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FRACTIONAL CALCULUS AND INTEGRAL TRANSFORMS OF INCOMPLETE τ-HYPERGEOMETRIC FUNCTION

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Abstract. In the present article, authors obtained certain fractional derivative and integral formulas involving incomplete τ-hypergeometric function introduced by Parmar and Saxena [14]. Some interesting special cases and consequences of our main results are also considered.

1. Introduction

Fractional calculus is one of the generalizations of classical calculus and it has been used successfully in various fields of Science and Technology. Many applications of fractional calculus can be found in other diverse fields, etc. See [4, 7–10, 15, 17, 18].

Integral transforms and fractional calculus formulae involving hypergeometric functions are interesting in themselves and play important roles in their diverse applications. A large number of integral transforms and fractional calculus formulae have been established by many authors ([1–3, 5]).

The incomplete gamma functions $\gamma(s, k)$ and $\Gamma(s, k)$ defined by (Parmar [14])

$$\gamma(s,k) = \int_0^k e^{-t} t^{s-1} dt, \ (\Re(s) > 0; k \geq 0)$$

and

$$\Gamma(s,k) = \int_k^\infty e^{-t} t^{s-1} dt, \ (\Re(s) > 0; k \geq 0).$$

These incomplete gamma functions $\gamma(s, k)$ and $\Gamma(s, k)$ satisfy the following decomposition formula:

$$\gamma(s,k) + \Gamma(s,k) = \Gamma(s); \ (\Re(s) > 0),$$

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where the gamma function $\Gamma(z)$ whose Euler's integral is given by (Rainville [19])

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt \quad (\Re(z)>0).$$

Incomplete Pochhammer symbols $(\lambda;k)_n$ and $[\lambda;k]_n$ $(\lambda,n\in\mathbb{C};\ k\geq 0)$ are defined as follows (Parmar [14])

$$(\lambda;k)_n=\frac{\gamma(\lambda+n,k)}{\Gamma(\lambda)},\ (\lambda,n\in\mathbb{C};k\geq 0)$$

and

$$[\lambda;k]_n=\frac{\Gamma(\lambda+n,k)}{\Gamma(\lambda)},\ (\lambda,n\in\mathbb{C};k\geq 0).$$

These incomplete Pochhammer symbols $(\lambda;k)_n$ and $[\lambda;k]_n$ satisfy the decomposition formula:

$$(\lambda;k)_n+[\lambda;k]_n=(\lambda)_n,$$

where the Pochhammer symbol $(\lambda)_n$ defined (for $\lambda\in\mathbb{C}$) as (Rainville [19])

$$(\lambda)_n=\begin{cases}1, & (n=0),\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & (n\in\mathbb{N})\\ \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, & (\lambda\in\mathbb{C}/\mathbb{Z}_0^-),\end{cases} \tag{1}$$

where $\mathbb{Z}_0^-$ denotes the set of non positive integers.

The incomplete τ-hypergeometric function ${}_2\Gamma_1^\tau(z)$ and ${}_2\gamma_1^\tau(z)$ defined in term of the incomplete Pochhammer symbols as (Parmar [14])

$$ {}_2\Gamma_1^\tau(z)={}_2\Gamma_1^\tau((a,k),b;c;z)=\left\{\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(z)^n}{n!}\right\}, \tag{2}$$

$$(k\geq 0;\tau>0;\Re(c)>\Re(b)>0 \text{ when } k=0).$$

$$ {}_2\gamma_1^\tau(z)={}_2\gamma_1^\tau((a,k),b;c;z)=\left\{\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a;k)_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(z)^n}{n!}\right\}, \tag{3}$$

$$(k\geq 0;\tau>0;\Re(c)>\Re(b)>0 \text{ when } k=0).$$

The above incomplete τ-hypergeometric function ${}_2\Gamma_1^\tau(z)$ and ${}_2\gamma_1^\tau(z)$ satisfy the decomposition formula:

$$ {}_2\Gamma_1^\tau((a,k),b;c;z)+{}_2\gamma_1^\tau((a,k),b;c;z)={}_2R_1^\tau(a,b;c;z),$$

where ${}_2R_1^\tau(a,b;c;z)$ is the generalized τ-hypergeometric function ([6, 25]).

The special cases of (2) and (3) when $\tau=1$ are easily seen to reduce to the known the incomplete Gauss hypergeometric functions [23, p. 664, Eq. (3.1), (3.2)],

$$ {}_2\gamma_1[(a,k),b;c;z]=\sum_{n=0}^{\infty}\frac{(a;k)_n\,(b)_n}{(c)_n}\frac{(z)^n}{n!} \tag{4}$$

and

$$ {}_2\Gamma_1[(a,k),b;c;z] = \sum_{n=0}^{\infty} \frac{[a;k]_n\,(b)_n}{(c)_n} \frac{(z)^n}{n!} \tag{5} $$

respectively. Also, the special cases of (2) and (3) when $\tau = 1$ and $k = 0$ is seen to yield the classical Gauss hypergeometric function defined as

$$ {}_2F_1\,(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!}. $$

The classical beta function (Srivastava and Choi [24]) defined as

$$ B(\alpha,\,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha)>0;\ \Re(\beta)>0), \\ \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} $$

The present investigation requires the concept of Hadamard product which can be used to decompose a newly-emerged function into two known functions.

