \mathbb{N}$. Then the following result holds:

$$(2.7) \quad \begin{aligned} &\int_0^\infty t^{\delta-1} K_\mu(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho, p) dt \\ &= \frac{2^{\delta-2}}{s^\delta \Gamma(\gamma)} {}^4\Psi_2 \left[\begin{matrix} (c, q), (\delta \pm \mu, \rho), (\gamma, 1) &; \\ & |(-\omega(\frac{2}{s})^\rho, p) \\ (\alpha + 1, \lambda), (c, 1) &; \end{matrix} \right]. \end{aligned}$$

Proof. From (1.11) and (1.15), we have

$$\begin{aligned} &\int_0^\infty t^{\delta-1} K_\mu(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho) dt \\ &= \int_0^\infty t^{\delta-1} K_\mu(st) \left(\sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\omega t^\rho)^n}{\Gamma(\alpha+\lambda n+1)n!} \right) dt. \end{aligned}$$

By changing the variable $t = \frac{z}{s}$ and interchanging of sum and integration, we have

$$\begin{aligned} &\int_0^\infty \left(\frac{z}{s} \right)^{\delta-1} K_\mu(z) \left(\sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\omega)^n (\frac{z}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \right) \frac{1}{s} dz \\ &= s^{-\delta} \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\omega)^n}{\Gamma(\alpha+\lambda n+1)n!} \int_0^\infty \left(\frac{z}{s} \right)^{\delta+\rho n-1} K_\mu(z) dz. \end{aligned}$$

Now, by using the following Mathai and Saxena ([10, p. 78]) formula

$$(2.8) \quad \int_0^\infty x^{\rho-1} K_\mu(x) dx = 2^{\rho-2} \Gamma\left(\frac{\rho \pm \mu}{2}\right),$$

in the above equation, we obtain

$$\int_0^\infty \left(\frac{z}{s} \right)^{\delta-1} K_\mu(z) \left(\sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\omega)^n (\frac{z}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \right) \frac{1}{s} dz$$

$$= \frac{1}{\Gamma(\gamma)} s^{-\delta} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{\Gamma(c - \gamma)} \frac{\Gamma(c + nq)(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha + \lambda n + 1)n!} 2^{\delta - \rho n - 2} \Gamma(\delta + \rho n \pm \mu)$$

which by using (1.10), we get the required result. \square

Theorem 2.4. Let $\alpha, \lambda, \delta, \rho, \gamma, \mu, \nu, c \in \mathbb{C}$ and $\Re(\alpha) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0, \Re(\rho) \geq 0, \Re(c) \geq 0, p \geq 0$ and $q \in \mathbb{N}$. Then the following result holds:

$$(2.9) \quad \begin{aligned} & \int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho, p) dt \\ & = \frac{1}{s^\delta \Gamma(\gamma)} {}_4\Psi_3 \left[\begin{array}{c} (c, q), (\frac{1}{2} \pm \nu + \delta, \rho), (\gamma, 1) \\ (\alpha + 1, \lambda), (1 - \mu + \delta, \rho), (c, 1) \end{array} ; \frac{(-\frac{\omega}{s^\rho}, p)}{} \right]. \end{aligned}$$

Proof. From (1.11) and (1.16), we have

$$\begin{aligned} & \int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho) dt \\ & = \int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-\omega t^\rho)^n}{\Gamma(\alpha + \lambda n + 1)n!} \right) dt. \end{aligned}$$

By changing the variable $st = z$ and interchanging of sum and integration in above equation yields

$$\begin{aligned} & \int_0^\infty e^{-\frac{z}{2}} \left(\frac{z}{s}\right)^{\delta-1} W_{\mu,\nu}(z) J_{\alpha,q}^{\lambda,\gamma,c}(t^\rho) \frac{1}{s} dz \\ & = s^{-\delta} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha + \lambda n + 1)n!} \left(\int_0^\infty e^{-\frac{z}{2}} \left(\frac{z}{s}\right)^{\delta+\rho n-1} W_{\mu,\nu}(z) dz \right). \end{aligned}$$

Now, by using the formula

$$(2.10) \quad \int_0^\infty e^{-\frac{s}{2}} s^{\beta-1} W_{\mu,\nu}(s) ds = \frac{\Gamma(\frac{1}{2} + \nu + \beta)\Gamma(\frac{1}{2} - \nu + \beta)}{\Gamma(1 - \mu + \beta)},$$