Definition (Pohlen [16])**.** If $f(t) := \sum_{k=0}^{\infty} \mu_k\, t^k$ and $g(t) := \sum_{k=0}^{\infty} \nu_k\, t^k$ are two power series whose radii of convergence are denoted by R_f and R_g respectively, then their Hadamard product is the power series defined by

$$ (f*g)(t) = \sum_{k=0}^{\infty} \mu_k \nu_k\, t^k. $$

The radius of convergence R of the Hadamard product series $(f*g)(t)$ satisfies $R_f.R_g \le R$.

Moreover, $f*g$ defines an entire function if either f or g is an entire function.

2. Fractional calculus and generalized incomplete τ-hypergeometric function

In this section, we established certain fractional calculus formulae for the generalized hypergeometric type function generalized incomplete τ-hypergeometric function has been introduced by Parmar [14]. Some interesting special cases of our main results are also considered.

For $x > 0$, $l, m, \xi \in \mathbb{C}$ and $\alpha > 0$, the left sided and right sided Saigo hypergeometric fractional integral operators are defined as (Saigo [20], Kiryakova [12])

$$ (I_{0,x}^{l,m,\xi} f(t))(x) = \frac{x^{-l-m}}{\Gamma(l)} \int_0^x (x-t)^{l-1}{}_2F_1\left(l+m,-\xi;l;1-\frac{t}{x}\right) f(t)\,dt, \tag{6} $$

$$ (J_{x,\infty}^{l,m,\xi} f(t))(x) = \frac{1}{\Gamma(l)} \int_x^{\infty} (t-x)^{l-1} t^{-l-m}{}_2F_1\left(l+m,-\xi;l;1-\frac{t}{x}\right) f(t)\,dt, \tag{7} $$

respectively.

The Riemann-Liouville $R^{l}_{0,x}(\cdot)$ and the Erdélyi-Kober $E^{l,\xi}_{0,x}(\cdot)$ fractional integral operators are particular cases of left sided Saigo fractional operator by means of the relationships:

$$(R^{l}_{0,x}f(t))(x) = (I^{l,-l,\xi}_{0,x}f(t))(x) = \frac{1}{\Gamma(l)}\int_0^x (x-t)^{l-1}f(t)\,dt,$$
$$(E^{l,\xi}_{0,x}f(t))(x) = (I^{l,0,\xi}_{0,x}f(t))(x) = \frac{x^{-l-\xi}}{\Gamma(l)}\int_0^x (x-t)^{l-1}t^{\xi}f(t)\,dt,$$

and also, the Weyl and the Erdélyi-Kober fractional operators as special cases of right sided Saigo fractional operator as follows:

$$(W^{l}_{x,\infty}f(t))(x) = (J^{l,-l,\xi}_{x,\infty}f(t))(x) = \frac{1}{\Gamma(l)}\int_x^\infty (t-x)^{l-1}f(t)\,dt,$$
$$(K^{l,\xi}_{x,\infty}f(t))(x) = (J^{l,0,\xi}_{x,\infty}f(t))(x) = \frac{x^{\xi}}{\Gamma(l)}\int_x^\infty (t-x)^{l-1}t^{-l-\xi}f(t)\,dt.$$

Following the right-hand sided and left-hand sided Saigo fractional integration of the type (6) and (7) for a power function are given by following lemmas (Saigo [20]):

Lemma 2.1. *Let $l, m, \xi, \lambda \in \mathbb{C}$, $x > 0$ and if $\Re(\lambda) > 0, \Re(\lambda - m + \xi) > 0$. Then*

$$(I^{l,m,\xi}_{0,x}t^{\lambda-1})(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\Gamma(\lambda+l+\xi)}x^{\lambda-m-1}. \tag{8}$$

Lemma 2.2. *Let $l, m, \xi, \lambda \in \mathbb{C}$, $x > 0$ and if $\Re(m-\lambda+1) > 0, \Re(\xi-\lambda+1) > 0$. Then*

$$(J^{l,m,\xi}_{x,\infty}t^{\lambda-1})(x) = \frac{\Gamma(m-\lambda+1)\Gamma(\xi-\lambda+1)}{\Gamma(1-\lambda)\Gamma(m+l-\lambda+\xi+1)}x^{\lambda-m-1}. \tag{9}$$

Let $l, l^{'}, m, m^{'}, \xi$ are complex numbers and $x > 0$. Then the left sided and right sided Saigo fractional derivative operators are defined as (Saigo [20]):

$$\left(D^{l,m,\xi}_{0+}f\right)(x) = \left(I^{-l,-m,l+\xi}f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I^{-l+\xi,-m-\xi,l+\xi-n}_{0+}f\right)(x), \tag{10}$$

$$(\Re(l) \geq 0, n = [\Re(l)] + 1),$$

and

$$\left(D^{l,m,\xi}_{0-}f\right)(x) = \left(I^{-l,-m,l+\xi}_{-}f\right)(x) = \left(-\frac{d}{dx}\right)^n \left(I^{-l+\xi,-m-\xi,l+\xi-n}_{-}f\right)(x), \tag{11}$$

$(\Re(l) \geq 0, n = [\Re(l)] + 1)$, respectively.

As a special cases, the Saigo fractional derivative operators contains the Riemann-Liouville $D^{l}_{0+}(\cdot)$ and Weyl fractional derivatives by means of the following relationships:

$$\begin{aligned}(D^{l,-l,\xi}_{0+}f)(x) &= (D^{l}_{0+}f)(x)\\ &= \left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-l)}\int_{0}^{x}\frac{f(t)dt}{(x-t)^{l-n+1}},\\ &(x>0,\ n=[\Re(l)]+1, l\in\mathbb{C}, \Re(l)\geq 0),\end{aligned} \tag{12}$$

$$\begin{aligned}(D^{l,-l,\xi}_{0^-}f)(x) &= (D^{l}_{-}f)(x)\\ &= \left(-\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-l)}\int_{x}^{\infty}\frac{f(t)dt}{(t-x)^{l-n+1}},\\ &(x>0,\ n=[\Re(l)]+1, l\in\mathbb{C}, \Re(l)\geq 0).\end{aligned} \tag{13}$$

Also the Saigo fractional derivative operators include the Erdélyi-Kober fractional derivative operators (Kiryakova [11]) as follows:

$$\begin{aligned}(D^{l,0,\xi}_{0+}f)(x) = (D^{+}_{\xi,l}f)(x) = \left(\frac{d}{dx}\right)^{n}(I^{-l+n,-l,-l+\xi-n}_{0+}f)(x),\\ (x>0,\ n=[\Re(l)]+1, l\in\mathbb{C}),\end{aligned} \tag{14}$$

$$\begin{aligned}(D^{l,0,\xi}_{0^-}f)(x) = (D^{-}_{\xi,l}f)(x) = \left(-\frac{d}{dx}\right)^{n}(I^{-l+n,-l,-l+\xi-n}_{-}f)(x),\\ (x>0,\ n=[\Re(l)]+1, l\in\mathbb{C}).\end{aligned} \tag{15}$$

Following the right-hand sided and left-hand sided Saigo fractional derivative of the type (14) and (15) for a power function are given by following lemmas (Saigo [20]):

Lemma 2.3. *Let $l,m,\xi,\lambda\in\mathbb{C}$, $\Re(l)\geq 0$, $x>0$ and $\Re(\lambda)>-\min[0,\Re(l+m+\xi)]$. Then*

$$D^{l,m,\xi}_{0+}\left(t^{\lambda-1}\right)(x) = \frac{\Gamma(\lambda)\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)}x^{\lambda+m-1}. \tag{16}$$

Lemma 2.4. *Let $l,m,\xi,\lambda\in\mathbb{C}$, $\Re(l)\geq 0$, $x>0$ and $\Re(\lambda)<1+\min[\Re(-m-n),\Re(l+\xi)]$. Then*

$$D^{l,m,\eta}_{0^-}\left(t^{\lambda-1}\right)(x) = \frac{\Gamma(1-\lambda-m)(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}x^{\lambda+m-1}. \tag{17}$$

2.1. Fractional integral formulae involving generalized incomplete τ-hypergeometric function

The fractional integral formulae for the generalized incomplete τ-hypergeometric function are given by the following results:

Theorem 2.5. *Let* $t > 0,\ k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > \max\{0, \Re(m - \xi)\}$. *Then, the following fractional integral formula holds*:

$$\begin{aligned} & I_{0,t}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;t)\right\}(x) \\ & = x^{\lambda-m-1}\frac{\Gamma(\lambda)\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\Gamma(\lambda+l+\xi)} \\ & \quad \times \left\{{}_2\Gamma_1^{\tau}((a,k),b;c;t)(x) * {}_2F_2(\lambda, \lambda-m+\xi; \lambda-m, \lambda+l+\xi; x)\right\}. \end{aligned} \tag{18}$$

Proof. For convenience, we denote the left-hand side of the result (18) by $\mathfrak{L}$. Using (2), we find

$$\mathfrak{L} = I_{0,t}^{l,m,\xi}[t^{\lambda-1}{}_2\Gamma_1^{\tau}((a;k),b;c;t)](x),$$

$$\mathfrak{L} = I_{0,t}^{l,m,\xi}[t^{\lambda-1}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(t)^n}{n!}](x),$$

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}I_{0,t}^{l,m,\xi}[t^{\lambda+n-1}](x).$$

Now using (8), we get

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}\frac{\Gamma(\lambda+n)\Gamma(\lambda+n-m+\xi)}{\Gamma(\lambda+n-m)\Gamma(\lambda+n+l+\xi)}x^{\lambda+n-m-1}.$$

Now, using (1), we get

$$\mathfrak{L} = x^{\lambda-m-1}\frac{\Gamma(\lambda)\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\Gamma(\lambda+l+\xi)}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{x^n}{n!}\frac{(\lambda)_n\,(\lambda-m+\eta)_n}{(\lambda-m)_n\,(\lambda+l+\xi)_n}.$$

Now, using equation (2) and applying Hadamard product series yields the desired result (18). ☐

Theorem 2.6. *Let* $t > 0,\ k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > \max\{0, \Re(m - \xi)\}$. *Then the following fractional integral formula holds*:

$$\begin{aligned} & I_{0,t}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;t)\right\}(x) \\ & = x^{\lambda-m-1}\frac{\Gamma(\lambda)\Gamma(\lambda-m+\xi)}{\Gamma(\lambda-m)\Gamma(\lambda+l+\xi)} \\ & \quad \times \left\{{}_2\gamma_1^{\tau}((a,k),b;c;t)(x) * {}_2F_2(\lambda, \lambda-m+\xi; \lambda-m, \lambda+l+\xi; x)\right\}. \end{aligned}$$

Theorem 2.7. *Let* $t > 0,\ k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > 1 + \min\{\Re(\xi), \Re(m)\}$.