(where $\Re(\beta \pm \nu) > -\frac{1}{2}$) in above equation (see Mathai and Saxena [10, p. 79]), we get

$$\begin{aligned} & \int_0^\infty e^{-\frac{z}{2}} \left(\frac{z}{s}\right)^{\delta-1} W_{\mu,\nu}(z) J_{\alpha,q}^{\lambda,\gamma,c}(t^\rho) \frac{1}{s} dz \\ & = \frac{s^{-\delta}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{\Gamma(c - \gamma)} \frac{\Gamma(c + nq)(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha + \lambda n + 1)n!} \frac{\Gamma(\frac{1}{2} \pm \nu + \delta + \rho n)}{\Gamma(1 - \mu + \delta + \rho n)} \end{aligned}$$

which by using (1.10), we get the required result. \square

3. Unified integral formulas of an extended Bessel function

In this section, we establish two generalized integral formulas containing extended Wright-Bessel function as defined in (1.11) which is represented in terms of Wright-type function defined in (1.10) by inserting with the suitable argument defined in (1.12).

Theorem 3.1. *Let $\alpha, \mu, \delta, v, \lambda, \gamma, c \in \mathbb{C}$ with $\Re(\delta) > 0, \Re(\alpha) > -1, \Re(\lambda) > 0, \Re(\gamma) > 0, \Re(c) > 0, \Re(\mu) > \Re(\delta) > 0, p \geq 0, q \in (0, 1) \cup \mathbb{N}$ and $z > 0$. Then the following result holds:*

$$(3.1) \quad \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} J_{\alpha, q}^{\lambda, \gamma, c} \left(\frac{y}{z + b + \sqrt{z^2 + 2bz}} \right) dz = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{matrix} (\mu+1; 1), (\mu-\delta, 1), (c, q), (\gamma, 1); \\ (\alpha+1, \lambda), (\mu, 1), (\delta+\mu+1, 1), (c, 1) \end{matrix} \middle| \left(-\frac{y}{b}, p \right) \right].$$

Proof. Let \mathfrak{L}_1 be the left hand side of (3.1) and applying (1.11) to the integrand of (3.1), we have

$$\mathfrak{L}_1 = \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} \times \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}}{B(\gamma, c-\gamma)\Gamma(\lambda n + \alpha + 1)n!} \left(-\frac{y}{z+b+\sqrt{z^2+2bz}} \right)^n dz.$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of Theorem 3.1, we have

$$(3.2) \quad \mathfrak{L}_1 = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma, c-\gamma)\Gamma(\lambda n + \alpha + 1)n!} \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-(\mu+n)} dz.$$

By considering the assumption given in Theorem 3.1, since $\Re(\mu+n) > \Re(\delta) > 0, \Re(\alpha) > -1$ and applying (1.12) to (3.2), we obtain

$$\mathfrak{L}_1 = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma, c-\gamma)\Gamma(\lambda n + \alpha + 1)n!} 2(\mu+n)b^{-(\mu+n)} \left(\frac{b}{2} \right)^\delta \frac{\Gamma(2\delta)\Gamma(\mu+n-\alpha)}{\Gamma(1+\mu+\delta+n)}.$$

Applying (1.6), we get

$$\mathfrak{L}_1 = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(c-\gamma)n!} \frac{y^n}{b^n} \frac{B_p(\gamma+nq, c-\gamma)\Gamma(c+nq)\Gamma(\mu+n+1)\Gamma(\mu+n-\delta)}{\Gamma(\alpha+\lambda n+1)\Gamma(\mu+n)\Gamma(\delta+\mu+n+1)}$$

which upon using (1.10), we get the required result. \square

Theorem 3.2. *Let $\alpha, \mu, \delta, v, \lambda, \gamma, c \in \mathbb{C}$ with $\Re(\delta) > 0, \Re(\alpha) > -1, \Re(\lambda) > 0, \Re(\gamma) > 0, \Re(c) > 0, \Re(\mu) > \Re(\delta) > 0, p \geq 0, q \in (0, 1) \cup \mathbb{N}$ and $z > 0$. Then*

the following result holds:

$$(3.3) \quad \int_0^\infty z^{\delta-1} (z + b + \sqrt{z^2 + 2bz})^{-\mu} J_{\alpha,q}^{\lambda,\gamma,c} \left(\frac{yz}{z+b+\sqrt{z^2+2bz}} \right) dz \\ = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{array}{c} (\mu+1;1), (2\delta,2), (c,q), (\gamma,1); \\ (\mu,1), (\alpha+1,\lambda), (\mu+\delta+1,2), (c,1) \end{array} \middle| -\left(\frac{y}{2},p\right) \right].$$