Then the following fractional integral formula holds:

(19)

$$J_{z,\infty}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_2F_2(1-\lambda-m,\right.$$
$$\left. 1-\lambda+l+\xi;1-\lambda,1-\lambda+\xi-m;\frac{1}{x})\right\}.$$

Proof. As in the proof of Theorem 2.7, taking the operator (7) and the result (9) into account, one can easily prove result (19). □

Theorem 2.8. *Let* $t>0$*,* $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$ *and* $\Re(\lambda)>1+\min\{\Re(\xi),\Re(m)\}$. *Then the following fractional integral formula holds*:

$$J_{t,\infty}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_2F_2(1-\lambda-m,\right.$$
$$\left. 1-\lambda+l+\xi;1-\lambda,1-\lambda+\xi-m;\frac{1}{x})\right\}.$$

Setting $m=0$ in Theorem 2.5 and Theorem 2.7, yield the results asserted by the following corollaries.

Corollary 2.9. *Let* $t>0$*,* $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$ *and* $\Re(\lambda)>\Re(-\xi)$. *Then the following Erdélyi-Kober fractional integral formula holds*:

$$E_{0,t}^{l,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= x^{\lambda-1}\frac{\Gamma(\lambda+\xi)}{\Gamma(l+\lambda+\xi)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda+\eta;\lambda+l+\xi;x)\right\}.$$

Corollary 2.10. *Let* $t>0$*,* $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$ *and* $\Re(\lambda)>\Re(-\xi)$. *Then the following Erdélyi-Kober fractional integral formula holds*:

$$E_{0,t}^{l,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= x^{\lambda-1}\frac{\Gamma(\lambda+\xi)}{\Gamma(l+\lambda+\xi)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda+\xi;\lambda+l+\xi;x)\right\}.$$

Corollary 2.11. *Let* $t>0$*,* $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ *and* $l,m,\xi\in\mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda),\Re(l)\}>0$ *and* $\Re(\lambda)<1+\Re(\xi)$. *Then the following fractional integral formula holds*:

$$K_{t,\infty}^{l,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$

$$= t^{\lambda-1}\frac{\Gamma(1-\lambda+\xi)}{\Gamma(1+l-\lambda-\xi}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1(1-\lambda+\eta;1-\lambda+l+\xi;\frac{1}{x})\right\}.$$

Corollary 2.12. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $l,m,\xi\in\mathbb{C}$ be parameters such that $\min\{\Re(\lambda),\Re(l)\}>0$ and $\Re(\lambda)<1+\Re(\xi)$. Then the following fractional integral formula holds:*

$$K_{t,\infty}^{l,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= z^{\lambda-1}\frac{\Gamma(1-\lambda+\xi)}{\Gamma(1+l-\lambda-\xi}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1(1-\lambda+\xi;1-\lambda+l+\xi;\frac{1}{x})\right\}.$$

By replacing m by $-l$ in Theorem 2.5 and Theorem 2.7, we obtain the Riemann-Liouville fractional integrals of the generalized incomplete τ-hypergeometric type functions given by the corollaries.

Corollary 2.13. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $l,m,\xi\in\mathbb{C}$ be parameters such that $\min\{\Re(\lambda),\Re(l)\}>0$. Then the following Riemann-Liouville fractional integral formula holds*:

$$R_{0,t}^{l}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= t^{\lambda+l-1}\frac{\Gamma(\lambda)}{\Gamma(\lambda+l)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda;\lambda+l,;x)\right\}.$$

Corollary 2.14. *Let $z>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $l,m,\xi\in\mathbb{C}$ be parameters such that $\min\{\Re(\lambda),\Re(l)\}>0$. Then the following Riemann-Liouville fractional integral formula holds*:

$$R_{0,t}^{l}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= t^{\lambda+l-1}\frac{\Gamma(\lambda)}{\Gamma(\lambda+l)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda;\lambda+l,;x)\right\}.$$

Corollary 2.15. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $l,m,\xi\in\mathbb{C}$ be parameters such that $\min\{\Re(\lambda),\Re(l)\}>0$. Then the following Weyl fractional integral formula holds*:

$$W_{x,\infty}^{l}\left[t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right](x)$$
$$= x^{\lambda+l-1}\frac{\Gamma(1-\lambda-l)}{\Gamma(1-\lambda)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1\left[\begin{array}{c}(1-\lambda-l),\\(1-\lambda),\end{array};\frac{1}{x}\right]\right\}.$$

Corollary 2.16. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $l,m,\xi\in\mathbb{C}$ be parameters such that $\min\{\Re(\lambda),\Re(l)\}>0$. Then the following Weyl fractional integral formula holds*:

$$W_{x,\infty}^{l}\left[t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right](x)$$

$$= x^{\lambda+l-1}\frac{\Gamma(1-\lambda-l)}{\Gamma(1-\lambda)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1\left[\begin{array}{c}(1-\lambda-l),\\(1-\lambda),\end{array};\frac{1}{x}\right]\right\}.$$

2.2. Fractional derivative formulae involving generalized incomplete τ-hypergeometric function

The fractional derivative formulae for the generalized incomplete τ-hypergeometric function are given by the following results:

Theorem 2.17. *Let* $t > 0,\ k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(m)\} > 0$ *and* $\Re(\lambda) > -\min\{0, \Re(l+m+\xi)\}$. *Then the following fractional derivative formula holds*:

(20)

$$D_{0+}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(\lambda)\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;x) * {}_2F_2(\lambda, \lambda+l+m+\xi;\right.$$
$$\left.\lambda+m, \lambda+\xi; x)\right\}.$$

Proof. For convenience, we denote the left-hand side of the result (20) by $\mathfrak{L}$. Using (2) and changing order of integration and summation, we find

$$\mathfrak{L} = D_{0+}^{l,m,\xi}[t^{\lambda-1}{}_2\Gamma_1^{\tau}((a;k),b;c;t)](x),$$
$$\mathfrak{L} = D_{0+}^{l,m,\xi}[t^{\lambda-1}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(t)^n}{n!}](x),$$
$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}D_{0^+}^{l,m,\xi}[z^{\lambda+n-1}](x).$$

Now, using (16), we get

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}\frac{\Gamma(\lambda+n)\Gamma(\lambda+n+l+m+\xi)}{\Gamma(\lambda+n+m)\Gamma(\lambda+n+\xi)}x^{\lambda+n+m-1}.$$

Now, using (1), we get

$$\mathfrak{L} = x^{\lambda+m-1}\frac{\Gamma(\lambda)\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{x^n}{n!}\frac{(\lambda)_n\,(\lambda+l+m+\xi)_n}{(\lambda+m)_n\,(\lambda+\xi)_n},$$

using equation (2) and applying the Hadamard product series yields the desired result (20). □

Theorem 2.18. *Let* $t > 0,\ k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > -\min\{0, \Re(l+m+\xi)\}$. *Then the following fractional derivative formula holds*:

$$D_{0+}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(\lambda)\Gamma(\lambda+l+m+\xi)}{\Gamma(\lambda+m)\Gamma(\lambda+\xi)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;x) * {}_2F_2(\lambda, \lambda+l+m+\xi;\right.$$
$$\left.\lambda+m, \lambda+\xi; x)\right\}.$$

Theorem 2.19. *Let $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \min[\Re(-m - \xi), \Re(l + \xi)]$. *Then the following fractional derivative formula holds*:

(21)
$$D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\Big\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_2F_2(1-\lambda-m,$$
$$1-\lambda+l+\xi; 1-\lambda, 1-\lambda+\xi-m; \frac{1}{x})\Big\}.$$

Proof. For convenience, we denote the left-hand side of the result (21) by $\mathfrak{L}$. Using (2) and changing order of integration and summation, we find
$$\mathfrak{L} = D_{0-}^{l,m,\xi}[t_2^{\lambda-1}\Gamma_1^{\tau}((a;k),b;c;\frac{1}{t})](x),$$
$$\mathfrak{L} = D_{0-}^{l,m,\xi}[t^{\lambda-1}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{t^n\,n!}](x),$$
$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}D_{0-}^{l,m,\xi}[t^{\lambda-n-1}](x).$$

Now, using (17), we get
$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{n!}\frac{\Gamma(1-\lambda+n-m)\Gamma(1-\lambda+n+l+\xi)}{\Gamma(1-\lambda+n)\Gamma(1-\lambda+n+\xi-m)}x^{\lambda-n+m-1}.$$

Now, using (1), we get
$$\mathfrak{L} = x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{1}{x^n\,n!}\frac{(1-\lambda-m)_n\,(1-\lambda+l+\xi)_n}{(1-\lambda)_n\,(1-\lambda+\xi-m)_n},$$

using equation (2) and applying the Hadamard product series yields the desired result (21). □

Theorem 2.20. *Let $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \min[\Re(-m - \xi), \Re(l + \xi)]$. *Then the following fractional derivative formula holds*:
$$D_{0-}^{l,m,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda+m-1}\frac{\Gamma(1-\lambda-m)\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda)\Gamma(1-\lambda+\xi-m)}\Big\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_2F_2(1-\lambda-m,$$
$$1-\lambda+l+\xi; 1-\lambda, 1-\lambda+\xi-m; \frac{1}{x})\Big\}.$$

Setting $m = 0$ in Theorem 2.17 and Theorem 2.19, yield the results asserted by the following corollaries.

Corollary 2.21. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > -\min\{0, \Re(\xi)\}$. *Then the right-hand side Erdélyi-Kober fractional derivative of the generalized incomplete* τ*-hypergeometric type functions is given by*

$$D_{0+}^{l,0,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= t^{\lambda-1}\frac{\Gamma(\lambda+l+\xi)}{\Gamma(\lambda+\xi)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda+l+\xi;\lambda+\xi;x)\right\}.$$

Corollary 2.22. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) > -\min\{0, \Re(\xi)\}$. *Then the right-hand side Erdélyi-Kober fractional derivative of the generalized incomplete* τ*-hypergeometric type functions is given by*

$$D_{0+}^{l,0,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= t^{\lambda-1}\frac{\Gamma(\lambda+l+\xi)}{\Gamma(\lambda+\xi)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda+l+\xi;\lambda+\xi;x)\right\}.$$