Proof. Let \mathfrak{L}_2 be the left hand side of (3.3) and applying (1.11) to the integrand of (3.3), we have

$$\mathfrak{L}_2 = \int_0^\infty z^{\delta-1} (z + b + \sqrt{z^2 + 2bz})^{-\mu} \\ \times \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}}{B(\gamma, c-\gamma)\Gamma(\alpha+\lambda n+1)n!} \left(-\frac{yz}{z+b+\sqrt{z^2+2bz}} \right)^n dz.$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of Theorem 3.2, we have

$$(3.4) \quad \mathfrak{L}_2 = \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma, c-\gamma)\Gamma(\alpha+\lambda n+1)n!} \\ \times \int_0^\infty z^{\delta+n-1} (z + b + \sqrt{z^2 + 2bz})^{-(\mu+n)} dz.$$

By considering the assumption given in Theorem 3.2, since $\Re(\beta) > \Re(\alpha) > 0$, $\Re(\alpha) > -1$ and applying (1.12) to (3.4), we obtain

$$\mathfrak{L}_2 = \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma, c-\gamma)\Gamma(\alpha+\lambda n+1)n!} 2(\mu+n)b^{\delta-\mu} \left(\frac{1}{2^{\delta+n}} \right) \frac{\Gamma(2\delta+2n)\Gamma(\mu-\delta)}{\Gamma(1+\mu+\delta+2n)} \\ = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)(-1)^n y^n}{2^n \Gamma(c-\gamma)} \frac{\Gamma(c+nq)\Gamma(2\delta+2n)\Gamma(\mu+n+1)}{\Gamma(\alpha+\lambda n+1)\Gamma(\mu+n)\Gamma(1+\mu+\delta+2n)}$$

which upon using (1.10), we get the required result. \square

4. Special cases

In this section, we present the extended form of generalized Mittag-Leffler functions which are the special cases of extended Wright-Bessel function defined in (1.11). Also, we prove four corollaries which are the special cases of obtained Theorems 3.1 and 3.2.

Case 1. If α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to an extended Mittag-Leffler function defined as:

$$(4.1) \quad J_{\alpha-1,q}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,q,c}(z) = \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)(c)_{nq}}{B(\gamma, c-\gamma)\Gamma(\lambda n+\alpha)n!}.$$

Case 2. If $q = 1$ and α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to an extended Mittag-Leffler function

$$(4.2) \quad J_{\alpha-1,1}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,c}(z) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)(c)_n}{B(\gamma, c-\gamma)\Gamma(\lambda n + \alpha)n!}.$$

Corollary 4.1. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral transforms holds:

$$(4.3) \quad \begin{aligned} & \int_0^1 z^{\mu-1} (1-z)^{\nu-1} E_{\lambda,\alpha}^{\gamma,c}(xz^\delta, p) dz \\ &= \frac{\Gamma(\nu)}{\Gamma(\gamma)} \cdot {}_3\Psi_3 \left[\begin{array}{cc|c} (c, q), (\mu, \delta), (\gamma, 1) & ; & |(x, p) \\ (\alpha, \lambda), (\mu + \nu, \delta), (c, 1) & ; & \end{array} \right]. \end{aligned}$$

Corollary 4.2. Assume that the conditions of Theorem 2.1 are satisfied. Then the following integral transforms holds:

$$(4.4) \quad \begin{aligned} & \int_0^1 z^{\mu-1} (1-z)^{\nu-1} E_{\lambda,\alpha}^{\gamma,c}(xz^\delta, p) dz \\ &= \frac{\Gamma(\nu)}{\Gamma(\gamma)} \cdot {}_3\Psi_3 \left[\begin{array}{cc|c} (c, 1), (\mu, \delta), (\gamma, 1) & ; & |(x, p) \\ (\alpha, \lambda), (\mu + \nu, \delta), (c, 1) & ; & \end{array} \right]. \end{aligned}$$