Corollary 2.23. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \min\{\Re(-\xi), \Re(l+\xi)\}$. *Then the left-hand side of Erdélyi-Kober fractional derivative of the generalized incomplete* τ*-hypergeometric type functions is given by*

$$D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda-1}\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1(1-\lambda+l+\xi;1-\lambda+\xi;\frac{1}{x})\right\}.$$

Corollary 2.24. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \min\{\Re(-\xi), \Re(l+\xi)\}$. *Then the left-hand side of Erdélyi-Kober fractional derivative of the generalized incomplete* τ*-hypergeometric type functions is given by*

$$D_{0-}^{l,0,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right\}(x)$$
$$= x^{\lambda-1}\frac{\Gamma(1-\lambda+l+\xi)}{\Gamma(1-\lambda+\xi)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1(1-\lambda+l+\xi;1-\lambda+\xi;\frac{1}{x})\right\}.$$

By replacing m with $-l$ in Theorems 2.17 and 2.19, and using the relations (12) and (13), we obtain the Riemann-Liouville fractional derivative of the generalized incomplete τ-hypergeometric type functions given by the following corollaries.

Corollary 2.25. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $l, m, \xi \in \mathbb{C}$ *be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$. *Then the right-hand side Riemann-Liouville fractional derivative of the generalized incomplete* τ*-hypergeometric type functions is given by*

$$D_{0+}^{l,-l,\xi}\left\{t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$

$$= x^{\lambda-l-1}\frac{\Gamma(\lambda)}{\Gamma(\lambda-l)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda;\lambda-l;x)\right\}.$$

Corollary 2.26. *Let $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$. *Then the Riemann-Liouville fractional derivative of the generalized incomplete τ-hypergeometric type functions is given by*

$$D_{0+}^{l,-l,\xi}\left\{t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;t)\right\}(x)$$
$$= x^{\lambda-l-1}\frac{\Gamma(\lambda)}{\Gamma(\lambda-l)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;x) * {}_1F_1(\lambda;\lambda-l;x)\right\}.$$

Corollary 2.27. *Let $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \Re(l)$. *Then the Weyl fractional derivative of the generalized incomplete τ-hypergeometric type function is given by*

$$D_{0-}^{l,-l,\xi}\left[t^{\lambda-1}{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right](x)$$
$$= x^{\lambda+l-1}\frac{\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\left\{{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1\left[\begin{array}{c} 1-\lambda+l, \\ 1-\lambda, \end{array};\frac{1}{x}\right]\right\}.$$

Corollary 2.28. *Let $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ and $l, m, \xi \in \mathbb{C}$ be parameters such that* $\min\{\Re(\lambda), \Re(l)\} > 0$ *and* $\Re(\lambda) < 1 + \Re(l)$. *Then the Weyl fractional derivative of the generalized incomplete τ-hypergeometric type function is given by*

$$D_{0-}^{l,-l,\xi}\left[t^{\lambda-1}{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{t})\right](x)$$
$$= x^{\lambda+l-1}\frac{\Gamma(1-\lambda+l)}{\Gamma(1-\lambda)}\left\{{}_2\gamma_1^{\tau}((a,k),b;c;\frac{1}{x}) * {}_1F_1\left[\begin{array}{c} 1-\lambda+l, \\ 1-\lambda, \end{array};\frac{1}{x}\right]\right\}.$$

3. Integral transform and generalized incomplete τ-hypergeometric function

The section deals with integral transforms formulae, which exhibit the connection between the Euler, Varma, Laplace and Whittaker integral transforms and the generalized incomplete τ-hypergeometric type function.

We begin by recalling the following beta transform of a function (Sneddon [21]):

$$B\{f(t) : l, m\} = \int_0^1 t^{l-1}(1-t)^{m-1}f(t)dt. \tag{22}$$

The Verma transform of a function $f(t)$ is defined by the following integral equation (Mathai *et al.* [13, p. 55]):

$$V(f,k,m;s) = \int_0^{\infty}(st)^{m-\frac{1}{2}}\exp(-\frac{1}{2}st)\ W_{k,m}(st)\ f(t)dt, (\Re(s) > 0), \tag{23}$$

where $W_{k,m}$ is the Whittaker function defined by (Mathai *et al.* [13, p. 55])

$$W_{k,m}(t) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2} - k - m)} M_{k,m}(t),$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$ is to be added and

$$M_{k,m}(t) = t^{m+\frac{1}{2}} \exp(-\frac{t}{2})\, {}_1F_1(\frac{1}{2} - k + m; 2m + 1; t).$$

It is interesting to observe that, for $k = -\nu + \frac{1}{2}$, the Verma transform defined by(23) reduces to the well-known Laplace transform of a function $f(t)$ (Sneddon [21]):

$$L\{f(t) : s\} = \int_0^\infty e^{-st} f(t)\, dt.$$

The following results exhibit the connection between the Euler, Varma, Laplace and Whittaker integral transforms and the generalized incomplete τ-hypergeometric type function.