Corollary 4.3. Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral transforms holds:

$$(4.5) \quad \begin{aligned} & \int_0^\infty z^{\mu-1} e^{-sz} E_{\lambda,\alpha}^{\gamma,q,c}(xz^\delta, p) dz \\ &= \frac{s^{-\mu}}{\Gamma(\gamma)} \cdot {}_3\Psi_2 \left[\begin{array}{cc|c} (c, q), (\mu, \delta), (\gamma, 1) & ; & |(\frac{x}{s^\delta}, p) \\ (\alpha, \lambda), (c, 1) & ; & \end{array} \right]. \end{aligned}$$

Corollary 4.4. Assume that the conditions of Theorem 2.2 are satisfied. Then the following integral transforms holds:

$$(4.6) \quad \begin{aligned} & \int_0^\infty z^{\mu-1} e^{-sz} E_{\lambda,\alpha}^{\gamma,c}(xz^\delta, p) dz \\ &= \frac{s^{-\mu}}{\Gamma(\gamma)} \cdot {}_3\Psi_2 \left[\begin{array}{cc|c} (c, 1), (\mu, \delta), (\gamma, 1) & ; & |(\frac{x}{s^\delta}, p) \\ (\alpha, \lambda), (c, 1) & ; & \end{array} \right]. \end{aligned}$$

Similarly, we can derives some corollaries by using the conditions of Theorems 2.3 and 2.4.

Corollary 4.5. Assume that the conditions of Theorem 3.1 are satisfied. Then the following integral formula holds:

$$\int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} E_{\lambda,\alpha}^{\gamma,q,c} \left(\frac{y}{z + b + \sqrt{z^2 + 2bz}} \right) dz$$

$$(4.7) \quad = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{array}{l} (\mu+1;1), (\mu-\delta,1), (c,q), (\gamma,1); \\ (\alpha,\lambda), (\mu,1), (\delta+\mu+1,1), (c,1) \end{array} \middle| (-\frac{y}{b}, p) \right].$$

Corollary 4.6. Assume that the conditions of Theorem 3.1 are satisfied. Then the following integral formula holds:

$$(4.8) \quad \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} E_{\lambda,\alpha}^{\gamma,c} \left(\frac{y}{z + b + \sqrt{z^2 + 2bz}} \right) dz \\ = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{array}{l} (\mu+1;1), (\mu-\delta,1), (c,1), (\gamma,1); \\ (\alpha,\lambda), (\mu,1), (\delta+\mu+1,1), (c,1) \end{array} \middle| (-\frac{y}{b}, p) \right].$$

Corollary 4.7. Assume that the conditions of Theorem 3.2 are satisfied. Then the following integral formula holds:

$$(4.9) \quad \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} E_{\lambda,\alpha}^{\gamma,q,c} \left(\frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz \\ = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{array}{l} (\mu+1;1), (2\delta,2), (c,q), (\gamma,1); \\ (\mu,1), (\alpha,\lambda), (\mu+\delta+1,2), (c,1) \end{array} \middle| -(\frac{y}{2}, p) \right].$$

Corollary 4.8. Assume that the conditions of Theorem 3.2 are satisfied. Then the following integral formula holds:

$$(4.10) \quad \int_0^\infty z^{\delta-1} \left(z + b + \sqrt{z^2 + 2bz} \right)^{-\mu} E_{\lambda,\alpha}^{\gamma,c} \left(\frac{yz}{z + b + \sqrt{z^2 + 2bz}} \right) dz \\ = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} {}_4\Psi_4 \left[\begin{array}{l} (\mu+1;1), (2\delta,2), (c,1), (\gamma,1); \\ (\mu,1), (\alpha,\lambda), (\mu+\delta+1,2), (c,1) \end{array} \middle| -(\frac{y}{2}, p) \right].$$

Concluding Remark. In this paper, we introduced certain integral transforms and representation formulas for newly defined extended Wright-Bessel function and express the obtained results in terms of an extended Wright-type hypergeometric function. If letting $p = 0$, then we have the results of the generalized Wright-Bessel function and Mittag-Leffler functions proved in ([7], [22]).

References

- [1] M. S. Abouzaid, A. H. Abusufian, and K. S. Nisar, *Some unified integrals associated with generalized Bessel-Maitland function*, Int. Bull. Math. Research **3** (2016), 18–23.
- [2] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159** (2004), no. 2, 589–602.
- [3] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press, Boca Raton, 2002.
- [4] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc. **51** (2014), no. 4, 995–1003.
- [5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, Toronto, London, 1953.