Theorem 3.1. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$. *Then the following beta transform formula holds*:

$$\begin{aligned} & B\{{}_2\Gamma_1^\tau((a,k), b; c; yt); l, m\} \\ & = B(l,m)\{{}_2\Gamma_1^\tau((a,k), b; c; y) *_1 F_1(l; l+m; y)\}, (|y| < 1). \end{aligned} \tag{24}$$

Further, it is assumed that the involved Euler (*Beta*) *transforms* (22) *of* ${}_2\Gamma_1^\tau((a,k), b; c; y)$ *exist.*

Proof. Applying Euler (beta) transform (22) on ${}_2\Gamma_1^\tau((a,k), b; c; yz)$, we get

$$\mathfrak{L} = \int_0^1 t^{l-1}(1-t)^{m-1}\, {}_2\Gamma_1^\tau((a,k), b; c; yt) dt,$$

$$\mathfrak{L} = \int_0^1 t^{l-1}(1-t)^{m-1} \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;k]_n\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(yt)^n}{n!} dt.$$

By changing the order of integration and summation using beta integral, we get

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;k]_n\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(y)^n}{n!} \int_0^1 t^{n+l-1}(1-t)^{m-1} dt,$$

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;k]_n\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(y)^n}{n!} B(l+n, m),$$

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;k]_n\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(y)^n}{n!} \frac{\Gamma(l+n)\Gamma(m)}{\Gamma(l+m+n)},$$

$$\mathfrak{L} = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} \frac{\Gamma(c)}{\Gamma(b)} \sum_{n=0}^\infty \frac{[a;k]_n\, \Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{(y)^n}{n!} \frac{\Gamma(l+n)\Gamma(l+m)}{\Gamma(l+m+n)\Gamma(l)},$$

$$\mathfrak{L} = B(l,m)\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(y)^n}{n!}\frac{(l)_n}{(l+m)_n}.$$

Now, using Hadamard product series, leads to the right hand side of (24). □

Theorem 3.2. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$. Then the following beta transform formula holds*:

$$\begin{aligned}&B\{{}_2\gamma_1^{\tau}((a,k),b;c;yt);l,m\}\\ &= B(l,m)\{{}_2\gamma_1^{\tau}((a,k),b;c;y) *_1 F_1(l;l+m;y)\},(|y|<1).\end{aligned}$$

Further, it is assumed that the involved Euler (Beta) transforms (22) of ${}_2\Gamma_1^{\tau}((a,k),b;c;y)$ exist.

Theorem 3.3. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $\Re(l)>0$. Then the following Verma transform formula holds true*:

(25)

$$\begin{aligned}&V\{t^{l-1}\,{}_2\Gamma_1^{\tau}((a,k),b;c;yt);s\}\\ &=\frac{1}{s^l}\frac{\Gamma(l)\Gamma(2m+l)}{\Gamma(m+l-k+\frac{1}{2})}\left[{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{y}{s}) * {}_2F_1(2m+l,l;m+l-k+\frac{1}{2};\frac{y}{s})\right],\end{aligned}$$

where both sides of (25) are assumed to exist.

Proof. Let $\mathfrak{L}$ be the left-hand side of (25). Then

$$\mathfrak{L} = \int_0^{\infty}(st)^{m-\frac{1}{2}}\exp(-\frac{1}{2}sz)W_{k,m}(sz).t^{l-1}\,{}_2\Gamma_1^{\tau}((a,k),b;c;yt)dt,$$

$$\mathfrak{L} = s^{m-\frac{1}{2}}\int_0^{\infty}t^{(m+l-\frac{1}{2})-1}\exp(-\frac{1}{2}st)\,W_{k,m}(st)\frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(yt)^n}{n!}.$$

By changing the order of integration and summation which may be verified under the conditions, we obtain

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(y)^n}{n!}s^{m-\frac{1}{2}}\int_0^{\infty}t^{(m+l-\frac{1}{2})-1}\exp(-\frac{1}{2}sz)\,W_{k,m}(st)dt,$$

$$\mathfrak{L} = \frac{\Gamma(c)}{\Gamma(b)}\sum_{n=0}^{\infty}\frac{[a;k]_n\,\Gamma(b+\tau n)}{\Gamma(c+\tau n)}\frac{(y)^n}{n!}\left\{\frac{1}{s^{l+m}}\frac{\Gamma(2m+l+n)\Gamma(l+n)}{\Gamma(\frac{1}{2}-k+m+l+n)}\right\}.$$

Now, using Hadamard product series, leads to the right hand side of (25). □

Theorem 3.4. *Let $t>0$, $k\geq 0;\tau>0;\Re(c)>\Re(b)>0$ and $\Re(l)>0$. Then the following Verma transform formula holds true*:

(26)

$$\begin{aligned}&V\{t^{l-1}\,{}_2\gamma_1^{\tau}((a,k),b;c;yt);s\}\\ &=\frac{1}{s^l}\frac{\Gamma(l)\Gamma(2m+l)}{\Gamma(m+l-k+\frac{1}{2})}\left[{}_2\gamma_1^{\tau}((a,k),b;c;\frac{y}{s}) * {}_2F_1(2m+l,l;m+l-k+\frac{1}{2};\frac{y}{s})\right],\end{aligned}$$

where both sides of (26) are assumed to exist.

Laplace transform is the special case of Verma transform. In fact, we have an interesting Laplace transform asserted by the following corollaries.