- [6] S. Jain and P. Agarwal, *A new class of integral relation involving general class of Polynomials and I-function*, Walailak J. Sci. Tech. **12** (2015), 1009–1018.
- [7] N. U. Khan and T. Kashmin, *Some integrals for the generalized Bessel-Maitland function*, Electron. J. Math. Analy. Appl. **4** (2016), no. 2, 139–149.
- [8] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow and New York, 1994.
- [9] A. M. Mathai and R. K. Saxena, *On linear combinations of stochastic variables*, Metrika **20** (1973), no. 3, 160–169.
- [10] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function Theory and Applications*, Springer New York Dordrecht Heidelberg London, 2010.
- [11] C. S. Meijer, *Über eine Erweiterung der Laplace-Transform*, Neder Wetensch Proc. **2** (1940), 269–278.
- [12] N. Menaria, S. D. Purohit, and R. K. Parmar, *On a new class of integrals involving generalized Mittag-Leffler function*, Sur. Math. Appl. **11** (2016), 1–9.
- [13] G. M. Mittag-Leffler, *Sur la nouvelle fonction $E_\alpha(x)$* , C. R. Acad. Soc. Paris **137** (1903), 554–558.
- [14] E. Mittal, R. M. Pandey, and S. Joshi, *On extension of Mittag-Leffler function*, Appl. Appl. Math. **11** (2016), no. 1, 307–316.
- [15] K. S. Nisar and S. R. Mondal, *Certain unified integral formulas involving the generalized modified k-Bessel function of first kind*, Commun. Korean Math. Soc. **32** (2017), no. 1, 47–53.
- [16] K. S. Nisar, R. K. Parmar, and A. H. Abusufian, *Certain new unified integrals associated with the generalized K-Bessel function*, Far East J. Math. Sci. **9** (2016), 1533–1544.
- [17] F. Oberhettinger, *Tables of Mellin Transforms*, Springer, New York, 1974.
- [18] M. A. Özarslan and B. Yilmaz, *The extended Mittag-Leffler function and its properties*, J. Inequal. Appl. **2014** (2014), 85, 10 pp.
- [19] R. S. Pathak, *Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations*, Proc. Nat. Acad. Sci. India Sect. A **36** (1966), 81–86.
- [20] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
- [21] S. C. Sharma and M. Devi, *Certain Properties of Extended Wright Generalized Hypergeometric Function*, Ann. Pure Appl. Math. **9** (2015), 45–51.
- [22] M. Singh, M. A. Khan, and A. H. Khan, *On some properties of a generalization of Bessel-Maitland function*, Inter. J. Math. Trend Tech. **14** (2014), 46–47.
- [23] I. N. Sneddon, *The use of Integral Transform*, Tata Mc Graw-Hill, New Delhi, 1979.
- [24] E. M. Wright, *The asymptotic expansion of the generalized Bessel function*, Proc. Lond. Math. Soc. **38** (1935), no. 2, 257–270.
- [25] ———, *The asymptotic expansion of integral functions defined by Taylor Series*, Philos. Trans. Roy. Soc. London Ser. A **238** (1940), 423–451.
- [26] ———, *The asymptotic expansion of the generalized hypergeometric function II*, Proc. London. Math. Soc. **46** (1935), no. 2, 389–408.

MUHAMMAD ARSHAD
 DEPARTMENT OF MATHEMATICS
 INTERNATIONAL ISLAMIC UNIVERSITY
 ISLAMABAD, PAKISTAN
Email address: marshad_@yahoo.com

SHAHID MUBEEN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SARGODHA
SARGODHA, PAKISTAN
Email address: smjhanda@gmail.com

KOTTAKKARAN SOOPPY NISAR
DEPARTMENT OF MATHEMATICS
COLLEGE OF ARTS AND SCIENCE-WADI ALDAWASER
11991, PRINCE SATTAM BIN ABDULAZIZ UNIVERSITY
ALKHARJ, KINGDOM OF SAUDI ARABIA
Email address: ksnisar1@gmail.com, n.sooppy@psau.edu.sa

GAUHAR RAHMAN
DEPARTMENT OF MATHEMATICS
INTERNATIONAL ISLAMIC UNIVERSITY
ISLAMABAD, PAKISTAN
Email address: gauhar55uom@gmail.com