Corollary 3.5. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $\Re(l) > 0$. *Then the following Laplace transform formula holds true*:

$$L\{t^{l-1}\,{}_2\Gamma_1^{\tau}((a,k),b;c;yt);s\} = \frac{\Gamma(l)}{s^l}\left[{}_2\Gamma_1^{\tau}((a,k),b;c;\frac{y}{s}) * {}_1F_0(l;-;\frac{y}{s})\right], \tag{27}$$

where both sides of (27) *are assumed to exist.*

Corollary 3.6. *Let* $t > 0$, $k \geq 0; \tau > 0; \Re(c) > \Re(b) > 0$ *and* $\Re(l) > 0$. *Then the following Laplace transform formula holds true*:

$$L\{t^{l-1}\,{}_2\gamma_1^{\tau}((a,k),b;c;yt);s\} = \frac{\Gamma(l)}{s^l}\left[{}_2\gamma_1^{\tau}((a,k),b;c;\frac{y}{s}) * {}_1F_0(l;-;\frac{y}{s})\right], \tag{28}$$

where both sides of (28) *are assumed to exist.*

Remark 3.7. It is noted that the all above results also can be obtained for incomplete hypergeometric functions (4) and (5) given by Srivastava *et al.* [23]. Fractional integral of incomplete hypergeometric functions (4) and (5) is given by Srivastava and Agarwal [22].

4. Concluding remarks

The generalized τ-hypergeometric type functions (2) and (3) have the advantage that most of the known and widely-investigated special functions are expressible in terms of the generalized incomplete τ-hypergeometric functions. Therefore, we conclude this paper with the remark that, the results deduced above are significant and can lead to yield numerous fractional calculus and integral transform formulae involving various special functions by the suitable specializations of arbitrary parameters in the results.

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References

[1] P. Agarwal, M. Chand, and E. T. Karimov, *Certain image formulas of genearlized hypergeometric functions*, Appl. Math. Comput. **266** (2015), 763–772.

[2] P. Agarwal and J. Choi, *Fractional calculus operators and their image formulas*, J. Korean Math. Soc. **53** (2016), no. 5, 1183–1210.

[3] P. Agarwal, J. Choi, K. B. Kachhia, J. C. Prajapati, and H. Zhou, *Some Integral Transforms and Fractional Integral Formulas for the Extended Hypergeometric Functions*, Commun. Korean Math. Soc. **31** (2016), no. 3, 591–601.

[4] A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Physics*, Springer, Vienna, 1997.

[5] J. Choi, K. B. Kachhia, J. C. Prajapati, and S. D. Purohit, *Some integral transforms involving extended generalized Gauss hypergeometric functions*, Commun. Korean Math. Soc. **31** (2016), no. 4, 779–790.

[6] L. Galue, A. Al-Zamel, and S. L. Kalla, *Further results on generalized hypergeometric functions*, Appl. Math. Comput. **136** (2003), no. 1, 17–25.

[7] R. Hilfer, *Applications of Fractional Calculus in Physics*, World scientific, Singapore, 2000.
[8] I. S. Jesus and J. A. T Machado, *Fractional control of heat diffusion systems*, Nonlinear Dynam. **54** (2008), no. 3, 263–282.
[9] K. B. Kachhia and J. C. Prajapati, *Solution of fractional partial differential equation aries in study of heat transfer through diathermanous material*, J. Interdisciplinary Math. **18** (2015), no. 1-2, 125–132.
[10] ———, *On generalized fractional kinetic equations involving generalized Lommel-Wright functions*, Alexandria Engineering J. **55** (2016), 2953–2957.
[11] V. Kiryakova, *Generalized Fractional Calculus and Applications*, Wiley and Sons Inc., New York, 1994.
[12] ———, *On two Saigo's fractional integral operators in the class of univalent functions*, Fract. Cal. Appl. Anal. **9** (2006), no. 2, 161–176.
[13] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Functions: Theory and Applications*, Springer, New York, 2010.
[14] R. K. Parmar and R. K. Saxena, *The incomplete generalized τ-hypergeometric and second τ-Appell functions*, J. Korean Math. Soc. **53** (2016), no. 2, 363–379.
[15] I. J. Podulbuny, *Fractional Differential Equations*, Academic Press, New York, 1999.
[16] T. Pohlen, *The Hadamard Product and Universal Power Series*, Dissertation, Universitåt Trier, 2009.
[17] J. C. Prajapati and K. B. Kachhia, *Fractional modeling of temperature distribution and heat flux in the semi infinite solid*, J. Fract. Cal. Appl. **5** (2014), no. 2, 38–43.
[18] J. C. Prajapati, K. B. Kachhia, and S. P. Kosta, *Fractional Calculus Approach to Study Temperature Distribution Within a Spinning Satellite*, Alexandria Engineering J. **55** (2016), 2345–2350.
[19] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx-New York, 1971.
[20] M. Saigo, *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. College of General Edu. Kyushu University **1** (1978), 135–143.
[21] I. N. Sneddon, *The Use of Integral Transform*, Tata McGraw-Hill, New Delhi, India, 1979.
[22] H. M. Srivastava and R. P. Agarwal, *Certain Fractional Integral Operators and the Generalized Incomplete Hypergeometric Functions*, Int. Appl. Math. **8** (2013), no. 2, 333–345.
[23] H. M. Srivastava, M. A. Chaudhary, and R. P. Agarwal, *The incomplete Pochhammer symbols and their applications to hypergeometric and relative functions*, Integral Transforms Spec. Funct. **23** (2012), no. 9, 659–683.
[24] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science, Amsterdam, The Netherlands, 2012.
[25] N. Virchenko, S. L. Kalla, and A. Al-Zamel, *Some results on a generalized hypergeometric function*, Integral Transforms Spec. Funct. **12** (2001), no. 1, 89–100.

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